Counting Branches in Trees Using Games

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Abstract

We study finite automata running over infinite binary trees. A run of such an automaton is usually said to be accepting if all its branches are accepting. In this article, we relax the notion of accepting run by allowing a certain quantity of rejecting branches. More precisely we study the following criteria for a run to be accepting:

(i) it contains at most finitely (resp. countably) many rejecting branches;

(ii) it contains infinitely (resp. uncountably) many accepting branches;

(iii) the set of accepting branches is topologically "big".

In all situations we provide a simple acceptance game that later permits to prove that the languages accepted by automata with cardinality constraints are always ω-regular. In the case (ii) where one counts accepting branches it leads to new proofs (without appealing to logic) of a result of Beauquier and Niwiński.

Keywords: Automaton on Infinite Trees, Two-Player Game, Cardinality Constraint, Topologically Large Set

1. Introduction

There are several natural ways of describing sets of infinite trees. One is *logic* where, with any formula, one associates the set of all trees for which the formula holds. Another option is using *finite automata*. Finite automata on infinite trees (that extends both automata on infinite words and on finite trees) were originally introduced by Rabin in [1] to prove the decidability of the monadic second order logic (MSOL) over the full binary tree. Indeed, Rabin proved that for any MSOL formula, one can construct a tree automaton such that it accepts a non empty language if and only if the original formula holds at the root of the full binary tree. These automata were also successfully used by Rabin in [2] to solve Church's synthesis problem [3], that asks for constructing a circuit based on a formal specification (typically expressed in MSOL) describing the desired input/output behaviour. His approach was to represent the set of all possible behaviours of a circuit by an infinite tree (directions code the inputs while node labels along a branch code the outputs) and to reduce the synthesis problem to emptiness of a tree automaton accepting all those trees coding circuits satisfying the specification. Since then, automata on infinite trees and their variants have been intensively studied and found many applications, in particular in logic. Connections between automata on infinite trees and logic are discussed *e.g.* in the excellent surveys [4, 5].

Roughly speaking a finite automaton on infinite trees is a finite memory machine that takes as input an infinite node-labelled binary tree and processes it in a top-down fashion as follows. It starts at the root of the tree in its initial state, and picks (possibly nondeterministically) two successor states, one per child, according to the current control state, the letter at the current node and the transition relation. Then the computation proceeds in parallel from both children, and so on. Hence, a run of the automaton on an input tree is a

labelling of this tree by control states of the automaton, that should satisfy the local constraints imposed by the transition relation. A branch in a run is accepting if the ω-word obtained by reading the states along the branch satisfies some acceptance condition (typically an ω-regular condition such as a Büchi or a parity condition). Finally, a tree is accepted by the automaton if there exists a run over this tree in which *every* branch is accepting. An ω-regular tree language is a tree language accepted by some tree automaton equipped with a parity condition.

A fundamental result of Rabin is that ω-regular tree languages form a Boolean algebra [1]. The main technical difficulty in establishing this result is to show the closure under complementation. Since the publication of this result in 1969, it has been a challenging problem to simplify this proof. A much simpler one was obtained by Gurevich and Harrington in [6] making use of two-player perfect information games for checking membership of a tree in the language accepted by the automaton[3]: Éloïse (a.k.a. *Automaton*) builds a run on the input tree while Abélard (a.k.a. *Pathfinder*) tries to exhibit a rejecting branch in the run. Another fruitful connection between automata and games is for emptiness checking. In a nutshell the emptiness problem for an automaton on infinite trees can be modelled as a game where Éloïse builds an input tree together with a run while Abélard tries to exhibit a rejecting branch in the run. Hence, the emptiness problem for tree automata can be reduced to solving a two-player parity game played on a *finite* graph. Beyond these results, the tight connection between automata and games is one of the main tools in automata theory [4, 8, 9].

There are several levers on which one can act to define alternative families of tree automata / classes of tree languages. A first lever is *local* with respect to the run: it is the condition required for a branch to be accepting, the reasonable options here being all classical ω-regular conditions (reachability, Büchi, parity...). A second one has to do with the set of runs. The usual definition is existential: a tree is accepted if there exists an accepting run on that tree. Other popular approaches are universality, alternation or probabilistic transition functions. A third lever is *global* with respect to the run: it is the condition required for a run to be accepting. The usual definition is that *all* branches must be accepting for the run to be accepting but one could relax this condition by specifying *how many* branches should be accepting/rejecting. One can do this either by *counting* the number of accepting branches (*e.g.* infinitely many, uncountably many) or by counting the number of rejecting branches (*e.g.* finitely many, at most countably many): this leads to the notion of automata with cardinality constraints [10, 11]. As these properties can be expressed in MSOL [12], the classes of languages accepted under these various restrictions are always ω-regular. However, this logical approach does not give a tractable transformation to standard parity or Büchi automata. Another option is to use a notion of topological "bigness" and to require for a run to be accepting that the set of accepting branches is big [13, 14]. Yet another option considered in [15, 16, 17] is to *measure* (in the usual sense of measure theory) the set of accepting branches and to put a constraint on this measure (*e.g.* positive, equal to one).

The idea of allowing a certain amount of rejecting branches in a run was first considered by Beauquier, Nivat and Niwiński in [10, 11], where it was required that the number of accepting branches in a run belongs to a specified set of cardinals Γ. In particular, they proved that if Γ consists of all cardinals greater than some γ, then one obtains an ω-regular tree language. Their approach was based on logic (actually they proved that a tree language defined by such an automaton can be defined by a Σ_1^1 formula hence, can also be defined by a *Büchi* tree automaton) while the one we develop here is based on designing acceptance games. There is also work on the logical side with decidability results but that do not lead to efficient algorithms [12].

Our main contributions are to introduce (automata with cardinality constraints on the number of rejecting branches; automata with topological bigness constraints) or revisit (automata with cardinality constraints on the number of accepting branches) variants of tree automata where acceptance for a run allows a somehow negligible set of rejecting branches. For each model, we provide a game counterpart by means of an equivalent acceptance game and this permits to retrieve the classical (and fruitful) connection between automata and game. It also permit to argue that languages defined by those classes are always ω-regular. Moreover, in the case where one counts accepting branches we show that the languages that we obtain are always accepted by a *Büchi* automaton, which contrasts with the case where one counts rejecting branches where we exhibit a counter-example for that property.

The paper is organised as follows. Section 2 recalls classical concepts while Section 3 introduces the main

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[3]Note that the idea of using games to prove this result was already proposed by Büchi in [7].

notions studied in the paper, namely automata with cardinality constraints and automata with topological bigness constraints. Then, Section 4 studies those languages obtained by automata with cardinality constraints on the number of rejecting branches while Section 5 is devoted to those languages obtained by automata with cardinality constraints on the number of accepting branches. Finally, Section 6 considers automata with topological bigness constraints.

2. Preliminaries

2.1. Words and Trees

An **alphabet** A is a (possibly infinite) set of letters. In the sequel A^* denotes the set of finite words over A, and A^ω the set of infinite words (or ω-words) over A. The **empty word** is written ε. The length of a word $u \in A^*$ is denoted by $|u|$. For any $k \geqslant 0$, we let $A^k = \{u \mid |u| = k\}$, $A^{\leqslant k} = \{u \mid |u| \leqslant k\}$ and $A^{\geqslant k} = \{u \mid |u| \geqslant k\}$. We let $A^+ = A^* \backslash \{\varepsilon\}$.

Let u be a finite word and v be a (possibly infinite) word. Then $u \cdot v$ (or simply uv) denotes the **concatenation** of u and v; the word u is a **prefix** of v, denoted $u \sqsubseteq v$, if there exists a word w such that $v = u \cdot w$. We denote by $u \sqsubset v$ the fact that u is a strict prefix of v (*i.e.* $u \sqsubseteq v$ and $u \neq v$). When u is a prefix of v we let $u^{-1}v$ denote the unique word w such that $v = uw$. For some finite word u and some integer $k \geqslant 0$, we denote by u^k the word obtained by concatenating k copies of u (with the convention that $u^0 = \varepsilon$).

In this paper we consider full binary node-labelled trees. Let A be an alphabet, then an **A-labelled tree** t is a (total) function from $\{0,1\}^*$ to A. In this context, an element $u \in \{0,1\}^*$ is called a **node**, and the node $u \cdot 0$ (*resp.* $u \cdot 1$) is the **left child** (*resp.* **right child**) of u. The node ε is called the **root**. The letter $t(u)$ is called the **label** of u in t.

A **branch** is an infinite word $\pi \in \{0,1\}^\omega$ and a node u belongs to a branch π if u is a prefix of π. For an A-labelled tree t and a branch $\pi = \pi_0 \pi_1 \cdots$ we define the **label** of π as the ω-word $t(\pi) = t(\varepsilon)t(\pi_0)t(\pi_0\pi_1)t(\pi_0\pi_1\pi_2)\cdots$.

2.2. Two-Player Perfect Information Turn-Based Games on Graphs

A **graph** is a pair $G = (V, E)$ where V is a (possibly infinite) set of **vertices** and $E \subseteq V \times V$ is a set of **edges**. For a vertex v, we denote by $E(v) = \{v' \mid (v, v') \in E\}$ the set of successors of v in G. In the rest of the paper (hence, this is implicit from now on), we only consider graphs that have no dead-end, *i.e.* such that $E(v) \neq \varnothing$ for all v.

An **arena** is a triple $\mathcal{G} = (G, V_{\mathbf{E}}, V_{\mathbf{A}})$ where $G = (V, E)$ is a graph and $V = V_{\mathbf{E}} \uplus V_{\mathbf{A}}$ is a partition of the vertices among two players, Éloïse and Abélard.

Éloïse and Abélard play in $\mathcal{G}$ by moving a pebble along edges. A **play** from an initial vertex v_0 proceeds as follows: the player owning v_0 (*i.e.* Éloïse if $v_0 \in V_{\mathbf{E}}$, Abélard otherwise) moves the pebble to a vertex $v_1 \in E(v_0)$. Then the player owning v_1 chooses a successor $v_2 \in E(v_1)$ and so on. As we assumed that there is no dead-end, a play is an infinite word $v_0 v_1 v_2 \cdots \in V^\omega$ such that for all $i \geqslant 0$ one has $v_{i+1} \in E(v_i)$. A **partial play** is a prefix of a play, *i.e.*, it is a finite word $v_0 v_1 \cdots v_\ell \in V^*$ such that for all $0 \leqslant i < \ell$ one has $v_{i+1} \in E(v_i)$.

A **strategy** for Éloïse is a function $\varphi : V^* V_{\mathbf{E}} \to V$ assigning, to every partial play ending in some vertex $v \in V_{\mathbf{E}}$, a vertex $v' \in E(v)$. Strategies of Abélard are defined likewise, and usually denoted ψ. In a given play $\lambda = v_0 v_1 \cdots$ we say that Éloïse (resp. Abélard) **respects a strategy** φ (resp. ψ) if whenever $v_i \in V_{\mathbf{E}}$ (resp. $v_i \in V_{\mathbf{A}}$) one has $v_{i+1} = \varphi(v_0 \cdots v_i)$ (resp. $v_{i+1} = \psi(v_0 \cdots v_i)$).

A **winning condition** is a subset $\Omega \subseteq V^\omega$ and a (two-player perfect information) **game** is a pair $\mathbb{G} = (\mathcal{G}, \Omega)$ consisting of an arena and a winning condition.

A play λ is **won** by Éloïse if and only if $\lambda \in \Omega$; otherwise λ is won by Abélard. A strategy φ is **winning** for Éloïse in $\mathbb{G}$ from a vertex v_0 if any play starting from v_0 where Éloïse respects φ is won by her. Finally, a vertex v_0 is **winning** for Éloïse in $\mathbb{G}$ if she has a winning strategy φ from v_0. Winning strategies and winning vertices for Abélard are defined likewise.

We now define three classical winning conditions.

- A **Büchi** winning condition is of the form $(V^*F)^\omega$ for a set $F \subseteq V$ of final vertices, *i.e.* winning plays are those that infinitely often visit vertices in F.

- A **co-Büchi** condition is of the form $V^*(V \backslash F)^\omega$ for a set $F \subseteq V$ of forbidden vertices, *i.e.* winning plays are those that visit only finitely often forbidden vertices.

- A **parity** winning condition is defined by a *colouring* function Col that is a mapping $\text{Col} : V \to C \subset \mathbb{N}$ where C is a *finite* set of **colours**. The parity winning condition associated with Col is the set

$$\Omega_{\text{Col}} = \{v_0 v_1 \cdots \in V^\omega \mid \liminf(\text{Col}(v_i))_{i \geqslant 0} \text{ is even}\}$$

i.e. a play is winning if and only if the smallest colour infinitely often visited is even.

Finally, a Büchi (resp. co-Büchi, parity) game is one equipped with a Büchi (resp. co-Büchi, parity) winning condition. For notation of such games we often replace the winning condition by the object that is used to defined it (*i.e.* F or Col).

2.3. Tree Automata, Regular Tree Languages and Acceptance Game

A **tree automaton** $\mathcal{A}$ is a tuple $\langle A, Q, q_{ini}, \Delta, \text{Acc} \rangle$ where A is the **input alphabet**, Q is the finite set of **states**, $q_{\text{ini}} \in Q$ is the **initial state**, $\Delta \subseteq Q \times A \times Q \times Q$ is the **transition relation** and $\text{Acc} \subseteq Q^\omega$ is the **acceptance condition**. An automaton is **complete** if, for all $q \in Q$ and $a \in A$ there is at least one pair $(q_0, q_1) \in Q^2$ such that $(q, a, q_0, q_1) \in \Delta$. In this work we always assume that the automata are complete and this is implicit from now. Note that we will discuss the impact of this restriction in the conclusion.

Given an A-labelled tree t, a **run** of $\mathcal{A}$ over t is a Q-labelled tree ρ such that

(i) the root is labelled by the initial state, *i.e.* $\rho(\varepsilon) = q_{\text{ini}}$;

(ii) for all nodes u, $(\rho(u), t(u), \rho(u \cdot 0), \rho(u \cdot 1)) \in \Delta$.

A branch $\pi \in \{0, 1\}^\omega$ is **accepting** in the run ρ if $\rho(\pi) \in \text{Acc}$, otherwise it is **rejecting**. A run ρ is **accepting** if *all* its branches are accepting. Finally, a tree t is **accepted** if there exists an accepting run of $\mathcal{A}$ over t. The set of all trees accepted by $\mathcal{A}$ (or the language *recognised* by $\mathcal{A}$) is denoted $L(\mathcal{A})$.

In this work we consider the following three classical acceptance conditions:

- A **Büchi** condition is given by a subset $F \subseteq Q$ of final states by letting $\text{Acc} = \text{Büchi}(F) = (Q^* F)^\omega$, *i.e.* a branch is accepting if it contains infinitely many final states.

