# Straight-line drawing of quadrangulations

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**Abstract.** This article introduces a straight-line drawing algorithm for quadrangulations, in the family of the face-counting algorithms. It outputs in linear time a drawing on a regular $W \times H$ grid such that $W + H = n - 1 - \Delta$, where $n$ is the number of vertices and $\Delta$ is an explicit combinatorial parameter of the quadrangulation.

## 1 Introduction

A *plane graph* $G$ (also called planar map) is a graph embedded in the plane, defined up to continuous deformation. A major issue in graph drawing is to compute a *straight-line drawing* of $G$, i.e., to place vertices of $G$ at points of a regular $W \times H$ grid ($W$ being called the width and $H$ the height of the grid), so that the drawing obtained by linking adjacent vertices by segments is a planar drawing of $G$. An extensive literature is devoted to straight-line drawing of triangulations, i.e., plane graphs with faces of degree 3, and more generally of 3-connected plane graphs. Essentially there are two classes of linear time algorithms: a) purely iterative algorithms based on a specific order of treatment of vertices, the drawing being globally updated at each step [5, 8, 10]; b) one-shot algorithms, where the plane graph is first endowed with a combinatorial structure (e.g. Schnyder woods for triangulations), which is used to compute the coordinates of vertices, usually using some face-counting operations [11, 2, 6]. The algorithms of the second class have the advantage that the coordinates of vertices are computed independently, so that they are easier to implement and to perform on a piece of paper. In addition, the grid size can be expressed in terms of combinatorial parameters of the graph.

Surprisingly, little attention has been given to straight-line drawing algorithms dealing specifically with quadrangulations, i.e., plane graphs with all faces of degree 4 (these are also called maximal bipartite plane graphs). The only reference we found is an article by Biedl and Brandenburg [1], where it is shown that a quadrangulation $Q$ can be triangulated into a triangulation $T$ of the 4-gon with no separating triangle, so that the algorithm of Miura et al. [10] can be called to embed $T$ (hence, also $Q$) on a $(\lceil n/2 \rceil - 1) \times \lfloor n/2 \rfloor$ grid. This algorithm is linear and has small grid size, but it does not really use a combinatorial structure specific to quadrangulations. In contrast, the algorithm we introduce exploits a *quadri-partition* of the inner edges of a quadrangulation, presented in Section 3. This combinatorial structure is also investigated in [7] using another terminology (with labels) and is closely related to a bicoloration of the edges of a quadrangulation [4]. We use the quadri-partition to triangulate partially $Q$ into a plane graph $G$ endowed with a so-called *transversal structure*, i.e., a bicoloration and orientation of the inner edges of $G$ with specific local conditions. Transversal structures were originally only defined for triangulations of a 4-gon [6, 9]. The definition is easily extended to *partially triangulated quadrangulations* in Section 2. Then, the transversal structure is used to draw $G$ (hence, also $Q$) with face-counting operations. To do this, we extend the algorithm of [6], which draws triangulations of the 4-gon, to partially triangulated quadrangulations. The obtained drawing of $Q$ verifies $W + H = n - 1 - \Delta$, where $n$ is the number of vertices and $\Delta$ is an explicit parameter of $Q$, see Theorem 1. Hence, the semi-perimeter is at most as large as in [1]. In addition, our algorithm has the advantage of being of the *one-shot* type.

## 2 Transversal structures

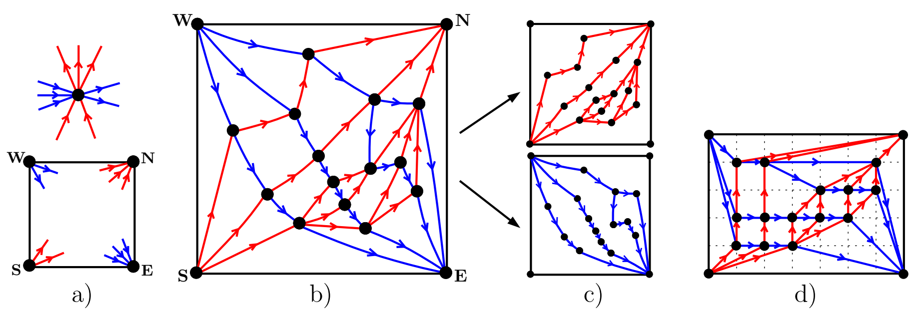


**Fig. 1.** The local conditions C1 and C2 for transversal structures (Fig. (a)), a partially triangulated quadrangulation $G$ endowed with a transversal structure (Fig. (b)), the associated red map and blue map (Fig. (c)), and the straight-line drawing of $G$ using TransversalDraw(G) (Fig. (d)).

