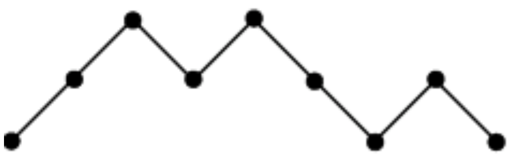
Distances in plane trees and planar maps

Eric Fusy

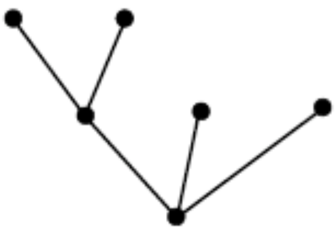
LIX, Ecole Polytechnique

Overview

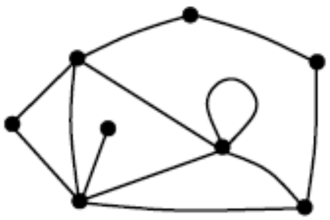
- Structures we study:



paths



trees



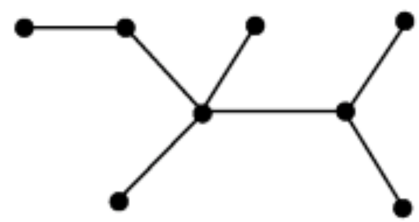
maps

- Distance-parameters
  - typical (depth, distance between 2 vertices)
  - extremal (height, radius, diameter)

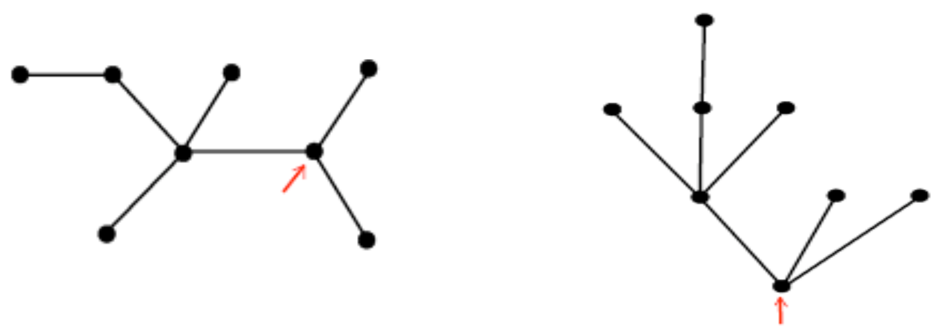
Part 1: distances in plane trees

Plane trees

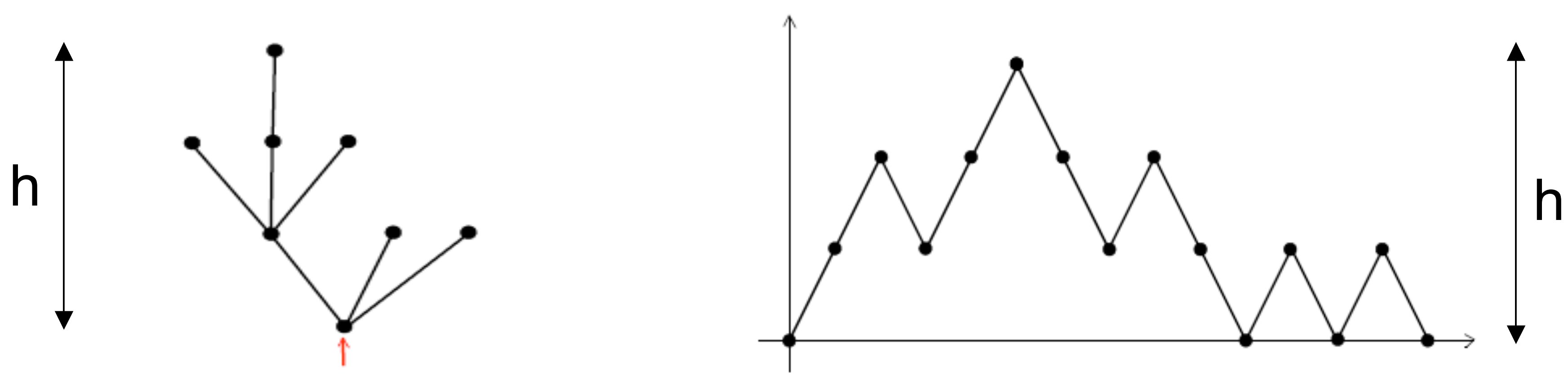
- Plane tree = tree embedded in the plane