- A **co-Büchi** condition is given by a subset $F \subseteq Q$ of forbidden states by letting $\text{Acc} = \text{coBüchi}(F) = Q^*(Q \backslash F)^\omega$, *i.e.* a branch is accepting if it contains finitely many forbidden states.

- A **parity** condition is given by a colouring mapping $\text{Col} : Q \to \mathbb{N}$ by letting

$$\text{Acc} = \text{Parity}(\text{Col}) = \{q_0 q_1 q_2 \cdots \mid \liminf(\text{Col}(q_i))_i \text{ is even}\}$$

i.e. a branch is accepting if the smallest colour appearing infinitely often is even.

These conditions are all examples of ω-regular acceptance conditions, *i.e.* Acc is an ω-regular set of ω-words over the alphabet Q (see *e.g.* [18] for a reference book on languages of infinite words).

Remark 1. *The parity condition is expressive enough to capture the general case of an arbitrary ω-regular condition* Acc. *Indeed, it is well known that* Acc, *when it is ω-regular, is accepted by a* deterministic *parity word automaton. By taking the synchronised product of this automaton with the tree automaton, we obtain a parity tree automaton accepting the same language (see* e.g. *[18]).*

When it is clear from the context, we may replace, in the description of $\mathcal{A}$, *Acc* by F for Büchi/co-Büchi condition (resp. Col for a parity condition), and we shall refer to the automaton as a Büchi/co-Büchi (resp. parity) tree automaton. A set L of infinite trees is an **ω-regular** tree language if there exists a parity tree automaton $\mathcal{A}$ such that $L = L(\mathcal{A})$. The class of ω-regular tree languages is robust, as illustrated by the following famous statement [1].

Theorem 1. *The class of ω-regular tree languages is a Boolean algebra.*

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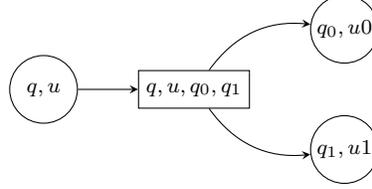
for any $(q, t(u), q_0, q_1) \in \Delta$

Figure 1: Local structure of the arena of the acceptance game $\mathbb{G}_{\mathcal{A},t}$.

Fix an automaton $\mathcal{A} = \langle A, Q, q_{\mathrm{ini}}, \Delta, \mathrm{Acc} \rangle$ and a tree t and define an acceptance game $\mathbb{G}_{\mathcal{A},t}$, *i.e.* a game where Éloïse wins if and only if there exists an accepting run of $\mathcal{A}$ on t, as follows.

Intuitively, a play in $\mathbb{G}_{\mathcal{A},t}$ consists in moving a pebble along a branch of t in a top-down manner: to the pebble is attached a state, and in a node u with state q, Éloïse picks a transition $(q, t(u), q_0, q_1) \in \Delta$, and then Abélard chooses to move down the pebble either to $u \cdot 0$ (and update the state to q_0) or to $u \cdot 1$ (and update the state to q_1).

Formally (see Figure 1 for an illustration[4]), let $G_{\mathcal{A},t} = (V_{\mathbf{E}} \uplus V_{\mathbf{A}}, E)$ with $V_{\mathbf{E}} = Q \times \{0,1\}^*$,

$$V_{\mathbf{A}} = \{(q, u, q_0, q_1) \mid u \in \{0,1\}^* \text{ and } (q, t(u), q_0, q_1) \in \Delta\} \subseteq Q \times \{0,1\}^* \times Q \times Q$$

and

$$E = \{((q,u), (q, u, q_0, q_1)) \mid (q, u, q_0, q_1) \in V_{\mathbf{A}}\} \cup \{((q, u, q_0, q_1), (u \cdot x, q_x)) \mid x \in \{0,1\} \text{ and } (q, u, q_0, q_1) \in V_{\mathbf{A}}\}$$

Then let $\mathcal{G}_{\mathcal{A},t} = (G_{\mathcal{A},t}, V_{\mathbf{E}}, V_{\mathbf{A}})$ and extend Col on $V_{\mathbf{E}} \cup V_{\mathbf{A}}$ by letting $\mathrm{Col}((q, u)) = \mathrm{Col}((q, u, q_0, q_1)) = \mathrm{Col}(q)$. Finally define $\mathbb{G}_{\mathcal{A},t}$ as the parity game $(\mathcal{G}_{\mathcal{A},t}, \mathrm{Col})$.

The next theorem is well-known (see *e.g.* [6, 8]) and its proof is obtained by remarking that strategies for Éloïse in $\mathbb{G}_{\mathcal{A},t}$ are in bijection with runs of $\mathcal{A}$ on t (with the winning strategies corresponding to the accepting runs).

Theorem 2. *One has* $t \in L(\mathcal{A})$ *if and only if Éloïse wins in* $\mathbb{G}_{\mathcal{A},t}$ *from* $(q_{\mathrm{ini}}, \varepsilon)$.

## 3. Automata with Cardinality Constraints and Automata with Topological Bigness Constraints

We now introduce the main notions studied in the paper, namely automata with cardinality constraints (studied in Section 4 and Section 5) and automata with topological bigness constraints (studied in Section 6).

3.1. Automata with Cardinality Constraints

We now relax the criterion for a run to be accepting. Recall that classically, a run is accepting if *every* branch in it is accepting. For a given automaton $\mathcal{A}$, we define the following four criteria (two for the case where one counts the number of accepting branches and two for the case where one counts the number of rejecting branches) for a run to be accepting. Note that the case where one counts accepting branches was already considered in [10, 11].

- *There are finitely many rejecting branches in the run.* A tree t is in $L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A})$ if and only if there is a run of $\mathcal{A}$ on t satisfying the previous condition.

- *There are at most countably many rejecting branches in the run.* A tree t is in $L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ if and only if there is a run of $\mathcal{A}$ on t satisfying the previous condition.

- *There are infinitely many accepting branches in the run.* A tree t is in $L_{\infty}^{\mathrm{Acc}}(\mathcal{A})$ if and only if there is a run of $\mathcal{A}$ on t satisfying the previous condition.

- *There are uncountably many accepting branches in the run.* A tree t is in $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$ if and only if there is a run of $\mathcal{A}$ on t satisfying the previous condition.

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[4]In pictures, we always depict by circles (resp. squares) the vertices controlled by Éloïse (resp. Abélard).
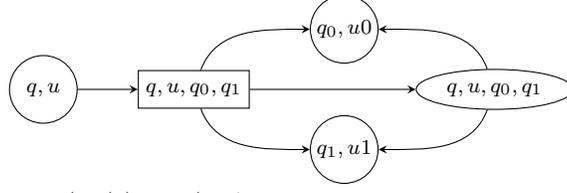
for any $(q, t(u), q_0, q_1) \in \Delta$

Figure 2: Local structure of $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$.

3.2. Automata with Topological Bigness Constraints

A notion of topological "bigness" and "smallness" is given by *large* and *meagre* sets respectively (see [19, 20] for a survey of the notion). The idea is to see the set of branches in a tree as a topological space by taking as basic open sets the *cones*. For a node $u \in \{0,1\}^*$, the cone Cone(u) is defined as $\{\pi \in \{0,1\}^\omega \mid u \sqsubseteq \pi\}$. A set of branches $B \subseteq \{0,1\}^\omega$ is **nowhere dense** if for all nodes u, there exists $v \in \{0,1\}^*$ such that no branch of B has uv as a prefix. It is **meagre** if it is the countable union of nowhere dense sets. Finally it is **large** if it is the complement of a meagre set.

For a given automaton $\mathcal{A}$, we define the following acceptance criterion: a run is accepting if and only if its set of accepting branches is large. Note that this is equivalent to require that the set of rejecting branches is meagre.

Finally, a tree t is in $L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$ if and only if there is a run of $\mathcal{A}$ on t satisfying the previous condition.

4. Counting Rejecting Branches

For the classes of automata where acceptance is defined by a constraint on the number of rejecting branches we show that the associated languages are ω-regular. For this, we adopt the following roadmap: first we design an acceptance game and then we note that it can be transformed into another *equivalent* game that turns out to be the (usual) acceptance game for some tree automaton.

Fix, for this section, a parity tree automaton $\mathcal{A} = \langle A, Q, q_{ini}, \Delta, \mathrm{Col} \rangle$ and recall that a tree t is in $L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ (resp. in $L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A})$) if and only if there is a run of $\mathcal{A}$ on t in which there are at most countably (resp. finitely) many rejecting branches.

4.1. The Case of Languages $L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$

Fix a tree t and define an acceptance game for $L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ as follows. In this game the two players move a pebble along a branch of t in a top-down manner: to the pebble is attached a state whose colour gives the colour of the configuration. Hence, (Éloïse's main) configurations in the game are elements of $Q \times \{0,1\}^*$. See Figure 2 for the local structure of the arena. In a node u with state q Éloïse picks a transition $(q, t(u), q_0, q_1) \in \Delta$, and then Abélard has two possible options:

(i) he chooses a direction 0 or 1; or

(ii) he lets Éloïse choose a direction 0 or 1.

Once the direction $i \in \{0,1\}$ is chosen, the pebble is moved down to $u \cdot i$ and the state is updated to q_i. A play is won by Éloïse if one of the following two situations occurs: either the parity condition is satisfied or Abélard has not let Éloïse infinitely often choose the direction. Call this game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$.

The next theorem states that it is an acceptance game for the language $L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$.

Theorem 3. *One has* $t \in L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ *if and only if Éloïse wins in* $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$ *from* $(q_{\mathrm{ini}}, \varepsilon)$.

Proof. Assume that Éloïse has a winning strategy φ in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$ from $(q_{\mathrm{ini}}, \varepsilon)$. With φ we associate a run ρ of $\mathcal{A}$ on t as follows. We inductively associate with any node u a partial play λ_u where Éloïse respects φ and that ends in a vertex of Éloïse of the form (q, u). For this we let $\lambda_\varepsilon = (q_{\mathrm{ini}}, \varepsilon)$. Now assume that we have

6

defined λ_u for some node u and let $\varphi(\lambda_u) = (q, t(u), q_0, q_1)$ be the transition Éloïse plays from λ_u when she respects φ. Then let i be the direction Éloïse would choose (again playing according to φ) if Abélard lets her pick the direction right after she played $(q, t(u), q_0, q_1)$: one defines λ_{ui} as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Abélard letting her choose the direction and Éloïse choosing direction i; and one defines $\lambda_{u(1-i)}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Abélard choosing direction $(1-i)$. Note that for $j \in \{0, 1\}$, λ_{uj} ends with the pebble on uj with the state q_j attached to it, equivalently in configuration (q_j, uj). We also refer to the node ui (*i.e.* the node that Éloïse has picked) as *marked*: note that any node has exactly one child that is marked (by convention the root is marked).

The run ρ is defined by letting $\rho(u)$ be the state attached to the pebble in the last configuration of λ_u. By construction, ρ is a valid run of $\mathcal{A}$ on t and moreover with any branch π in ρ one can associate a play λ_π in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$ from $(q_{\mathrm{ini}}, \varepsilon)$ where Éloïse respects φ (one simply considers the limit of the *increasing* sequence of partial plays λ_u where u ranges over those nodes along branch π).

It remains to show that the number of rejecting branches is at most countable. Now consider a rejecting branch π. By construction π is rejecting if and only if λ_π does not fulfil the parity condition. As φ is winning so is λ_π. Hence, it means that in λ_π Abélard does not let Éloïse choose infinitely often the direction (indeed, this is the only way for λ_π to be winning as we assumed π is rejecting, which implies that λ_π does not satisfy the parity condition). Equivalently, π contains finitely many marked nodes (marked nodes corresponding precisely to those steps where Éloïse chooses the direction). Hence, with any rejecting branch π, one can associate the last marked node u_π in it. And if $\pi \neq \pi'$ one has $u_\pi \neq u_{\pi'}$: indeed, at the point where π and π' first differ, one of the nodes is marked from the property that every node has exactly one child that is marked. Hence, the number of rejecting branches is countable as the map $\pi \mapsto u_\pi$ is injective and as the number of nodes in a tree is countable. This permits to conclude that ρ is an accepting run – in the sense of $L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ – of $\mathcal{A}$ on t.

Conversely, assume that Éloïse has no winning strategy. It follows from Borel determinacy [21] that Abélard has a winning strategy ψ in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$ from $(q_{\mathrm{ini}}, \varepsilon)$. Let us prove that any run ρ of $\mathcal{A}$ on t contains uncountably many rejecting branches. For this, fix a run ρ of $\mathcal{A}$ on t. With any sequence $\alpha = \alpha_1 \alpha_2 \cdots \in \{0, 1\}^\omega$ we associate a strategy φ_α of Éloïse in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$. The strategy φ_α of Éloïse consists in describing the run ρ and to propose direction α_i when it is the i-th time that Abélard lets her choose the direction. More formally, when the pebble is on node u with state q (we will trivially have $q = \rho(u)$ as an invariant) she picks the transition $(\rho(u), t(u), \rho(u0), \rho(u1))$; moreover if Abélard lets her choose the direction, she picks α_{i+1} where i is the number of times Abélard let her choose the direction since the beginning of the play.

As we assumed that ψ is winning, the (unique) play obtained when she plays φ_α and when he plays ψ is loosing for Éloïse: such a play defines a branch π_α in ρ, and this branch is a rejecting one. Now, for any $\alpha \neq \alpha'$ one has $\pi_\alpha \neq \pi_{\alpha'}$: indeed, at some point α and α' differs and, as infinitely often Abélard lets Éloïse choose the directions, the branches π_α and $\pi_{\alpha'}$ will differ as well. But as there are uncountably many different sequences α, it leads an uncountable number of rejecting branches in ρ. Hence, ρ is rejecting. □

Consider the game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$ and modify it so that Éloïse is now announcing in advance which direction she would choose if Abélard let her do so. This new game is equivalent to the previous one (meaning that she has a winning strategy in one game if and only if she also has one in the other game). As this new game can easily be modified to obtain an *equivalent* acceptance game for the classical acceptance condition (as described in Section 2.3) one concludes that the languages of the form $L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ are always ω-regular.