Let $G$ be a plane graph with quadrangular outer face, the four outer vertices in cw order being denoted by $N$, $E$, $S$ and $W$ (like North, East, South, West). A *transversal structure* is an orientation and partition of the inner edges of $G$ into red and blue edges, satisfying the following conditions (see Figure 1(a)):

**C1: (Inner vertices)** Each inner vertex of $G$ is incident in cw order to: a non-empty interval of outgoing red edges, a non-empty interval of outgoing blue edges, a non-empty interval of ingoing red edges, and a non-empty interval of ingoing blue edges.

**C2: (Border vertices)** The inner edges incident to $N$, $E$, $S$, $W$ are ingoing red, ingoing blue, outgoing red, outgoing blue, respectively.

Transversal structures have been defined in [6, 9] in the case where all inner faces are triangles. However, as we will see, other plane graphs can be endowed with a transversal structure. Even if we do not need a precise condition of existence, let us mention that a necessary condition is that all inner faces have degree 3 or 4. Such a plane graph is called a *partially triangulated quadrangulation*. Let $G$ be a partially triangulated quadrangulation endowed with a tranversal structure. Then, it is easily shown (adapting the proof of [6, Prop.1]) that the red edges of $G$ form a bipolar orientation with source $S$ and sink $N$; and the blue edges form a bipolar orientation with source $W$ and sink $E$. (We recall that a bipolar orientation is an acyclic orientation with a unique vertex having only outgoing edges, called *source*, and a unique vertex having only ingoing edges, called *sink*.) As developed in [6] for triangulated graphs, this property gives rise to a straight-line drawing algorithm TransversalDraw(G), where the red edges are used to give abscissas and the blue edges are used to give ordinates. The *red map* (blue map) of $G$ is the plane graph $G_{\mathrm{r}}$ ($G_{\mathrm{b}}$) obtained by deleting all blue edges (red edges, respectively) of $G$, see Figure 1(c). Given an inner vertex $v$ of $G$, the *rightmost ingoing red path* of $v$ is the path $P_S(v) = (v_0 = v, v_1, \ldots, v_k = S)$ from $v$ to $S$ where, for each $i \in [0..k-1]$, $(v_i, v_{i+1})$ is the rightmost ingoing red edge of $v_i$, i.e., the unique ingoing edge of $v_i$ whose ccw consecutive edge around $v_i$ is outgoing. The *leftmost outgoing red path* of $v$ is the path $P_N(v) = (v_0 = v, v_1, \ldots, v_l = N)$ where, for $i \in [0..l-1]$, $(v_i, v_{i+1})$ is the leftmost outgoing red edge of $v_i$. The *separating red path* $\mathcal{P}_{\mathrm{r}}(v)$ of $v$ is the concatenation of $P_N(v)$ and $P_S(v)$. Hence, $\mathcal{P}_{\mathrm{r}}(v)$ is a path from $S$ to $N$. We define similarly the *separating blue path* $\mathcal{P}_{\mathrm{b}}(v)$ of $v$, which goes from $W$ to $E$.

TransversalDraw(G):
1. Take a regular grid of size $W \times H$, where $W$ ($H$) is the number of inner faces of the red map $G_{\mathrm{r}}$ (of the blue map $G_{\mathrm{b}}$, respectively).
2. Place the outer vertices $S$, $W$, $N$, $E$ at the corners $(0,0)$, $(0,H)$, $(W,H)$ and $(W,0)$, respectively.
3. For each inner vertex $v$ of $G$, let $x$ be the number of inner faces of $G_{\mathrm{r}}$ on the left of $\mathcal{P}_{\mathrm{r}}(v)$ and let $y$ be the number of inner faces of $G_{\mathrm{b}}$ on the right of $\mathcal{P}_{\mathrm{b}}(v)$. Place $v$ at the point of coordinates $(x, y)$.
4. Link each pair of adjacent vertices by a segment.