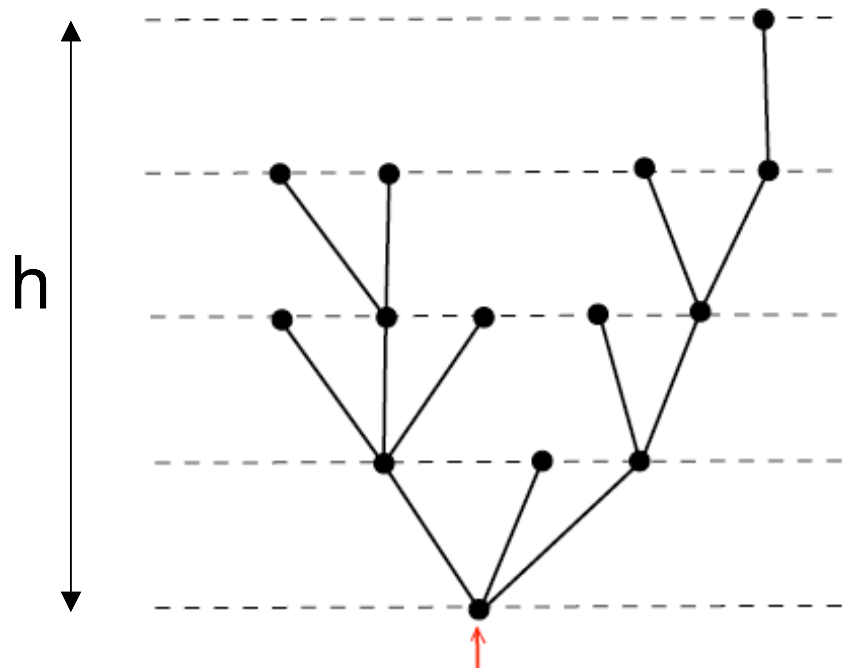
- Rooted Plane tree = plane tree + marked corner

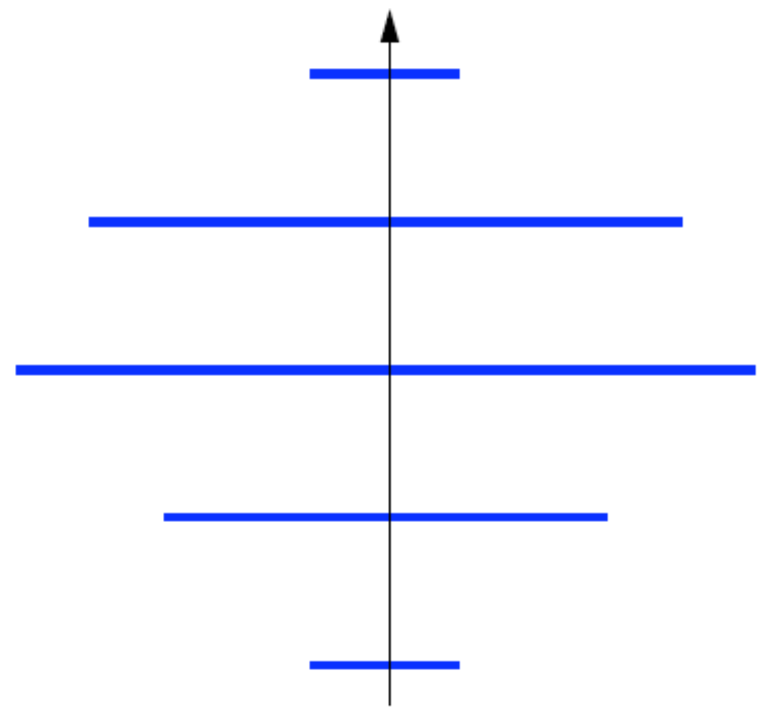


- Rooted plane tree <-> Dyck path



Profile of a plane tree





- Overview:
  - show (using cyclic lemma) that h ≈ 2 · Typical Level
  - show limit profile (Rayleigh law)