Theorem 4. *Let $\mathcal{A} = \langle A, Q, q_{ini}, \Delta, \mathrm{Col} \rangle$ be a parity tree automaton using d colours. Then there exists a parity tree automaton $\mathcal{A}' = \langle A, Q', q'_{ini}, \Delta', \mathrm{Col}' \rangle$ such that $L_{\leqslant \mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A}) = L(\mathcal{A}')$. Moreover $|Q'| = \mathcal{O}(d|Q|)$ and $\mathcal{A}'$ uses $d + 1$ colours.*

Proof. Define $\mathbb{G}_{\mathcal{A},t}^{\prime \mathrm{Rej} \leqslant \mathrm{Count}}$ as the game obtained from $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$ by asking Éloïse to say which direction she would choose before Abélard possibly lets her this option. Éloïse has a winning strategy in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Rej} \leqslant \mathrm{Count}}$ if and only if she has a winning strategy in $\mathbb{G}_{\mathcal{A},t}^{\prime \mathrm{Rej} \leqslant \mathrm{Count}}$ (strategies being essentially the same in both games). The way she indicates the direction can be encoded in the control state: just duplicate the control states (with a classical version and a starred version of each state) and when she wants to pick a transition *e.g.* $(q, t(u), q_0, q_1)$ and direction 1, she just moves to configuration (q, u, q_0, q_1^*) in the new game. Now the winning

condition can be rephrased as either the parity condition is satisfied or finitely many configuration of the form (q^*, u) are visited. Now this later game can be transformed into a standard acceptance game for ω-regular language (as defined in Section 2.3) by the following trick. One adds to states an integer where one stores the smallest colour seen since the last starred state was visited (this colour is easily updated); whenever a starred state is visited the colour is reset to the colour of the state. Now unstarred states are given an even colour that is greater than all colour previously used (hence, it ensures that if finitely many starred states are visited Éloïse wins) and starred states are given the colour that was stored (hence, if infinitely many starred states are visited we retrieve the previous parity condition). It should then be clear that the later game is a classical acceptance game, showing that $L^{\mathrm{Rej}}_{\leqslant \mathrm{Count}}(\mathcal{A})$ is ω-regular.

The construction of $\mathcal{A}'$ is immediate from the final game and the size is linear in $d|Q|$ due to the fact that one needs to compute the smallest colour visited between two starred states.

More formally[5], we assume that $\mathcal{A}$ uses colours $0, 1, \ldots, d-1$ and we let $\mathcal{A}' = \langle A, Q', q'_{ini}, \Delta', \mathrm{Col}' \rangle$ where:

- $Q' = \{q_i, q_i^* \mid q \in Q \text{ and } 0 \leqslant i \leqslant d-1\}$;

- $q'_{ini} = q_{ini, \mathrm{Col}(q_{ini})}$;

- for any $q \in Q$ and any $0 \leqslant i \leqslant d-1$ one lets $\mathrm{Col}'(q_i^*) = i$ and $\mathrm{Col}(q_i) = d+1$ if $d+1$ is even and $\mathrm{Col}(q_i) = d$ otherwise; and

- Δ' consists of the following tuples for every $(q, a, r, s) \in \Delta$ and every $0 \leqslant i \leqslant d-1$

  - $(q_i, a, r^*_{\min\{i, \mathrm{Col}(r)\}}, s_{\min\{i, \mathrm{Col}(s)\}})$,
  - $(q_i, a, r_{\min\{i, \mathrm{Col}(r)\}}, s^*_{\min\{i, \mathrm{Col}(s)\}})$,
  - $(q_i^*, a, r^*_{\mathrm{Col}(r)}, s_{\mathrm{Col}(s)})$, and
  - $(q_i^*, a, r_{\mathrm{Col}(r)}, s^*_{\mathrm{Col}(s)})$.

It is then easily seen that the acceptance game $\mathbb{G}_{\mathcal{A}', t}$ of $\mathcal{A}'$ as defined in Section 2.3 is essentially the same as the above game $\mathbb{G}'^{\mathrm{Rej} \leqslant \mathrm{Count}}_{\mathcal{A}, t}$. $\qquad \square$

4.2. The Case of Languages $L^{\mathrm{Rej}}_{\mathrm{Fin}}(\mathcal{A})$

The following lemma (whose proof is straightforward) characterises *finite* sets of branches by noting that for such a set there is a finite number of nodes belonging to at least two branches in the set.

Lemma 1. *Let Π be a set of branches. Then Π is finite if and only if the set $W = \{u \in \{0,1\}^* \mid \exists \pi_0 \neq \pi_1 \in \Pi \text{ s.t. } u \sqsubseteq \pi_0 \text{ and } u \sqsubseteq \pi_1\}$ is finite. Equivalently, Π is finite if and only if there exists some $\ell \geqslant 0$ such that for all $u \in \{0,1\}^{\geqslant \ell}$ there is at most one $\pi \in \Pi$ such that $u \sqsubseteq \pi$.*

Now, fix a tree t and define an acceptance game for $L^{\mathrm{Rej}}_{\mathrm{Fin}}(\mathcal{A})$ as follows. In this game (we refer the reader to Figure 3 for the local structure of the arena for game $\mathbb{G}^{\mathrm{RejFin}}_{\mathcal{A}, t}$) the two players move a pebble along a branch of t in a top-down manner: as in the classical case the players first select a transition and then a direction. The colour of the current state gives the colour of the configuration. There are three modes in this game: *wait* mode, *path* mode and *check* mode and the game starts in *wait* mode. Hence, (Éloïse's main) configurations in the game are elements of $Q \times \{0,1\}^* \times \{wait, path, check\}$.

Regardless of the mode, in a node u with state q Éloïse picks a transition $(q, t(u), q_0, q_1) \in \Delta$, and for each direction in $i \in \{0,1\}$ she proposes the next mode m_i in $\{wait, path, check\}$ (we describe below what are the possible options depending on the current mode). Then Abélard chooses a direction $j \in \{0,1\}$, the pebble is moved down to $u \cdot j$, the state is updated to q_j and the mode changes to m_j. The possible modes that Éloïse can propose depend on the current mode in the following manner.

- In *wait* mode she can propose any modes m_i in $\{wait, path, check\}$ but if one proposed mode m_i is *path* then the other mode m_{1-i} must be *check*.
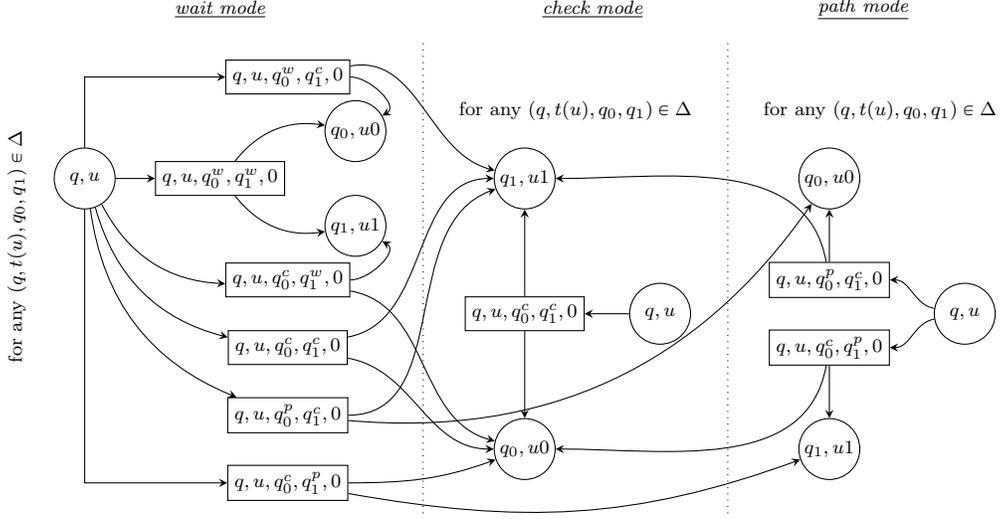

Figure 3: Local structure of the arena of the acceptance game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{RejFin}}$. We use superscripts to indicate which modes have been proposed by Éloïse.

- In *check* mode the proposed modes must be *check* (*i.e.* once the mode is *check* it no longer changes).

- In *path* mode one proposed mode must be *path* and the other must be *check*.

A play is won by Éloïse if one of the two following situations occurs.

- The *wait* mode is eventually left and the parity condition is satisfied.

- The mode is eventually always equal to *path*.

In particular a play in which the mode is *wait* forever is lost by Éloïse. Note that the latter winning condition can easily be reformulated as a parity condition. Call this game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{RejFin}}$.

The next theorem states that it is an acceptance game for the language $L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A})$.

Theorem 5. *One has $t \in L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A})$ if and only if Éloïse wins in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{RejFin}}$ from $(q_{\mathrm{ini}}, \varepsilon, wait)$.*

Proof. Assume that Éloïse has a winning strategy φ in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{RejFin}}$ from $(q_{\mathrm{ini}}, \varepsilon, wait)$. With φ we associate a run ρ of $\mathcal{A}$ on t as follows. We inductively define for any node $u \in \{0,1\}^*$ a partial play λ_u where Éloïse respects φ. For this we let $\lambda_\varepsilon = (q_{\mathrm{ini}}, \varepsilon, wait)$. Now assume that we defined λ_u for some node $u \in \{0,1\}^*$ and let $(q, t(u), q_0, q_1)$ be the transition Éloïse plays from λ_u when she respects φ. Then for each $i \in \{0,1\}$ one defines $\lambda_{u \cdot i}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Abélard choosing direction i (we update the mode accordingly to the choice of Éloïse when respecting φ in λ_u).

The run ρ is defined by letting, for any $u \in \{0,1\}^*$, $\rho(u)$ be the state attached to the pebble in the last configuration of λ_u. By construction, ρ is a run of $\mathcal{A}$ on t. Moreover with any branch π one can associate a play λ_π in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{RejFin}}$ from $(q_{\mathrm{ini}}, \varepsilon)$ where Éloïse respects φ (one simply considers the limit of the *increasing* sequence of partial plays λ_u where u ranges over nodes along branch π).

First, note that there exists some $\ell \geqslant 0$ such that, for all $u \in \{0,1\}^{\geqslant \ell}$, λ_u ends in a vertex where the mode is not *wait*. Indeed, if this was not the case, one could construct an infinite branch π such that, for all nodes u in π, λ_u ends with a vertex in mode *wait* (recall that the only way to be in *wait* mode is to be in that mode from the very beginning) and therefore the corresponding play λ_π would be loosing, which contradicts the fact that Éloïse respects her winning strategy φ in play λ_π. Now, consider some node $u \in \{0,1\}^\ell$. If the final vertex in λ_u is in mode *check* one easily verifies that any branch that goes through u is accepting (because the corresponding play is winning hence, satisfies the parity condition). If the final vertex in λ_u is in mode

path one easily checks that among all branches that go through u, there is exactly one branch π such that λ_π eventually stays in mode *path* forever (and this branch may not satisfy the parity condition) while all other branches eventually stay in mode *check* forever (and satisfy the parity condition). Therefore, the number of rejecting branches is finite.

Conversely, assume that there is a run ρ of $\mathcal{A}$ on t that contains finitely many rejecting branches. Call this set of branches Π. Thanks to Lemma 1, there exists some $\ell \geqslant 0$ such that for all $w \in \{0,1\}^{\geqslant \ell}$ there is at most one $\pi \in \Pi$ such that $w \sqsubseteq \pi$. Using ρ we define a strategy φ for Éloïse in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{RejFin}}$ as follows. In any configuration (q, u) (regardless of the mode) the strategy is to play the transition $(q, t(u), \rho(u0), \rho(u1))$. Then there are several cases for determining how the mode is updated.

- In some configuration (q, u) with u of length strictly smaller than ℓ the mode remains in *wait*.

- In some configuration (q, u) with u of length equal to ℓ the strategy proposes to update the mode to *path* for direction $i \in \{0, 1\}$ such that $u \cdot i \sqsubseteq \pi$ for some branch $\pi \in \Pi$, and to *check* otherwise. Note that due to the definition of ℓ, there is at most one direction i in which the mode becomes *path*.

- In some configuration (q, u) with u of length strictly greater than ℓ if the mode is *check* it will remain to *check* in both direction. Otherwise (*i.e.* the mode is *path*) the strategy proposes to update the mode to *path* for direction $i \in \{0, 1\}$ such $u \cdot i \sqsubseteq \pi$ for some branch $\pi \in \Pi$, and to *check* otherwise. Note that in the latter case, there is exactly one direction i in which the mode is *path*.

Remark that no play where Éloïse respects φ stays in *wait* mode forever. Moreover, with any λ where Éloïse respects φ one can associate a branch in the run ρ and this branch is rejecting if and only if λ stays eventually in mode *path* forever. Hence, any play where the mode is not infinitely often *path* satisfies the parity condition (because the corresponding branch in ρ does so). Hence, φ is winning. $\qquad\square$

From Theorem 5 and the local structure of the arena of game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{RejFin}}$ one easily concludes that any language of the form $L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A})$ is ω-regular.

Theorem 6. *Let* $\mathcal{A} = \langle A, Q, q_{ini}, \Delta, \mathrm{Col} \rangle$ *be a parity tree automaton using d colours. Then, there exists a parity tree automaton* $\mathcal{A}' = \langle A, Q', q'_{ini}, \Delta', \mathrm{Col}' \rangle$ *such that* $L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A}) = L(\mathcal{A}')$. *Moreover,* $|Q'| = \mathcal{O}(|Q|)$ *and* $\mathcal{A}'$ *uses d colours.*

Proof. Consider the local structure of game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{RejFin}}$ as described in Figure 3. The way one defines $\mathcal{A}'$ is fairly simple: for any state on $\mathcal{A}$ and any mode, one gets a new state in $\mathcal{A}'$, and the transition function Δ' directly follows from the way we update the modes. The states in *wait* mode all get the same odd minimal colour (hence, if they are never left Éloïse looses), the states in path mode all get the same even minimal colour (hence, if they are never left Éloïse wins), and the states in check mode get the colour they had in $\mathcal{A}$. Hence, we do not need to add any extra colour (except in the case where $\mathcal{A}$ uses only one colour but in this very degenerated case one can simply take $\mathcal{A}' = \mathcal{A}$). $\qquad\square$

4.3. Languages $L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ *and* $L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A})$ *vs. Büchi Tree Languages*

One can wonder, as it will be later the case (see Section 5) for languages of the form $L_\infty^{\mathrm{Acc}}(\mathcal{A})$ or $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$, whether a Büchi condition is enough to accept (with the classical semantics) a language of the form $L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ (resp. $L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A})$). The next Proposition answers negatively.

Proposition 1. *There is a co-Büchi deterministic tree automaton $\mathcal{A}$ such that for any* Büchi *tree automaton* $\mathcal{A}'$, $L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A}) \neq L(\mathcal{A}')$ *and* $L_{\mathrm{Fin}}^{\mathrm{Rej}}(\mathcal{A}) \neq L(\mathcal{A}')$.

Proof. We choose for $\mathcal{A}$ the same automaton that was used by Rabin in [2] to derive a similar statement where one replaces $L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$ by $L(\mathcal{A})$ and we generalise the proof of this result as given in [4, Example 6.3].

Let L be the set of $\{a, b\}$-labelled trees such that the number of branches that contain infinitely many b's is at most countable. Obviously there is a *deterministic* co-Büchi automaton $\mathcal{A}$ such that $L = L_{\leqslant\mathrm{Count}}^{\mathrm{Rej}}(\mathcal{A})$. Indeed, consider an automaton $\mathcal{A}$ with two states, one forbidden and the other one non-forbidden, and that from any state, goes (for both children) in the forbidden state whenever he was in a b-labelled node and otherwise goes (for both children) in the non-forbidden state.