**Proposition 1.** *Given a partially triangulated plane graph $G$ endowed with a transversal structure,* TransversalDraw(G) *outputs a straight-line drawing of $G$ in linear time. The semi-perimeter satisfies $W + H = n - 1 - \Delta$, where $n$ is the number of vertices and $\Delta$ is the number of quadrangular faces of $G$.*

*Proof.* The correctness proof of TransversalDraw, given in [6, Theo.3] for the case of a triangulation of a 4-gon, adapts straightforwardly. The equality $W + H = n - 1 - \Delta$ is an easy consequence of Euler's relation. □

## 3 Edge partition of a quadrangulation

Let $Q$ be a quadrangulation, the four outer vertices in cw order denoted by $N$, $E$, $S$, and $W$. As all faces of $Q$ have even degree, the vertices of $Q$ can be greedily bicolored in black or white so that adjacent vertices have different colors. The bicoloration is unique up to the choice of the first vertex, hence there is a unique bicoloration such that $N$ and $S$ are black. Given $Q$ endowed with this bicoloration, the *primal map* of $Q$ is the plane graph $M$ whose vertices are the black vertices of $Q$, two black vertices being adjacent in $M$ if they are incident to the same face of $Q$. Hence, there is an edge of $M$ for each face of $Q$. Conversely, $Q$ is called the *angular map* of $M$ because each edge of $Q$ is associated with an angle of $M$. Observe also that each black vertex of $Q$ corresponds to a vertex of $M$ and each white vertex of $Q$ corresponds to a face of $M$, see Figure 2(b). As we consider quadrangulations with no double edge, it is well known that the primal map is 2-connected, i.e., the deletion of one vertex does not disconnect $M$. As detailed in [4], for any edge $(s, t)$ of $M$, there exists a bipolar orientation of $M$ with source $s$ and sink $t$, and there is a simple *sweeping algorithm* to compute such a bipolar orientation in linear time [3].

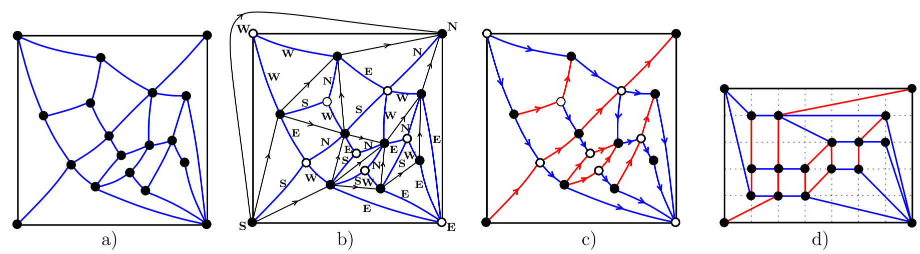


**Fig. 2.** A quadrangulation $Q$ (Fig. (a)), endowed with an angular partition of the inner edges (Fig. (b)), the associated uncomplete transversal structure (Fig. (c)), and the straight-line drawing of $Q$ (Fig. (d)).

Given a bipolar orientation $X$ of $M$ with source $S$ and sink $N$, an angle $(e, e')$ of two edges of $M$ around a black vertex $v$ —with $e'$ following $e$ in cw order around $v$— is called an angle *of type* $N$, $S$, $W$, $E$ if $(e, e')$ are (ingoing, ingoing), (outgoing, outgoing), (ingoing, outgoing), (outgoing, ingoing), respectively. Accordingly, the inner edges of $Q$ are partitioned into $N$-edges, $S$-edges, $W$-edges and $E$-edges, depending on the type of their associated angle. This partition is called the *angular partition* of $Q$ associated with $X$, see Figure 2(b). An important property of a plane bipolar orientation is that the edges incident to an inner vertex are partitioned into a non-empty interval of ingoing edges and a non-empty interval of outgoing edges; and, dually, each face $f$ of $M$ has two particular vertices $S_f$ and $N_f$ such that the contour of $f$ consists of two non-empty oriented paths both going from $S_f$ to $N_f$, called *left lateral path* and *right lateral path* of $f$, respectively. Hence, each inner black vertex $v$ of $Q$ is incident to one $W$-edge $e_W$ and one $E$-edge $e_E$, which are separated by a possibly empty interval of $N$-edges in the cw sector between $e_E$ and $e_W$ and a possibly empty interval of $S$-edges in the cw sector between $e_W$ and $e_E$. Dually, each white vertex of $Q$, corresponding to a face $f$ of $M$, is incident to one $N$-edge $e_N$ (connected to $N_f$) and one $S$-edge $e_S$ (connected to $S_f$), which are separated by a possibly empty interval of $W$-edges in the cw sector between $e_N$ and $e_S$ and a possibly empty interval of $E$-edges in the cw sector between $e_S$ and $e_N$. Observe also that all inner edges of $Q$ incident to $N$, $E$, $S$, and $W$ are $N$-edges, $E$-edges, $S$-edges, and $W$-edges respectively, see Figure 2(b).