Cyclic lemma to count Dyck paths

- **Def**: quasi-bridge = walk ending at $\{y = -1\}$

Cyclic lemma to count Dyck paths

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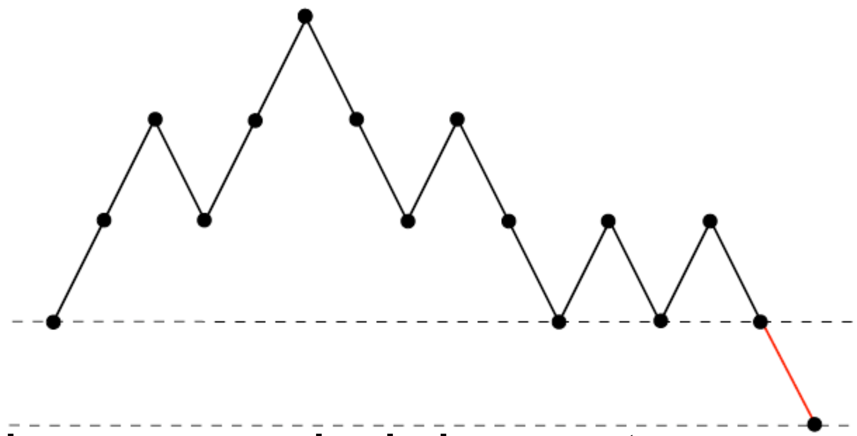

Dyck path + appended down-step

Cyclic lemma to count Dyck paths

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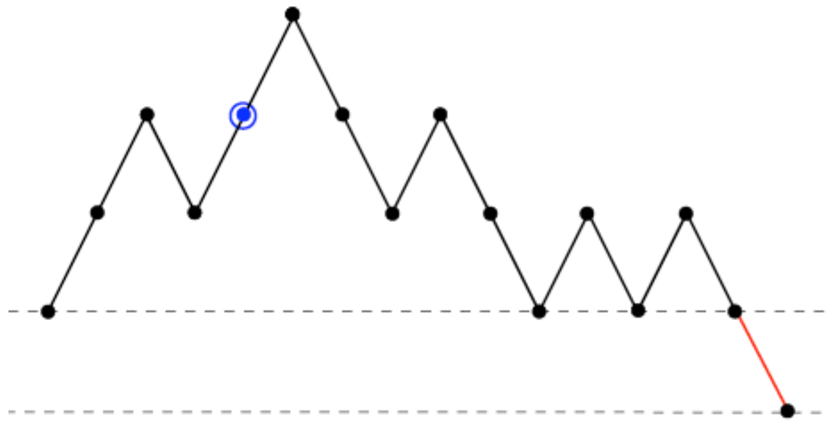
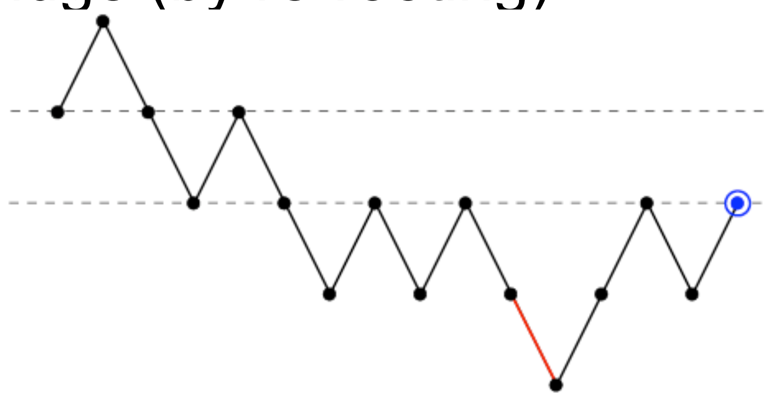


Dyck path + appended down-step + marked point

↕

Quasi-bridge (by re-rooting)