Assume, by contradiction, that there is some Büchi tree automaton $\mathcal{A}' = \langle \{a, b\}, Q, q_{ini}, \Delta, F \rangle$ such that $L_{\leqslant\text{Count}}^{\text{Rej}}(\mathcal{A}) = L(\mathcal{A}')$. Note that we will not treat the case where $L_{\text{Fin}}^{\text{Rej}}(\mathcal{A}) = L(\mathcal{A}')$ as it is identical. Let $n = |Q|$ and let t be the $\{a, b\}$-labelled tree such that $t(u) = b$ if and only if $u \in (0^+1)^k$ for some $1 \leqslant k \leqslant n$, *i.e.* label b occurs when a right successor is taken after a sequence of left successors, however allowing at most n right turns. Clearly, $t \in L$ as every branch contains finitely many b-labelled nodes. Let ρ be an accepting run of $\mathcal{A}'$ on t.

The goal is to exhibit three nodes u, v and w such that:

1. u is a strict prefix of both v and w.

2. v is not a prefix of w and *vice versa*;

3. $\rho(u) = \rho(v) = \rho(w)$ is a final state;

4. $t(u) = t(v) = t(w) = a$;

5. on the path segment from u to v there is at least one node labelled b;

6. on the path segment from u to w there is at least one node labelled b.

Once this is done we can form a new tree t' (and an associated run ρ' of $\mathcal{A}'$ on t') by iterating the finite path segment from u (inclusive) to v (exclusive) and from u (inclusive) to w (exclusive) indefinitely, copying also the subtrees which have their roots on these path segments. More formally, consider the two-hole context $C_t[\bullet, \bullet]$ (resp. $C_\rho[\bullet, \bullet]$) obtained by placing holes at v and w in t (resp. in ρ) and the two-hole context $D_t[\bullet, \bullet]$ (resp. $D_\rho[\bullet, \bullet]$) obtained by placing holes at $u^{-1}v$ and $u^{-1}w$ in the subtree $t_{/u}$ of t rooted at u (resp. $\rho_{/u}$). The tree t' is equal to $C_t[t'', t'']$ where t'' is the unique tree satisfying the equation $t'' = D_t[t'', t'']$. Similarly ρ' is equal to $C_\rho[\rho'', \rho'']$ where ρ'' is the unique tree satisfying the equation $\rho'' = D_\rho[\rho'', \rho'']$.

This process exhibits a binary tree like structure[6] inside t' (resp. ρ') such that any branch in t' contains infinitely many b-labelled nodes, hence $t' \notin L$ (indeed, there will be uncountably many branches in this binary tree like structure). But this leads to a contradiction as ρ' is easily seen to be accepting while being a run on t'.

We now explain why we can find nodes u, v and w as above. We first claim that for any node $u \in 0^+(10^+)^k$, for some $0 \leqslant k < n$, there are two nodes $v_u, w_u \in u0^+10^+$ such that (i) u is a strict prefix of v_u and w_u; (ii) v_u is not a prefix of w_u and *vice versa*; and (iii) $\rho(v_u) = \rho(w_u)$ is a final state. Indeed, for any $i > 0$, the branch $u0^i10^\omega$ is accepting in ρ and therefore there exists some index $k_i > 0$ such that $\rho(u0^i10^{k_i}) \in F$; thus, by the pigeon hole principle, there exists $i \neq j$ such that $\rho(u0^i10^{k_i}) = \rho(u0^j10^{k_j}) \in F$ and therefore one can choose $v_u = u0^i10^{k_i}$ and $w_u = u0^j10^{k_j}$.

Now define a sequence of nodes $u_0 \sqsubset u_1 \sqsubset u_2 \sqsubset \cdots \sqsubset u_n$ as follows: we let $u_0 \in 0^+$ to be such that $\rho(u_0) \in F$ (such a node exists as the branch 0^ω is accepting in ρ); and for all $1 \leqslant i \leqslant n$, $u_i = v_{u_{i-1}}$. In particular, one has $\rho(u_i) \in F$ for all $0 \leqslant i \leqslant n$ and therefore, by the pigeon hole principle there exists $0 \leqslant i < j \leqslant n$ such that $\rho(u_i) = \rho(u_j)$. Now, to finish our construction, we simply let $u = u_i$, $v = u_j$ and $w = w_{u_{j-1}}$. This concludes the proof. $\square$

5. Counting Accepting Branches

We now consider the case where acceptance is defined by a constraint on the number of accepting branches and we show that the associated languages are ω-regular. It leads to new proofs, that rely on games rather than on logic, of the results in [11].

Fix, for this section, a parity tree automaton $\mathcal{A} = \langle A, Q, q_{ini}, \Delta, \text{Col} \rangle$ and recall that a tree t is in $L_\infty^{\text{Acc}}(\mathcal{A})$ (resp. $L_{Uncount}^{\text{Acc}}(\mathcal{A})$) if and only if there is a run of $\mathcal{A}$ on t that contains infinitely (resp. uncountably) many accepting branches.

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[6] Actually it is what we refer to as an *accepting-pseudo binary tree* in Section 5.2.1.

5.1. The Case of Languages $L_\infty^{\mathrm{Acc}}(\mathcal{A})$

The key idea behind defining an acceptance game for $L_\infty^{\mathrm{Acc}}(\mathcal{A})$ for some tree t is to exhibit a *pseudo comb* in a run of $\mathcal{A}$ over t. In a nutshell, a pseudo comb consists of an infinite branch U and a collection V of accepting branches each of them sharing some prefix with U. One easily proves that a run contains infinitely many accepting branches if and only if it contains a pseudo comb.
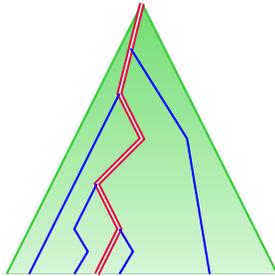

Figure 4: A pseudo comb ($\mathbb{U}, V$)

More formally, a **pseudo comb** (see Figure 4 for an illustration) is a pair of subset (U, V) of nodes with $U, V \subseteq \{0,1\}^*$ such that:

- U and V are disjoint.

- U is a branch: $\varepsilon \in U$, for all $u \in U$ one has $|\{u0, u1\} \cap U| = 1$ and if $u \neq \varepsilon$ its parent is in U as well.

- V is a set of nodes such that

    (i) for all $v \in V$, one has $|\{v0, v1\} \cap V| = 1$;
    (ii) for all $v \in V$, $v \in (U \cup V) \cdot \{0,1\}$.

- For infinitely many $u \in U$ there exists some $v \in V$ such that either $v = u0$ or $v = u1$.

The following folklore lemma characterises infinite sets of branches in the full binary tree (recall that we say that a node w belongs to a branch π if and only if $w \sqsubset \pi$).

Lemma 2. *Let Π be a set of branches. Then Π is infinite if and only if the set $W = \{w \mid \exists \pi \in \Pi \text{ s.t. } w \text{ belongs to } \pi\}$ contains a pseudo comb (U, V), i.e. $U \cup V \subseteq W$.*

Proof. If W contains a pseudo comb it directly implies that Π is infinite, so we focus on the other implication.

There exists an increasing sequence (for the prefix relation $\sqsubseteq$) $(u_i)_{i \geqslant 0}$ of nodes such that for all $i \geqslant 0$ infinitely many branches in Π go through u_i and for both directions $j = 0$ and $j = 1$, at least one branch in Π goes through $u_i \cdot j$. The existence of this sequence is by an immediate induction.

Define U as the set of prefixes of elements in the sequence $(u_i)_{i \geqslant 0}$: U is a branch as the sequence $(u_i)_{i \geqslant 0}$ is increasing.

For all i, pick a branch V_i that goes through u_i but not through u_{i+1} (it exists by definition of u_i). Then to obtain a pseudo comb (U, V) such that $U \cup V \subseteq W$, it suffices to define $V = (\bigcup_{i \geqslant 0} V_i) \backslash U$. $\square$

Now, fix a tree t and define an acceptance game for $L_\infty^{\mathrm{Acc}}(\mathcal{A})$. There are two modes in the game (See Figure 5 for the local structure of the arena): *path* mode and *check* mode and the game starts in *path* mode. Hence, (Éloïse's) configurations in the game are elements of $Q \times \{0,1\}^* \times \{path, check\}$. In *path* mode, in a node u with state q Éloïse picks a transition $(q, t(u), q_0, q_1) \in \Delta$, and she chooses a direction $i \in \{0,1\}$. Then Éloïse has two options. Either she moves down the pebble to $u \cdot i$ and updates the state to be q_i. Or she proposes Abélard to change to *check* mode: if he accepts, the pebble is moved down to $u \cdot (1 - i)$ and the state is updated to $q_{(1-i)}$; if he refuses, the pebble is moved down to $u \cdot i$ and the state is updated to q_i (and the game stays in *path* mode).

In *check* mode Éloïse plays alone: in a node u with state q she picks a transition $(q, t(u), q_0, q_1) \in \Delta$, and she chooses a direction $i \in \{0,1\}$; then the pebble is moved down to $u \cdot i$ and the state is updated to q_i. Note that it is not possible to switch from *check* mode back to *path* mode.

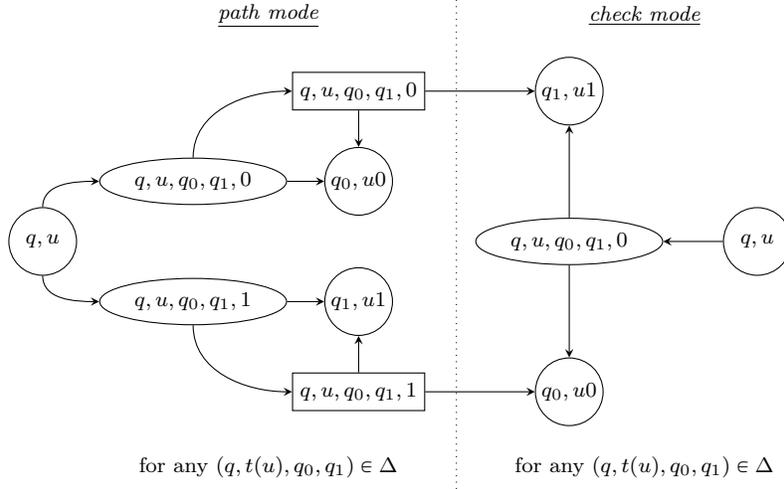A play is won by Éloïse if one of the two following situations occurs.

for any $(q, t(u), q_0, q_1) \in \Delta$ for any $(q, t(u), q_0, q_1) \in \Delta$

Figure 5: Local structure of the arena of the acceptance game $\mathbb{G}^{\mathrm{Acc}\,\infty}_{\mathcal{A},t}$.

- Eventually the players have switched to *check* mode and the parity condition is satisfied.

- Éloïse proposed infinitely often Abélard to switch the mode but he always refused.

Call this game $\mathbb{G}^{\mathrm{Acc}\,\infty}_{\mathcal{A},t}$. Intuitively *path* mode is used to define the U part of the pseudo-comb (U, V) and each of the branches in V is inspected in *check* mode. Note that in *check* mode, Éloïse plays alone and hence only checks the existence of an accepting branch. However, as the automata we consider are assumed to be complete hence, there always exists some run containing this accepting branch.

The next theorem states that it is an acceptance game for the language $L^{\mathrm{Acc}}_\infty(\mathcal{A})$.

Theorem 7. *One has* $t \in L^{\mathrm{Acc}}_\infty(\mathcal{A})$ *if and only if Éloïse wins in* $\mathbb{G}^{\mathrm{Acc}\,\infty}_{\mathcal{A},t}$ *from* $(q_{\mathrm{ini}}, \varepsilon, path)$.

Proof. Assume that Éloïse has a winning strategy φ in $\mathbb{G}^{\mathrm{Acc}\,\infty}_{\mathcal{A},t}$ from configuration $(q_{\mathrm{ini}}, \varepsilon, path)$. With φ we associate a run ρ of $\mathcal{A}$ on t and a pseudo comb (U, V) as follows. We inductively associate with any node $u \in U$ a partial play λ_u where Éloïse respects φ and that is always in *path* mode; and we inductively associate with any node $v \in V$ a partial play λ_v where Éloïse respects φ and where the mode has eventually been switched to *check* mode. For this we let $\varepsilon \in U$ and $\lambda_\varepsilon = (q_{\mathrm{ini}}, \varepsilon, path)$. Now assume that we defined λ_u for some node $u \in U$ and let $(q, t(u), q_0, q_1)$ be the transition and let i be the direction Éloïse plays from λ_u when she respects φ. Then we have two possible situations depending whether, right after playing $(q, t(u), q_0, q_1)$ and still respecting φ, Éloïse proposes Abélard to switch to *check* mode.

- If she does so we let $u \cdot (1-i)$ belong to V and we define $\lambda_{u \cdot (1-i)}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$ and direction i, followed by Éloïse proposing Abélard to switch the mode and Abélard accepting (hence, moving down the pebble in direction $(1-i)$ and attaching state $q_{(1-i)}$ to it). We let $u \cdot i$ belong to U and we define $\lambda_{u \cdot i}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$ and direction i, followed by Éloïse proposing Abélard to switch the mode and Abélard refusing (hence, moving down the pebble in direction i and attaching state q_i to it).

- If Éloïse does not propose Abélard to switch the mode we do not let $u \cdot (1-i)$ belong to V. And we let $u \cdot i$ belong to U and we define $\lambda_{u \cdot i}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$ and direction i, followed by Éloïse not proposing Abélard to switch the mode (hence, moving down the pebble in direction i and attaching state q_i to it).

The run ρ is defined by letting, for any $w \in U \cup V$, $\rho(w)$ be the state attached to the pebble in the last configuration of λ_w. For those $w \notin U \cup V$ we define $\rho(w)$ so that the resulting run is valid, which is always

13

possible as we only consider complete automata. By construction, ρ is a run of $\mathcal{A}$ on t and (U,V) is a pseudo comb. Moreover with any branch π that can be built as an initial sequence of nodes in U followed by an infinite sequence of nodes in V one can associate a play λ_π in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,\infty}$ from $(q_{\mathrm{ini}}, \varepsilon, path)$ where Éloïse respects φ (one simply considers the limit of the *increasing* sequence of partial plays λ_v where v ranges those nodes nodes in V along branch π). By construction π is accepting as λ_π fulfils the parity condition. Hence, by Lemma 2 we conclude that ρ contains infinitely many accepting branches, meaning that $t \in L_\infty^{\mathrm{Acc}}(\mathcal{A})$.