## 4 The algorithm

Let $Q$ be a quadrangulation endowed with an angular partition of the inner edges, associated with a bipolar orientation $X$ of the primal map $M$. The *uncomplete transversal structure* is the orientation and bicoloration of the inner edges of $Q$ where the $N$-edges and $S$-edges are colored red, the $W$-edges and $E$-edges are colored blue, the $S$-edges and $E$-edges are oriented from their black to their white vertex, and the $N$-edges and $W$-edges are oriented from their white to their black vertex. Clearly, the conditions of a transversal structure are satisfied, except that some of the four intervals of edges around an inner vertex may be empty. Precisely, if a black vertex $v$ of $M$ has $i \geq 1$ ingoing edges and $j \geq 1$ outgoing edges, then $v$ is incident in cw order to an outgoing blue edge (the $E$-edge $e_E$), an interval of $(i-1)$ ingoing red edges, an ingoing blue edge (the $W$-edge $e_W$), and an interval of $(j-1)$ outgoing red edges. Dually, if a face of $M$ has a left lateral path of length $i \geq 1$ and a right lateral path of length $j \geq 1$, then the associated white vertex of $Q$ is incident in cw order to an outgoing red edge (the $N$-edge $e_N$), an interval of $(j-1)$ outgoing blue edges, an ingoing red edge (the $S$-edge $e_S$), and an interval of $(i-1)$ ingoing blue edges.

We now describe an algorithm PartTriang(Q), which adds colored oriented edges to $Q$ so as to obtain a partially triangulated plane graph $G$ endowed with a transversal structure. An edge of $M$ is said to be *undeletable* if it is the unique outgoing edge of its origin or the unique ingoing edge of its extremity (or both). An edge of $M$ different from the edge $(S, N)$ is said to be *transitive* if it connects the two poles $N_f$ and $S_f$ of a face of $M$. Notice that an edge of $M$ can not be undeletable and transitive. The plane graph $G = $ PartTriang($Q$) is obtained by adding to $Q$ the undeletable edges of $M$, colored red and oriented as in $X$; and by adding the edges dual to the transitive edges, colored blue and oriented from left to right, i.e., for each face of $Q$ associated with a transitive edge $e$ of $M$, a blue edge is added connecting the two white vertices of the face and oriented from the left of $e$ to the right of $e$. It is easily checked that the edge bicoloration and orientation of $G$ is a transversal structure, see the transition between Figure 2(b)–(c) and Figure 1(b).

**Theorem 1.** *Let $Q$ be a quadrangulation endowed with an angular partition of its inner edges. The algorithm that computes* G=PartTriang(Q)*, then calls* TransversalDraw(G)*, and finally deletes the edges added from $Q$ to $G$, is a straight-line drawing algorithm for quadrangulations with linear time complexity. All horizontal and vertical lines of the grid are occupied by at least one vertex.*

*The semi-perimeter $W+H$ of the grid satisfies*

$$W + H = n - 1 - \Delta,$$

*where $n$ is the number of vertices and $\Delta$ is the number of alternating faces of $Q$, i.e., faces whose contour consists of two red edges and two blue edges that alternate. Alternating faces are strictly convex in the drawing.*

*Proof.* The result follows from Proposition 1 and from the fact that the faces of $Q$ that are not split into two triangles are the alternating faces. These faces are strictly convex because of the property (stated in [6] and also holding here) that red edges of TransversalDraw(G) are geometrically directed from bottom to top and weakly directed from left to right, while blue edges are geometrically directed from left to right and weakly directed from top to bottom. Finally, it is easily proved —adapting [6, Prop.12]— that a vertical (horizontal) line not occupied by any vertex corresponds to an edge $e = (v, v')$ of the red map (blue map, respectively) which is neither the rightmost ingoing edge of $v'$ nor the leftmost outgoing edge of $v$. Clearly, such edges do not exist in $Q$ and do not appear during the partial triangulation of $Q$. □

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