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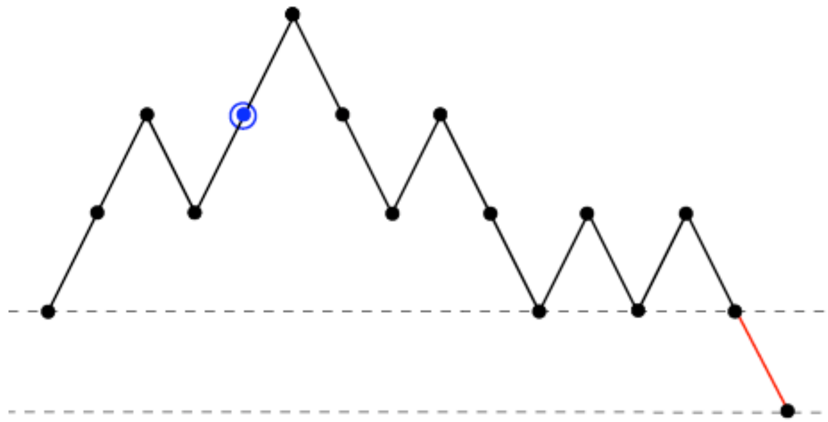

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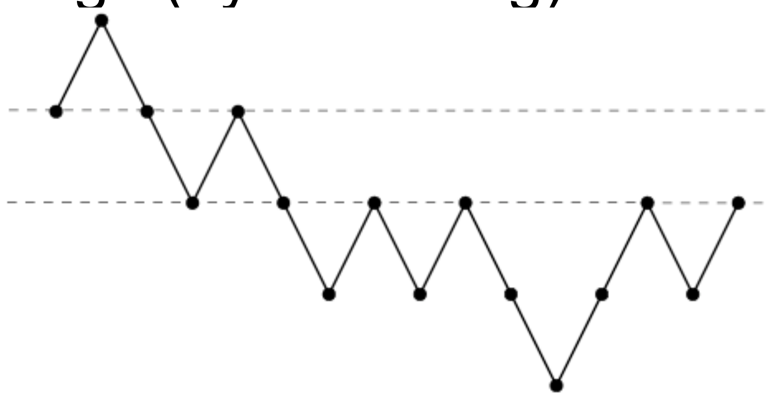

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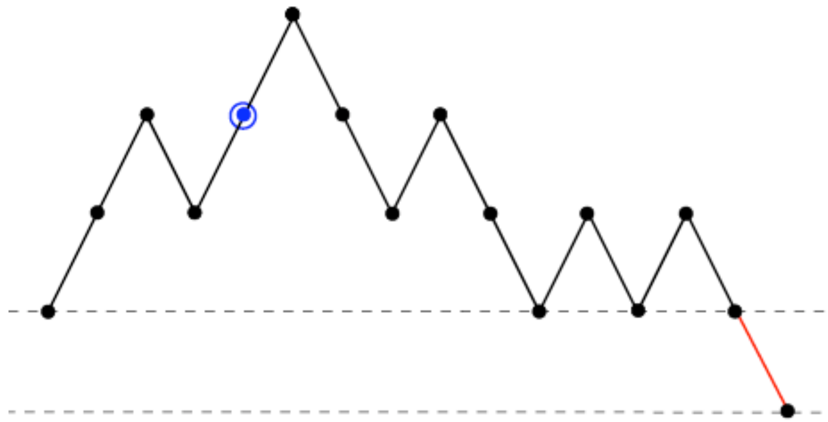