Conversely, assume that Éloïse does not have a winning strategy in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,\infty}$ from $(q_{\mathrm{ini}}, \varepsilon, path)$. By Borel determinacy, Abélard has a winning strategy ψ in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,\infty}$ from $(q_{\mathrm{ini}}, \varepsilon, path)$. By contradiction, assume that there is a run ρ of $\mathcal{A}$ on t that contains infinitely many accepting branches. By Lemma 2, it follows that ρ contains a pseudo comb (U,V) such that any branch that can be built as an initial sequence of nodes in U followed by an infinite sequence of nodes in V is an accepting branch. From ρ and (U,V) we define a strategy φ of Éloïse in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,\infty}$ from $(q_{\mathrm{ini}}, \varepsilon, path)$ as follows. Strategy φ uses as a memory either a node $u \in U$ if the play is in *path* mode or a node $v \in V$ if the play is in *check* mode; initially the memory is $u = \varepsilon$. Now assume that the pebble is in some node $u \in U$ with state q attached to it (one will inductively check that $\rho(u) = q$). Then there are two possibilities.

- Both $u0$ and $u1$ belong to $U \cup V$: the strategy φ indicates that Éloïse chooses transition $(q, t(u), \rho(u0), \rho(u1))$ and direction i where $ui \in U$ and proposes Abélard to switch to *check* mode. Then the memory is updated to $u \cdot j$ where $j = i$ if the mode is unchanged and $j = 1 - i$ otherwise.

- If $u0$ (resp. $u1$) belongs to U but $u1$ (resp. $u0$) does not belong to V: strategy φ indicates that Éloïse chooses the transition $(q, t(u), \rho(u0), \rho(u1))$ and chooses the direction 0 (resp. 1) and does not propose Abélard to switch to *check* mode. Then the memory is updated to $u0$ (resp. $u1$).

Now assume that the pebble is in some node $v \in V$ with state q attached to it: one will inductively check that $\rho(v) = q$ and that the mode is *check*. Call i the (unique) direction such that vi belongs to V: then the strategy φ indicates that Éloïse chooses transition $(q, t(v), \rho(v0), \rho(v1))$ and then moves down the pebble to direction vi. In particular, once a node in V is reached, strategy φ ensures that the pebble always stays in V for the rest of the play.

Now consider the (unique) play λ where Éloïse respects her strategy φ while Abélard respects his strategy ψ. As we assumed ψ to be a winning strategy for Abélard, λ is won by him. Now, as (U,V) is a pseudo comb and by definition of φ, it follows that λ only goes through nodes in $U \cup V$, and if λ only goes through nodes in U then Éloïse proposes infinitely often to Abélard to switch to *check* mode. Hence, as λ is winning for Abélard one concludes that eventually the mode is switched in λ and that the resulting play does not fulfil the parity condition. Now, it is easily seen that with λ one associates a branch π in the run ρ and that this branch can be built as an initial sequence of nodes in U followed by an infinite sequence of nodes in V (indeed, at some point the mode is switched to *check* and from that point the play stays in nodes from V forever). Now, by definition of a pseudo comb, it follows that the branch π is accepting in ρ which means that it satisfies the parity condition, and therefore so does λ, which brings a contradiction with the fact that it is won by Abélard. $\qquad\square$

One can modify $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,\infty}$ so that to obtain an *equivalent* game that has the form of a classical acceptance game. From this follows the fact that the languages of the form $L_\infty^{\mathrm{Acc}}(\mathcal{A})$ are indeed ω-regular. As the new game can be seen to be obtained from a Büchi automaton, this also permits to lower the acceptance condition.

Theorem 8. *Let* $\mathcal{A} = \langle A, Q, q_{ini}, \Delta, \mathrm{Col} \rangle$ *be a parity tree automaton using d colours. Then there exists a* Büchi *tree automaton* $\mathcal{A}' = \langle A, Q', q'_{ini}, \Delta', \mathrm{Col}' \rangle$ *such that* $L_\infty^{\mathrm{Acc}}(\mathcal{A}) = L(\mathcal{A}')$. *Moreover* $|Q'| = \mathcal{O}(d|Q|)$.

Proof. Start from game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,\infty}$ and observe that if one duplicates the control states (with a classical version and a starred version of each state) and add a Boolean flag Éloïse can indicate the direction she wants to follow and whether she proposes to switch to *check* mode: *e.g.* if she wants to choose transition $(q, t(u), q_0, q_1)$ and direction 1 and not change the mode, she just moves to configuration $(q, u, q_0, q_1^*, \bot)$ in the new game; if she wants to choose transition $(q, t(u), q_0, q_1)$ and direction 0 and offer Abélard the option to switch to *check* mode she moves to configuration $(q, u, q_0^*, q_1, \top)$. Now, we allow Abélard to choose any direction but if the corresponding state is not starred then either one goes to a dummy winning configuration for Éloïse if the Boolean was $\bot$ and otherwise one changes the mode to *check*. We indicate the *check* mode in the control

state and we use the same trick to let Éloïse impose the choice of the branch (if Abélard does not follow her choice one ends up in the previous dummy configuration). It should be clear that the resulting game is an equivalent acceptance game when equipped with the following winning condition: Éloïse wins if either the dummy configuration is reached, or infinitely many configuration with Boolean ⊤ are visited but the play is always in *path* mode or the play is eventually in *check* mode and the parity condition holds. Now, the two first criteria are Büchi criteria while the third one is *a priori* a parity condition. But as Éloïse plays alone in *check* mode, she can indicate at some point that the smallest infinitely visited colour will be some (even) integer and that no other smaller colour will latter be visited: hence, if one stores the colour, go to a final state whenever it is visited and to a rejecting state if some smaller colour occurs, then one obtains a Büchi condition. All together (combining the Büchi conditions in the usual way) one obtains an equivalent Büchi classical acceptance game, showing that $L^{\mathrm{Rej}}_{\leqslant\mathrm{Count}}(\mathcal{A})$ is ω-regular and accepted by a Büchi automaton.

The construction of $\mathcal{A}'$ is immediate from the final game and the size is linear in $d|Q|$ due to the fact that one needs to remember the smallest colour for the *check* mode. □

5.2. The Case of Languages $L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$

We now discuss the case of languages of the form $L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$. For this we start with some key objects (*accepting pseudo binary trees* and *k-pseudo binary trees*) that are used to characterise runs with uncountably many accepting branches. Then, we describe two acceptance games: the first one is very simple while the second one is more involved but later permits to lower the acceptance condition to Büchi when showing that the languages $L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$ are accepted by tree automata with the classical semantics.

5.2.1. Accepting-Pseudo Binary Tree & k-Pseudo Binary Tree

The key idea behind defining an acceptance game for $L^{\mathrm{Acc}}_{Uncount}(\mathcal{A})$ for some tree t is to exhibit an *accepting pseudo binary tree* in a run of $\mathcal{A}$ over t. In a nutshell, an accepting pseudo binary tree is an infinite set U of nodes with a tree-like structure between them and such that any branch that has infinitely many prefixes in U is accepting.

We now formally define *accepting-pseudo binary trees* and *k-pseudo binary trees* that characterise those runs that contains uncountably many accepting branches (Lemma 3 and Lemma 4 below).
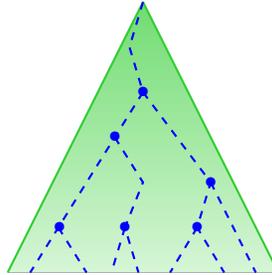

Figure 6: An accepting-pseudo binary tree U: nodes in U are marked by the symbol • and all blue branches are accepting.

Let $\mathcal{A} = \langle A, Q, q_{ini}, \Delta, \mathrm{Col}\rangle$ be a parity tree automaton and let ρ be a run of $\mathcal{A}$ on some tree t.

An **accepting-pseudo binary tree** in ρ (see Figure 6 for an illustration) is a subset $U \subseteq \{0,1\}^*$ of nodes such that

(i) for all $u \in U$ there are $v, w \in U$ such that $v = u0v'$ and $w = u1w'$ for some v' and $w' \in \{0,1\}^*$;

(ii) for all $v, w \in U$ the longest common prefix u of v and w belongs to U;

(iii) any branch π that goes through infinitely many nodes in U is accepting.

We now give a stronger notion than accepting-pseudo binary tree. For this, let k be some even colour. A **k-pseudo binary tree** in ρ (see Figure 7 for an illustration) is a subset $U \subseteq \{0,1\}^*$ of nodes such that

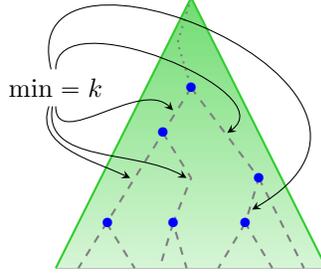(i) for all $u \in U$ there are $v, w \in U$ such that $v = u0v'$ and $w = u1w'$ for some v' and $w' \in \{0,1\}^*$;

Figure 7: A k-pseudo binary tree U: nodes in U are marked by symbol $\bullet$.

  (ii) for all $v, w \in U$ the largest common prefix u of v and w belongs to U;

 (iii) for all $u, v \in U$ such that $u \sqsubset v$, one has $\min\{\mathrm{Col}(\rho(w)) \mid u \sqsubseteq w \sqsubseteq v\} = k$.

The following lemma characterises runs that contain an uncountable sets of accepting branches. Its proof is a direct consequence of [11, Lemma 2]. But for the sake of completeness, we give a proof here.

Lemma 3. *Let ρ be a run. Then ρ contains uncountably many accepting branches if and only if it contains a k-pseudo binary tree for some even colour k.*

Proof. Clearly if ρ contains a k-pseudo binary tree for some even colour k, ρ contains uncountably many accepting branches. It remains to prove the converse.

Let ρ be a run with uncountably many accepting branches. For an even colour k and a node u in ρ, we denote by $\Pi_{k,u}$ the set of branches π of ρ such that:

- π goes through u (*i.e.* $u \sqsubseteq \pi$),

- the smallest colour appearing infinitely often in π is k,

- no colour strictly smaller than k appears below u (*i.e.* for all $u \sqsubseteq v \sqsubseteq \pi$, $\rho(v) \geqslant k$).

As the set of all accepting branches in ρ is equal to the (countable) union of all the $\Pi_{k,u}$, there exists an even colour k_0 and a node u_0 such that Π_{k_0,u_0} is uncountable. In the following, we write Π for Π_{k_0,u_0}.

Claim. For all nodes $u \sqsupseteq u_0$ such that uncountably many branches of Π go through u, there exists a node w_u such that:

- w_u is a strict descendant of u (*i.e.* $u \sqsubset w_u$),

- $\min\{\mathrm{Col}(\rho(v)) \mid u \sqsubseteq v \sqsubseteq w_u\} = k_0$,

- there are uncountably many branches of Π going though $w_u 0$ and through $w_u 1$.

If we assume that the claim holds, we can construct a k_0-pseudo binary tree by considering the smallest set U such that w_{u_0} belongs to U and such that for all v, $v \in U$ implies $w_{v0} \in U$ and $w_{v1} \in U$.

Hence, it remains to show that the claim holds. Let $u \sqsupseteq u_0$ be a node such that uncountably many branches of Π go through u. Consider the sets:

$$X := \{v \sqsupseteq u \mid \min\{\mathrm{Col}(\rho(v)) \mid u \sqsubseteq w \sqsubseteq v\} = k_0$$
$$\text{and } \textit{uncountably} \text{ many branches of } \Pi \text{ go through } v\ \}$$
$$Y := \{v \sqsupseteq u \mid \min\{\mathrm{Col}(\rho(v)) \mid u \sqsubseteq w \sqsubseteq v\} = k_0$$
$$\text{and } \textit{countably} \text{ many branches of } \Pi \text{ go through } v\ \}$$

A branch $\pi \in \Pi$ that goes through u must contain a node in $X \cup Y$. There are two cases:

16

1. either it contains a node in Y,

2. or after some position all nodes in π belong to X.

As there are at most countably many branches satisfying Case 1, there must be uncountably many branches satisfying Case 2. In turn this implies that there exists $v \in X$ such that both $v0$ and $v1$ also belong to X and we can simply take w_u to be such a v. Indeed if it was not the case, there would be exactly one branch of Π satisfying Case 2 and going through any given vertex in X: the set of branches satisfying Case 2 would be at most countable. $\qquad\square$

As any k-pseudo binary tree is an accepting-pseudo binary tree, we directly have the following Lemma from Lemma 3.

Lemma 4. *Let ρ be a run. Then ρ contains uncountably many accepting branches if and only if it contains an accepting-pseudo binary tree.*

5.2.2. The Acceptance Game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$

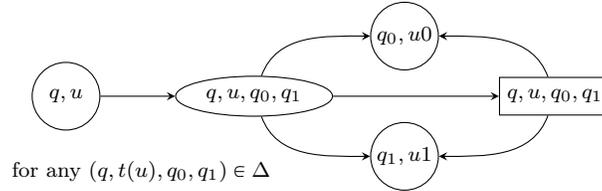for any $(q, t(u), q_0, q_1) \in \Delta$

Figure 8: Local structure of the arena of the acceptance game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$.

Fix a tree t and define an acceptance game for $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$. In this game (see Figure 8 for the local structure of the arena) the two players move a pebble along a branch of t in a top-down manner: to the pebble is attached a state, and the colour of the state gives the colour of the configuration. Hence, (Éloïse's main) configurations in the game are elements of $Q \times \{0, 1\}^*$. In a node u with state q Éloïse picks a transition $(q, t(u), q_0, q_1) \in \Delta$, and then Éloïse has two options. Either she chooses a direction 0 or 1 or she lets Abélard choose a direction 0 or 1. Once the direction $i \in \{0, 1\}$ is chosen, the pebble is moved down to $u \cdot i$ and the state is updated to q_i. A play is won by Éloïse if and only if

(1) the parity condition is satisfied and

(2) Éloïse lets Abélard infinitely often choose the direction during the play.

Call this game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$.

The next theorem states that it is an acceptance game for the language $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$.

Theorem 9. *Éloïse wins in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$ from $(q_{\mathrm{ini}}, \varepsilon)$ if and only if $t \in L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$.*

Proof. In the following proof for a set $X \subseteq \{0, 1\}^*$ we denote by $Pref(X)$ the set of prefixes of elements in X, *i.e.* $\mathrm{Pref}(X) = \{u \mid \exists v \in X \text{ s.t. } u \sqsubseteq v\}$.

Assume that Éloïse has a winning strategy φ in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$ from $(q_{\mathrm{ini}}, \varepsilon)$. With φ we associate a run ρ of $\mathcal{A}$ on t and an accepting-pseudo binary tree U as follows. We inductively define U and $\mathrm{Pref}(U)$ and associate with any node $u \in \mathrm{Pref}(U)$ a partial play λ_u where Éloïse respects φ. Remark that even if $\mathrm{Pref}(U)$ is uniquely determined by U we independently define them, making sure that they are indeed compatible. For this we let $\varepsilon \in \mathrm{Pref}(U)$ and we set $\lambda_\varepsilon = (q_{\mathrm{ini}}, \varepsilon)$.

Now assume that we have defined λ_u for some node $u \in \mathrm{Pref}(U)$. Then let $(q, t(u), q_0, q_1)$ be the transition Éloïse plays from λ_u when she respects φ. Then we have two possible situations depending whether, right after playing $(q, t(u), q_0, q_1)$ and still respecting φ, Éloïse chooses the direction or lets Abélard make that choice. If she chooses the direction, let i be this direction: then one lets $ui \in \mathrm{Pref}(U)$ and defines $\lambda_{u \cdot i}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Éloïse

choosing direction i. If she lets Abélard choose the direction, one lets u belong to U and lets both $u0$ and $u1$ belongs to $\mathrm{Pref}(U)$ and defines $\lambda_{u \cdot i}$ for $i \in \{0, 1\}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Éloïse letting Abélard choose the direction and Abélard picking direction i. Note that for any $ui \in \mathrm{Pref}(U)$, $\lambda_{u \cdot i}$ ends with the pebble on $u \cdot i$ with state q_i attached to it, equivalently in configuration (q_i, ui).

The run ρ is defined by letting, for any $u \in \mathrm{Pref}(U)$, $\rho(u)$ be the state attached to the pebble in the last configuration of λ_u. For those $u \notin \mathrm{Pref}(U)$ we define $\rho(u)$ so that the resulting run is valid, which is always possible as we only consider complete automata. By construction, ρ is a run of $\mathcal{A}$ on t. Moreover, with any branch π consisting only of nodes in $\mathrm{Pref}(U)$, one can associate a play λ_π in $\mathbb{G}_{\mathcal{A}, t}^{\mathrm{Acc\ Uncount}}$ from $(q_{\mathrm{ini}}, \varepsilon)$ where Éloïse respects φ (one simply considers the limit of the *increasing* sequence of partial plays λ_u where u ranges over nodes along branch π). As λ_π is winning it follows easily that U is a pseudo binary tree (indeed, condition (i) and (ii) from the definition of an accepting-pseudo binary tree are immediate, while condition (iii) follows from the fact that λ_π is winning). Hence, from Lemma 4 we conclude that ρ contains uncountably many accepting branches, meaning that $t \in L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$.

Conversely, assume that there is a run ρ of $\mathcal{A}$ on t that contains uncountably many accepting branches. By Lemma 4, it follows that ρ contains an accepting-pseudo binary tree U.

From ρ and U we define a strategy φ of Éloïse in $\mathbb{G}_{\mathcal{A}, t}^{\mathrm{Acc\ Uncount}}$ from $(q_{\mathrm{ini}}, \varepsilon)$ as follows. Strategy φ uses as a memory a node $v \in \mathrm{Pref}(U)$, and initially $v = \varepsilon$. Now assume that the pebble is on some node v with state q attached to it (one will inductively check that $v \in \mathrm{Pref}(U)$ and that $\rho(v) = q$). Then we have two possibilities.

- Assume $v \in U$. Both $v0$ and $v1$ belong to $\mathrm{Pref}(U)$: strategy φ indicates that Éloïse chooses transition $(q, t(v), \rho(v0), \rho(v1))$ and let Abélard choose the direction, say i. Then the memory is updated to $v \cdot i$.

- Assume $v \notin U$. Hence, $v \cdot i$ belong to $\mathrm{Pref}(U)$ for only one $i \in \{0, 1\}$: strategy φ indicates that Éloïse chooses transition $(q, t(v), \rho(v0), \rho(v1))$ and chooses direction i. Then the memory is updated to $v \cdot i$.

Now consider a play λ where Éloïse respects her strategy φ. It is easily seen that with λ one associates a branch π in the run ρ and that this branch goes only through nodes in $\mathrm{Pref}(U)$. From this observation and from the definition of an accepting-pseudo binary tree, we conclude that λ is winning for Éloïse (it satisfies the parity condition as π does and in λ Éloïse lets Abélard choose the direction infinitely often, namely whenever her memory v belongs to U). Hence, we conclude that strategy φ is winning from $(q_{\mathrm{ini}}, \varepsilon)$. $\square$

5.2.3. The Acceptance Game $\widetilde{\mathbb{G}}_{\mathcal{A}, t}^{\mathrm{Acc\ Uncount}}$

One can modify $\mathbb{G}_{\mathcal{A}, t}^{\mathrm{Acc\ Uncount}}$ to obtain an *equivalent* game that has the form of a classical acceptance game. From this follows the fact that the languages of the form $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$ are indeed ω-regular. Nevertheless, using a more involved game than $\mathbb{G}_{\mathcal{A}, t}^{\mathrm{Acc\ Uncount}}$ one can obtain a stronger result where the acceptance condition is lowered to a Büchi condition. We now describe this game.

Fix a tree t and define an acceptance game for $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$. There are two modes in the game (See Figure 9 for the local structure of the arena): *wait* mode and *check* mode and the game starts in *wait* mode. Moreover the *check* mode is parametrised by a colour k. Again, the two players move a pebble along a branch of t in a top-down manner. Hence, (main) configurations in the game are elements of $Q \times \{0, 1\}^* \times \{wait, check^0, \ldots, check^{2\ell}\}$ where $\{0, \ldots, 2\ell\}$ are the even colours used by $\mathcal{A}$. In *wait* mode Éloïse plays alone: in a node u with state q she picks a transition $(q, t(u), q_0, q_1) \in \Delta$, and she chooses a direction $i \in \{0, 1\}$; then the pebble is moved down to $u \cdot i$ and the state is updated to q_i. When moving the pebble down she can decide to switch the mode to some $check^k$ (for any *even* colour k). Once entered $check^k$ mode the play stays in that mode forever and goes as follows. In a node u with state q Éloïse picks a transition $(q, t(u), q_0, q_1) \in \Delta$, and then she has two possible options. Either she chooses a direction 0 or 1 or she lets Abélard choose a direction 0 or 1. Once the direction $i \in \{0, 1\}$ is chosen, the pebble is moved down to $u \cdot i$ and the state is updated to q_i. A play is won by Éloïse if and only if

(1) it eventually enters some $check^k$ mode and

(2) it goes infinitely often through configurations in $\{(q, u, check^k) \mid \mathrm{Col}(q) = k\}$,

(3) it never visits a configuration in $\{(q, u, check^k) \mid \mathrm{Col}(q) < k\}$,
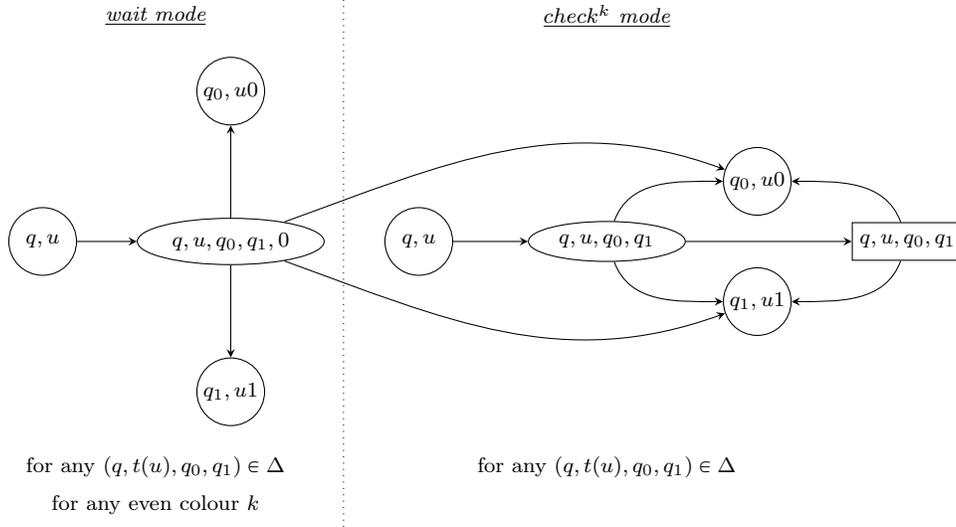
Figure 9: Local structure of the arena of the acceptance game $\widetilde{\mathbb{G}}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$.

(4) Éloïse lets Abélard infinitely often choose the direction during the play, and between two such situations the smallest colour visited is always k.

The next theorem states that it is an acceptance game for the language $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$. Note that its proof is a refinement of the one of Theorem 9.

Theorem 10. *One has $t \in L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$ if and only if Éloïse wins in $\widetilde{\mathbb{G}}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$ from $(q_{\mathrm{ini}}, \varepsilon, wait)$.*

Proof. In the following proof for a set $X \subseteq \{0,1\}^*$ we denote by $Pref(X)$ the set of prefixes of elements in X, *i.e.* $\mathrm{Pref}(X) = \{u \mid \exists v \in X \text{ s.t. } u \sqsubseteq v\}$.

Assume that Éloïse has a winning strategy φ in $\widetilde{\mathbb{G}}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$ from $(q_{\mathrm{ini}}, \varepsilon, wait)$. With φ we associate a run ρ of $\mathcal{A}$ on t and a k-pseudo binary tree U (for some k to be defined later) as follows. We inductively define U and $\mathrm{Pref}(U)$ (even if $\mathrm{Pref}(U)$ is uniquely determined by U we independently define them, making sure that they are indeed compatible) and associate with any node $u \in \mathrm{Pref}(U)$ a partial play λ_u where Éloïse respects φ. For this we let $\varepsilon \in \mathrm{Pref}(U)$ and we let $\lambda_\varepsilon = (q_{\mathrm{ini}}, \varepsilon, wait)$.

Now assume that we have defined λ_u for some node $u \in \mathrm{Pref}(U)$ and that the mode in λ_u is always *wait*. Then let $(q, t(u), q_0, q_1)$ be the transition and let i be the direction Éloïse plays from λ_u when she respects φ. If she does not change the mode, then one lets $ui \in \mathrm{Pref}(U)$ and defines $\lambda_{u \cdot i}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Éloïse choosing direction i and keeping the mode to *wait*. If she changes the mode to $check^k$, then one lets $ui \in \mathrm{Pref}(U)$ and defines $\lambda_{u \cdot i}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Éloïse choosing direction i and changing the mode to $check^k$ (this k is the one such that U is a k-pseudo binary tree U).

Now assume that we have defined λ_u for some node $u \in \mathrm{Pref}(U)$ and that the mode in λ_u has been switched from *wait* to $check^k$. Then let $(q, t(u), q_0, q_1)$ be the transition Éloïse plays from λ_u when she respects φ. Then we have two possible situations depending whether, right after playing $(q, t(u), q_0, q_1)$ and still respecting φ, Éloïse chooses the direction or lets Abélard make that choice. If she chooses the direction, let i be this direction: then one lets $ui \in \mathrm{Pref}(U)$ and defines $\lambda_{u \cdot i}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Éloïse choosing direction i. If she lets Abélard choose the direction, one lets u belongs to U and lets both $u0$ and $u1$ belongs to $\mathrm{Pref}(U)$ and defines $\lambda_{u \cdot i}$ for $i \in \{0, 1\}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1)$, followed by Éloïse letting Abélard choose the direction and Abélard picking direction i. Note that for any $ui \in \mathrm{Pref}(U)$, $\lambda_{u \cdot i}$ ends with the pebble on $u \cdot i$ with state q_i attached to it, equivalently in configuration (q_i, ui).

The run ρ is defined by letting, for any $u \in \mathrm{Pref}(U)$, $\rho(u)$ be the state attached to the pebble in the last configuration of λ_u. For those $u \notin \mathrm{Pref}(U)$ we define $\rho(u)$ so that the resulting run is valid, which is always

possible as we only consider complete automata. By construction, ρ is a run of $\mathcal{A}$ on t. Moreover with any branch π consisting only of nodes in $\mathrm{Pref}(U)$ one can associate a play λ_π in $\widetilde{\mathbb{G}}_{\mathcal{A},t}^{\mathrm{Acc\,Uncount}}$ from $(q_{\mathrm{ini}}, \varepsilon, wait)$ where Éloïse respects φ (one simply considers the limit of the *increasing* sequence of partial plays λ_u where u ranges over nodes along branch π). As λ_π is winning it follows easily that U is a k-pseudo binary tree (indeed, condition (i) and (ii) from the definition of a k-pseudo binary tree are immediate, while condition (iii) follows from the definition of the winning condition and of the fact that λ_π is winning). Moreover π is accepting as the smallest colour infinitely often visited is k. As there are uncountably many branches π consisting only of nodes in $\mathrm{Pref}(U)$ we conclude that ρ contains uncountably many accepting branches, meaning that $t \in L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$.

Conversely, assume that there is a run ρ of $\mathcal{A}$ on t that contains uncountably many accepting branches. By Lemma 3, it follows that ρ contains a k-pseudo binary tree U. Let $X = \{x \in \mathrm{Pref}(U) \mid \rho(x) > k\}$: then by definition of a k-pseudo binary tree we conclude that X is finite and has a minimal element for the prefix relation (with the convention that if X is empty this minimum is set to be the root ε); call r this minimum. Note that there is also a minimum element u_0 in U (for the prefix relation) and that $r \sqsubset u_0$.

From ρ and U we define a strategy φ of Éloïse in $\widetilde{\mathbb{G}}_{\mathcal{A},t}^{\mathrm{Acc\,Uncount}}$ from $(q_{\mathrm{ini}}, \varepsilon, wait)$ as follows. Strategy φ uses as a memory a node $v \in \mathrm{Pref}(U)$ and initially $v = \varepsilon$; moreover as long as $v \sqsubseteq r$ the play will be in *wait* mode. Now assume that the pebble is on some node v with state q attached to it (one will inductively check that $v \in \mathrm{Pref}(U)$ and that $\rho(v) = q$). Then we have several possibilities.

- The mode is *wait* (i.e. $v \sqsubseteq r \sqsubset u_0$): strategy φ indicates that Éloïse chooses transition $(q, t(v), \rho(v0), \rho(v1))$, goes to direction i where i is such that $vi \sqsubseteq u_0$, and stay in mode *wait* except if $v = r$ where the mode is switched to $check^k$.

- The mode is $check^k$ and $v \in U$. Both $v0$ and $v1$ belong to $\mathrm{Pref}(U)$: strategy φ indicates that Éloïse chooses transition $(q, t(v), \rho(v0), \rho(v1))$ and let Abélard choose the direction, say i. Then the memory is updated to $v \cdot i$.

- The mode is $check^k$ and $v \notin U$. Hence, $v \cdot i$ belong to $\mathrm{Pref}(U)$ for only one $i \in \{0, 1\}$: strategy φ indicates that Éloïse chooses transition $(q, t(v), \rho(v0), \rho(v1))$ and chooses direction i. Then the memory is updated to $v \cdot i$.

Now consider a play λ where Éloïse respects her strategy φ. It is easily seen that with λ one associates a branch π in the run ρ and that this branch goes only through nodes in $\mathrm{Pref}(U)$. From this observation and from the definition of a k-pseudo binary tree, we conclude that λ is winning for Éloïse hence, that strategy φ is winning from $(q_{\mathrm{ini}}, \varepsilon, wait)$. $\qquad\square$

Remark 2. *In the winning condition we can remark the following (the first two points are obvious, the third one requires to adapt the proof of Theorem 10 that then can simplify the construction of a Büchi automaton accepting $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$ (see Theorem 11 below). Condition (1) is in fact implied by Condition (2). Condition (3) can be enforced by removing the edges of the game graph that violate it. Condition (4) can be replaced by:*

(5) Éloïse lets Abélard infinitely often choose the direction during the play.

while preserving the validity of Theorem 10. Indeed, one can remark that if we require condition (5) instead of Condition (4) and follows the same lines as in the proof of Theorem 10 in the direct implication (the converse implication is unchanged) the pseudo binary tree we build from a winning strategy φ for Éloïse may no longer be a k-pseudo binary tree; however, thanks to Condition (2), one can easily extract a k-pseudo binary tree from it.

5.2.4. Languages of the Form $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$ Are Büchi Regular

Thanks to Theorem 10 we can easily prove that any language of the form $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A})$ can be accepted by a *Büchi* automaton.

Theorem 11. *Let $\mathcal{A} = \langle A, Q, q_{ini}, \Delta, \mathrm{Col} \rangle$ be a parity tree automaton using d colours. Then there exists a Büchi tree automaton $\mathcal{A}' = \langle A, Q', q_{ini}', \Delta', \mathrm{Col}' \rangle$ such that $L_{Uncount}^{\mathrm{Acc}}(\mathcal{A}) = L(\mathcal{A}')$. Moreover $|Q'| = \mathcal{O}(d|Q|)$.*

Proof. One can easily transform game $\widetilde{\mathbb{G}}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$ to obtain an equivalent game that is the acceptance game of some tree automaton $\mathcal{A}'$ with the classical semantics. The construction is very similar to the one we had for the other cases and we omit the details here. It simply suffices to notice that the winning condition in $\widetilde{\mathbb{G}}_{\mathcal{A},t}^{\mathrm{Acc\ Uncount}}$ is a conjunction of Büchi conditions, hence can be rephrased as a Büchi condition (up to adding some flags). $\qquad\square$

6. Checking Topological Largeness of Accepting Branches

We now consider the case of automata with topological bigness constraints and we prove that languages of the form $L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$ are always ω-regular (Theorem 13). This acceptance condition is referred to as the best model of a *fair adversary* in [13], and finite games where Éloïse plays against such an adversary have been studied and solved in [14]. We first characterise large set of branches (Lemma 5), then based on this, we define an acceptance game for $L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$ and finally we transform it to obtain an *equivalent* game that has the form of a classical acceptance game from which we extract an equivalent automaton with a classical semantic.

The Banach-Mazur theorem gives a game characterisation of large and meagre sets of branches (see for instance [22, 20]). The **Banach-Mazur game** on a tree t, is a two-player game where Abélard and Éloïse choose alternatively a node in the tree, forming a branch: Abélard chooses first a node and then Éloïse chooses a descendant of the previous node and Abélard chooses a descendant of the previous node and so on forever. In this game it is **always Abélard that starts** a play.

Formally a **play** is an infinite sequence $u_1, u_2, \ldots$ of words in $\{0,1\}^+$, and the branch associated with this play is $u_1 u_2 \cdots$. A **strategy** for Éloïse is a mapping $\varphi : (\{0,1\}^+)^+ \to \{0,1\}^+$ that takes as input a finite sequence of words, and outputs a word. A play $u_1, u_2, \ldots$ **respects** φ if for all $i \geqslant 1$, $u_{2i} = \varphi(u_1, \ldots, u_{2i-1})$. We define $Outcomes(\varphi)$ as the set of plays that respect φ and $\mathcal{B}(\varphi)$ as the set branches associated with the plays in $Outcomes(\varphi)$.

The Banach-Mazur theorem (see[7] *e.g.* [20, Theorem 4]) states that a set of branches B is large if and only if there exists a strategy φ for Éloïse such that $\mathcal{B}(\varphi) \subseteq B$.

Furthermore a folk result (see *e.g.* [20, Theorem 9]) about Banach-Mazur games states that when B is Borel[8] one can look only at "simple" strategies, defined as follows. A **decomposition-invariant strategy** is a mapping $f : \{0,1\}^* \to \{0,1\}^+$ and we associate with f the strategy φ_f defined by $\varphi_f(u_1, \ldots, u_k) = f(u_1 \cdots u_k)$. Finally, we define $Outcomes(f) = Outcomes(\varphi_f)$ and $\mathcal{B}(f) = \mathcal{B}(\varphi_f)$. The folk result states that for any Borel set of branches B, there exists a strategy φ such that $Outcomes(\varphi) \subseteq B$ if and only if there exists a decomposition-invariant strategy f such that $\mathcal{B}(f) \subseteq B$.

Call a set of nodes $W \subseteq \{0,1\}^*$ **dense** if $\forall u \in \{0,1\}^*$, $\exists w \in W$ such that $u \sqsubseteq w$. Given a dense set of nodes W, the set of **branches supported by W**, $\mathcal{B}(W)$ is the set of branches π that have infinitely many prefixes in W. Using the existence of decomposition-invariant winning strategies in Banach-Mazur games, the following lemma characterises large sets of branches.

Lemma 5. *A Borel subset of branches $B \subseteq \{0,1\}^\omega$ is large if and only if there exists a dense set of nodes $W \subseteq \{0,1\}^*$ such that $\mathcal{B}(W) \subseteq B$.*

Proof. Assume that B is large and let f be a decomposition-invariant strategy for Éloïse in the associated Banach-Mazur game (recall that we assumed B to be Borel). Consider the set:

$$W = \{vf(v) \mid v \in \{0,1\}^*\}.$$

The set W is dense (as for all $v \in \{0,1\}^*$, $v \sqsubset vf(v) \in W$). We claim that $\mathcal{B}(W)$ is included in B. Let π be a branch in $\mathcal{B}(W)$. As π has infinitely many prefixes in W, there exists a sequence of words $u_1, u_2, \cdots$ such that $u_1 f(u_1) \sqsubset u_2 f(u_2) \sqsubset \cdots \sqsubset \pi$. As the lengths of the u_i are strictly increasing, there exists a sub-sequence $(v_i)_{i \geqslant 1}$ of $(u_i)_{i \geqslant 1}$ such that for all $i \geqslant 1$, $v_i f(v_i) \sqsubset v_{i+1}$. Now, consider the play in the Banach-Mazur game where Abélard first moves to v_1 and then Éloïse responds by going to $v_1 f(v_1)$. Then Abélard moves to v_2

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[7]In [20] the players of the Banach-Mazur game are called 0 and 1 and Player 0 corresponds to Abélard while player 1 corresponds to Éloïse. Hence, when using a statement from [20] for our setting one has to keep this in mind as well as the fact that one must replace the winning condition by its complement (hence, replacing "meagre" by "large").

[8]This statement holds as soon as the Banach-Mazur games are determined and hence, in particular for Borel sets.

(which is possible as $v_1 f(v_1) \sqsubset v_2$) and Éloïse moves to $v_2 f(v_2)$. And so on. In this play Éloïse respects the strategy f and therefore wins. Hence, the branch π associated to this play belongs to B.

Conversely let W be a dense set of nodes such that $\mathcal{B}(W) \subseteq B$. To show that B is large, we define a decomposition-invariant strategy f for Éloïse in the associated Banach-Mazur game. For all nodes u we pick v of W such that u is a strict prefix of v (since W is dense there must always exist such a v). Let $v = uu'$ and fix $f(u) = u'$. A play where Éloïse respects f goes through infinitely many nodes in W (as f always points to an element in W). Hence, the branch associated with the play belongs to $\mathcal{B}(W) \subseteq B$ which shows that f is winning for Éloïse. $\qquad\square$

We aim to define, for a given tree an acceptance game witnessing membership into $L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$. In this game, Éloïse describes a run ρ together with a dense set U of nodes while Abélard tries either to prove that U is not dense or that there is a rejecting branch in $\mathcal{B}(U)$. The way Éloïse describes a run is as usual (she proposes valid transitions); the way she describes U is by (1) indicating explicitly when a node is in U and; (2) at each node giving a direction i that should lead (by iteratively following the directions) to a node in U. Abélard chooses the direction: if it does not select i and does not go to a node in U the colour is a large even one (preventing him not to follow Éloïse forever); if he chooses i but does not go to a node in U the colour is a large odd one (forcing Éloïse to describe a dense set U); and if he chooses i and goes to a node in U the colour is the smallest one seen since the last visit to a node in U (and it is computed in the game).

Before formally constructing the game we need the following lemma. A **_direction mapping_** is a mapping $d : \{0,1\}^* \to \{0,1\}$, and given a set of nodes U, we say that d **_points to U_** if for every node v there exists $i_1, \ldots, i_k \in \{0,1\}$ such that $vi_1 \cdots i_k \in U$ and for all $1 \leqslant j \leqslant k$, $i_j = d(vi_1 \cdots i_{j-1})$.

Lemma 6. *A set of nodes U is dense if and only if there exists a direction mapping that points to U.*

Proof. Assume that U is dense. We define $d(v)$ by induction on v as follows. Let v such that $d(v)$ is not yet defined, we pick a node $vi_1 \cdots i_k \in U$ (there must exists one since U is dense), and for all $j \leqslant k$ we define

$$d(vi_1 \cdots i_{j-1}) = i_j.$$

The mapping is defined on every node and satisfies the requirement by definition. The other implication is straightforward (for all nodes v, there exists $vi_1 \cdots i_k \in U$). $\qquad\square$

Fix a tree t and define an acceptance game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,Large}$ for $L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$ as follows. The game is played along a tree, Éloïse chooses the transitions of the automaton and Abélard chooses the directions. Furthermore, at each node Éloïse proposes a direction that Abélard may or may not follow, and possibly marks some of the children of the current state. In Éloïse's vertices, we keep track of informations about the choice of Abélard in his previous move differentiating three possible situations:

($\star$) Abélard has picked a child that Éloïse has marked,

($\circ$) Abélard has not picked a marked child, but he has followed the direction that Éloïse has given,

($\square$) Abélard has not picked a marked child and has not followed the direction given by Éloïse.

Therefore Éloïse's vertices will be of the form $(q, u, symb)$ with q a state, u a node, and $symb \in \{\star, \circ, \square\}$, and we define the colour of this vertex as the colour of q, and Abélard's state will be of the form (q, u, q_0, q_1, i, S) where $(q, t(u), q_0, q_1)$ is a transition of the automaton, $i \in \{0,1\}$ is the direction that Éloïse has proposed in the previous turn and $S \subseteq \{0,1\}$ describes which children of u she marked (see Figure 10 for the local structure of the game).

The acceptance condition for Éloïse is described as follows. She wins a play if and only if one of the following occurs.

- There are infinitely many $\star$-vertices and the smallest colour appearing infinitely often is even

- Eventually there are no more $\star$-vertices but there are infinitely often $\square$-vertices, *i.e.* Abélard stop visiting marked nodes and avoids infinitely often the directions given by Éloïse.
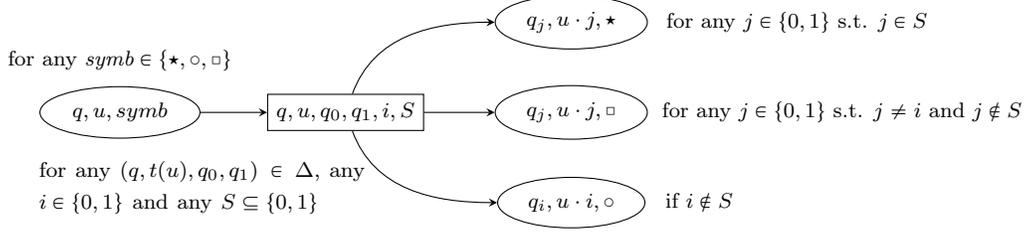
Figure 10: Local structure of the arena of the acceptance game $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,Large}$.

Call $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,Large}$ this game.

Intuitively a strategy of Éloïse is a run of the automaton over the tree, along with a set U of marked nodes and directions on each of the nodes, and Abélard chooses a branch along the tree. If at some point Abélard follows forever the directions given by Éloïse without going through a marked node, then Abélard wins. If Abélard goes infinitely often through a marked node, then the smallest colour seen infinitely often is the one of the branch in the run of Éloïse, therefore Éloïse wins if this branch is accepting. These two remarks intuitively mean that if Éloïse has a winning strategy, then the set U of marked nodes implied by this strategy must be a dense set and $\mathcal{B}(U)$ must consist only of accepting branches of the run, therefore the set of accepting branches of the run is large.

On the other hand, if there exists a run whose set of accepting branches is large, there exists a dense set of nodes U such that all branches in $\mathcal{B}(U)$ are accepting (Lemma 5), and directions on each nodes that leads to nodes in U (Lemma 6). If Éloïse plays according to them, she wins the game. Indeed, if Abélard follows infinitely often the nodes in U, then the branch is an accepting branch and therefore Éloïse wins the game. His only option to avoid the nodes of U is to infinitely often go in the opposite direction than the one given by Éloïse, in which case Éloïse also wins.

The next theorem states that $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,Large}$ is an acceptance game for $L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$.

Theorem 12. *One has $t \in L_{\mathrm{Large}}^{\mathrm{Acc}}(\mathcal{A})$ if and only if Éloïse wins in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,Large}$ from $(q_{\mathrm{ini}}, \varepsilon, \circ)$.*

Proof. Assume that Éloïse has a winning strategy φ in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,Large}$ from $(q_{\mathrm{ini}}, \varepsilon, \circ)$. With φ we associate a run ρ of $\mathcal{A}$ on t as follows. We inductively associate with any node u a partial play λ_u where Éloïse respects φ. For this we let $\lambda_\varepsilon = (q_{\mathrm{ini}}, \varepsilon, \circ)$. Now assume that we defined λ_u for some node u and let $(q, t(u), q_0, q_1, i, S)$ be the transition Éloïse plays from λ_u when she respects φ.

For $j \in \{0, 1\}$, one defines $\lambda_{u \cdot j}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1, i, S)$, followed by Abélard choosing direction j (*i.e.* we extend λ_u by the unique successor that respects φ and then let we Abélard choose direction j). Note that for $j \in \{0, 1\}$, $\lambda_{u \cdot j}$ ends in configuration $(q_j, uj, symb)$ for some $symb \in \{\star, \circ, \square\}$.

The run ρ is defined by letting $\rho(u)$ be the state q in the last configuration $(q, u, symb)$ of λ_u. By construction, ρ is a valid run of $\mathcal{A}$ on t and moreover with any branch π in ρ one can associate a play λ_π in $\mathbb{G}_{\mathcal{A},t}^{\mathrm{Acc}\,Large}$ from $(q_{\mathrm{ini}}, \varepsilon, \circ)$ where Éloïse respects φ (one simply considers the limit of the *increasing* sequence of partial plays λ_u where u ranges over those nodes along branch π). By construction π is accepting if and only if λ_π fulfils the parity condition.

We define $s(u)$ as the symbol $symb$ in the last configuration $(q, u, symb)$ of λ_u. Furthermore, we define a direction mapping d and a set of nodes U as follows: for all u, $d(u) = i$ with $\varphi(\lambda_u) = (q, u, q_0, q_1, i, S)$; and for all u, $u \in U$ if and only if $s(u) = \star$. Notice that if $d(u) = i$ then $s(u \cdot i) = \circ$ or $s(u \cdot i) = \star$.

Given a branch $\pi = i_1 i_2 \cdots$ we define $s(\pi)$ as the infinite sequence of $s(\varepsilon)s(i_1)s(i_1 i_2) \cdots$, and $\mathrm{Col}(\pi)$ as the smallest colours appearing infinitely often in $\rho(\pi)$. Note that since Éloïse wins the play λ_π, $\star$ appears infinitely often in $s(\pi)$ and $\mathrm{Col}(\pi)$ is even, or $\star$ does not appear infinitely often in $s(\pi)$ but $\square$ does.

First let us show that d points to U. Suppose by contradiction that this is not the case, *i.e.* there exists a branch $\pi = u i_1 i_2 \cdots$, with $i_j = d(u i_1 \cdots i_{j-1})$ for all $j \geqslant 1$, such that for all $k \geqslant 1$, $u i_1 \cdots i_k \notin U$. Then for all $k \geqslant 1$, $s(u i_1 \cdots i_k) = \circ$, therefore λ_π is loosing. This raises a contradiction since φ is a winning strategy.

Now, let us show that all branches in $\mathcal{B}(U)$ are winning in ρ. Let $\pi \in \mathcal{B}(U)$. Then by definition, $\star$ appears infinitely often in $s(\pi)$. Then since λ_π is winning we have that $\mathrm{Col}(\pi)$ is even, then π is an accepting branch

23

in ρ.

Conversely let ρ be a run whose set of accepting branches is large. From Lemma 6 there exist a direction mapping d and a set of nodes U such that d points to U, and every branch $\pi \in \mathcal{B}(U)$ is accepting in ρ. Define the strategy φ of Éloïse as follows. For all partial play λ ending in $(\rho(u), u, symb)$

$$\varphi(\lambda) = (\rho(u), u, \rho(u0), \rho(u1), d(u), \{j \mid uj \in U\}),$$

and for all other plays, we do not give any restriction on $\varphi(\lambda)$ (assuming that the automaton is complete, Éloïse can always play something). Let us show that φ is a winning strategy for Éloïse.

As for the other direction, we inductively associate with any node u a partial play λ_u where Éloïse respects φ. For this we let $\lambda_\varepsilon = (q_{\text{ini}}, \varepsilon, \circ)$. Now assume that we defined λ_u for some node u and let $(q, t(u), q_0, q_1, i, S)$ be the transition Éloïse plays from λ_u when she respects φ. For $j \in \{0, 1\}$, one defines $\lambda_{u \cdot j}$ as the partial play obtained by extending λ_u by Éloïse choosing transition $(q, t(u), q_0, q_1, i, S)$, followed by Abélard choosing direction j. Note that for $j \in \{0, 1\}$, $\lambda_{u \cdot j}$ ends in configuration $(q_j, uj, symb)$ for some $symb \in \{\star, \circ, \square\}$.

Moreover with any branch π in ρ one can associate a play λ_π in $\mathbb{G}_{\mathcal{A}, t}^{\text{Acc } Large}$ from $(q_{\text{ini}}, \varepsilon, \circ)$ where Éloïse respects φ (one simply considers the limit of the *increasing* sequence of partial plays λ_u where u ranges over those nodes along branch π). By construction π is accepting in ρ if and only if λ_π fulfils the parity condition. Furthermore observe that any play that respects φ is equal to λ_π for some branch π. Again, we define $s(u)$ as the symbol $symb$ in the last configuration $(q, u, symb)$ of λ_u. Observe that if $s(u \cdot i) = \circ$ for some node u and $i \in \{0, 1\}$ then $i = d(u)$.

Let λ_π be a play that respects φ. Note that Éloïse wins the play λ_π if and only if $\star$ appears infinitely often in $s(\pi)$ and $\text{Col}(\pi)$ is even, or $\star$ does not appear infinitely often in $s(\pi)$ but $\square$ does. First observe that $u \in U$ if and only if $s(u) = \star$. If $\star$ appears infinitely often in $s(\pi)$ then π is in $\mathcal{B}(U)$ therefore it is accepting, thus λ_π is winning. If $\star$ does not appear infinitely often in $s(\pi)$ let u and $i_1, i_2, \ldots$ be such that $\pi = u i_1 i_2 \cdots$ and for all k, $s(u i_1 \cdots i_k) \neq \star$. Assume by contradiction that $\square$ does not appears infinitely often in $s(\pi)$. Therefore there exists ℓ such that for all $k \geqslant \ell$, $s(u i_1 \cdots i_k) = \circ$, thus $i_k = d(u i_1 \cdots i_{k-1})$. Thus π is a branch where at some point d is followed, but no node in U is eventually reached, which means that d does not point to U hence, raises a contradiction.

Therefore λ_π is a winning play, thus φ is a winning strategy. $\qquad\square$

Thanks to Theorem 12 we can now easily prove that languages of the form $L_{\text{Large}}^{\text{Acc}}(\mathcal{A})$ are always ω-regular.

Theorem 13. *Let $\mathcal{A} = \langle A, Q, q_{ini}, \Delta, \text{Col} \rangle$ be a parity tree automaton using d colours. Then there exists a parity tree automaton $\mathcal{A}' = \langle A, Q', q'_{ini}, \Delta', \text{Col}' \rangle$ such that $L_{\text{Large}}^{\text{Acc}}(\mathcal{A}) = L(\mathcal{A}')$. Moreover $|Q'| = \mathcal{O}(d|Q|)$ and $\mathcal{A}'$ uses $d + 2$ colours.*

Proof. The game $\mathbb{G}_{\mathcal{A}, t}^{\text{Acc } Large}$ can be transformed into a standard acceptance game for ω-regular language (as defined in Section 2.3) by the following trick (this is the same as the one for $\mathbb{G}_{\mathcal{A}, t}^{\prime \text{Rej} \leqslant \text{Count}}$). One adds to states an integer where one stores the smallest colour seen since the last $\star$-state was visited (this colour is easily updated); whenever a starred state is visited the colour is reset to the colour of the state. Now $\square$-states are given an even colour e that is greater or equal than all colour previously used (hence, it ensures that if finitely many $\star$-states but infinitely many $\square$-states are visited then Éloïse wins), $\circ$-states are given the odd colour $e + 1$ (hence it ensures that if at some points only $\circ$-states are visited, Éloïse looses) and starred states are given the colour that was stored (hence, if infinitely many starred states are visited we retrieve the previous parity condition). It should then be clear that the latter game is a classical acceptance game, showing that $L_{\text{Large}}^{\text{Acc}}(\mathcal{A})$ is ω-regular.

The construction of $\mathcal{A}'$ is immediate from the final game and the size is linear in $d|Q|$ due to the fact that one needs to compute the smallest colour visited between to starred states. $\qquad\square$

7. Conclusion and Perspectives

In this paper we proved, using a game-theoretic approach, that the languages defined by automata with cardinality constraints as well as those defined by automata with topological bigness constraints are always ω-regular. Moreover, in the case where the cardinality constraint is on the number of accepting branches we showed that the languages collapse to those defined by classical Büchi tree automata, which contrasts with the case where the cardinality constraint is expressed on the number of rejecting branches.

One advantage of our approach is that it preserves the well-known connection between automata and games and that it permits to have tractable transformations (which might not be the case when using *e.g.* the more general logical approach of [12]).

One technical restriction of our work is that we required our automata to be complete. This restriction was made for ease of presentation and can easily be removed thanks to the following trick (pointed to us by an anonymous reviewer). It suffices that we modify the various constructions of acceptance games we consider as follows. Every time Éloïse determines the direction in the acceptance game Abélard has now the option to either accept her choice or to challenge her. In the former case the play continues as usual. But if Abélard challenges Éloïse, we enter a new phase where she has to prove (in the same flavour as in the classical setting described in Section 2.3) that she can construct a run (not necessarily accepting) on the subtree: if she is able to do so she wins otherwise Abélard wins.

A first perspective has to do with applications. Indeed, while the present work is at the theoretical level only, it permits to observe that classical questions (membership, emptiness) for the classes of languages under consideration are tractable (at least not harder than for classical ω-regular languages). Hence, it would be interesting to find examples where automata with cardinality constraints or with topological bigness constraints are a relevant way to express properties of systems: in particular we believe that it can be an alternative to qualitative tree languages [15, 16] when specifying properties of non-deterministic system where one wants to allow a negligible set of bad executions.

A second perspective has to do with games played on graphs. A generalisation of two-player games is given by stochastic games (see *e.g.* [23]) where a third, uncontrollable and unpredictable, player is added to the two usual antagonistic players. When the two first players aim to represent respectively a program and its (potentially malicious) environment, the third player can be seen as an abstraction of nature. Usually the nature player is equipped with a stochastic semantics and typical (qualitative) questions are whether the first player has a strategy that, against any strategy of the second player, almost-surely satisfies an objective (against all possible behaviours of the nature player). In other words, one seeks for a strategy that against any strategy of the second player produces a set of plays such that the losing plays form a negligible subset, negligible being understood in a measure theoretic sense (namely as being of measure 0). Following the ideas developed in this paper, one can consider qualitative questions on two-player games with a third nature player where one uses either counting or topology to define which sets of plays should be considered as being negligible. This seems a promising approach as we showed in our recent work [24].

A third perspective has to do with logic. One question is whether there is a natural logic that exactly captures the classes of automata studied in this paper. Obviously this should be a (strict) fragment of the monadic second order logic. A second question is to study how the automata models considered in the present work compare/can be combined with the quantifiers studied in the work of Bárány, Kaiser and Rabinovich [12] and in the recent study initiated by Michalewski and Mio in [25].

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References

[1] M. O. Rabin, Decidability of second-order theories and automata on infinite trees, Transactions of the American Mathematical Society 141 (1969) 1–35.

[2] M. O. Rabin, Automata on infinite objects and Church's problem, American Mathematical Society, Providence, R.I., 1972, conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 13.

[3] A. Church, Logic, arithmetic and automata, in: Proceedings of the International Congress of Mathematicians, 1962, pp. 23–35.

[4] W. Thomas, Languages, automata, and logic, in: G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Language Theory, Vol. III, Springer-Verlag, 1997, pp. 389–455.

[5] M. Y. Vardi, T. Wilke, Automata: from logics to algorithms, in: Logic and Automata: History and Perspectives, Amsterdam University Press, 2007, pp. 629–736.

[6] Y. Gurevich, L. Harrington, Trees, automata, and games, in: Proceedings of the Fourteenth Annual ACM Symposium on the Theory of Computing (STOC'82), ACM, 1982, pp. 60–65.

[7] J. R. Büchi, Using determinancy of games to eliminate quantifiers, in: Proceedings of the first International Conference on Fundamentals of Computation Theory (FCT'77), Vol. 56 of Lecture Notes in Computer Science, Springer-Verlag, 1977, pp. 367–378.

[8] E. Grädel, W. Thomas, T. Wilke (Eds.), Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001], Vol. 2500 of Lecture Notes in Computer Science, Springer-Verlag, 2002.

[9] C. Löding, Infinite games and automata theory, in: K. R. Apt, E. Grädel (Eds.), Lectures in Game Theory for Computer Scientists, Cambridge University Press, 2011, pp. 38–73.

[10] D. Beauquier, M. Nivat, D. Niwiński, About the effect of the number of successful paths in an infinite tree on the recognizability by a finite automaton with Büchi conditions, in: Proceedings of the 8th International Conference on Fundamentals of Computation Theory (FCT'91), Vol. 529 of Lecture Notes in Computer Science, Springer-Verlag, 1991, pp. 136–145.

[11] D. Beauquier, D. Niwiński, Automata on infinite trees with path counting constraints, Information and Computation 120 (1) (1995) 117 – 125.

[12] V. Bárány, L. Kaiser, A. Rabinovich, Expressing cardinality quantifiers in monadic second-order logic over trees, Fundamenta Informaticae 100 (2010) 1–18.

[13] H. Völzer, D. Varacca, E. Kindler, Defining fairness, in: Proceedings of the 16th International Conference on Concurrency Theory (CONCUR 2005), Vol. 3653 of Lecture Notes in Computer Science, Springer-Verlag, 2005, pp. 458–472.

[14] E. Asarin, R. Chane-Yack-Fa, D. Varacca, Fair adversaries and randomization in two-player games, in: Proceedings of the 13th International Conference on Foundations of Software Science and Computational Structures (FOSSACS'10), Vol. 6014 of Lecture Notes in Computer Science, Springer-Verlag, 2010, pp. 64–78.

[15] A. Carayol, A. Haddad, O. Serre, Qualitative tree languages, in: Proceedings of the 26th IEEE Symposium on Logic in Computer Science (LiCS 2011), IEEE Computer Society, 2011, pp. 13–22.

[16] A. Carayol, A. Haddad, O. Serre, Randomisation in automata on infinite trees, ACM Transactions on Computational Logic 15 (3) (2014) 24:1–24:33.

[17] N. Fijalkow, S. Pinchinat, O. Serre, Emptiness of alternating tree automata using games with imperfect information, in: Proceedings of the 33rd International Conference on Foundations of Software Technology and Theoretical Computer Science (FST&TCS 2013), Vol. 24 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2013, pp. 299–311.

[18] D. Perrin, J.-É. Pin, Infinite Words, Vol. 141 of Pure and Applied Mathematics, Elsevier, 2004.

[19] H. Völzer, D. Varacca, Defining fairness in reactive and concurrent systems, Journal of the Association for Computing Machinery (ACM) 59 (3) (2012) 13.

[20] E. Grädel, Banach-Mazur Games on Graphs, in: Proceedings of the 28th International Conference on Foundations of Software Technology and Theoretical Computer Science (FST&TCS 2008), Vol. 2 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2008, pp. 364–382.

[21] D. A. Martin, Borel determinacy, Annals of Mathematics 102 (2) (1975) 363–371.

[22] A. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, Springer-Verlag, 1995.

[23] K. Chatterjee, Stochastic ω-regular games, Ph.D. thesis, University of California Berkeley (2007).

[24] A. Carayol, O. Serre, How good is a strategy in a game with nature?, in: Proceedings of the 30th Annual IEEE Symposium on Logic in Computer Science (LiCS 2015), IEEE Computer Society, 2015, to appear.

[25] H. Michalewski, M. Mio, Baire category quantifier in monadic second order logic, in: Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP 2015), Vol. 9135 of Lecture Notes in Computer Science, Springer-Verlag, 2015, pp. 362–374.