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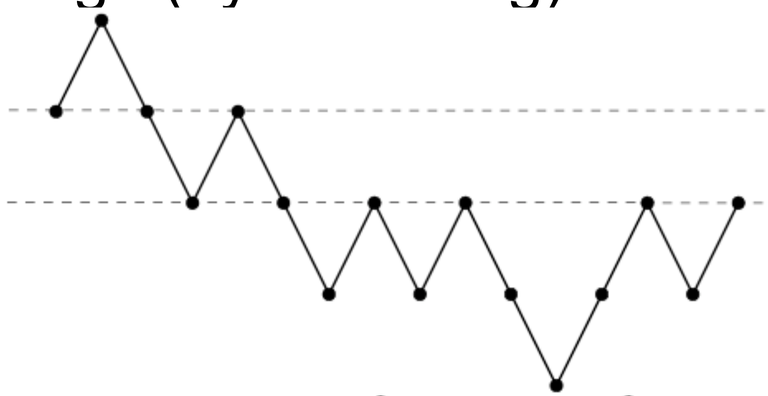
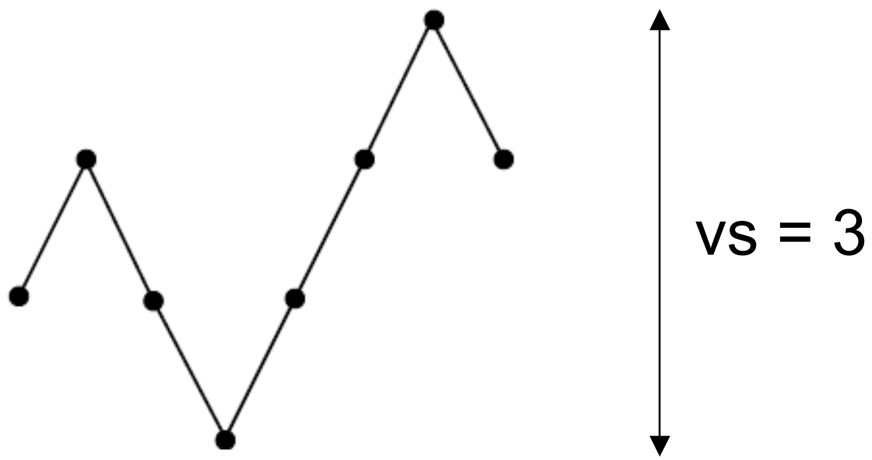


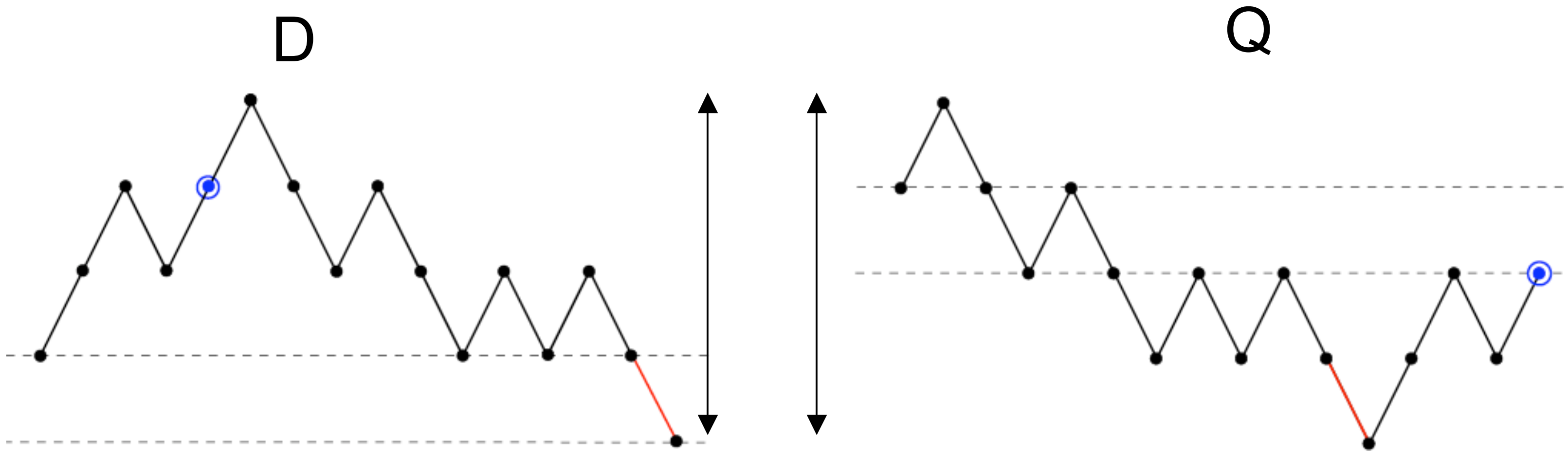
$$\Rightarrow \quad D_n \cdot (2n+1) = \binom{2n+1}{n} \quad \Rightarrow \quad D_n = \frac{(2n)!}{n!(n+1)!}$$

Vertical span of a path

Def: vertical span := MaxOrdinate - MinOrdinate

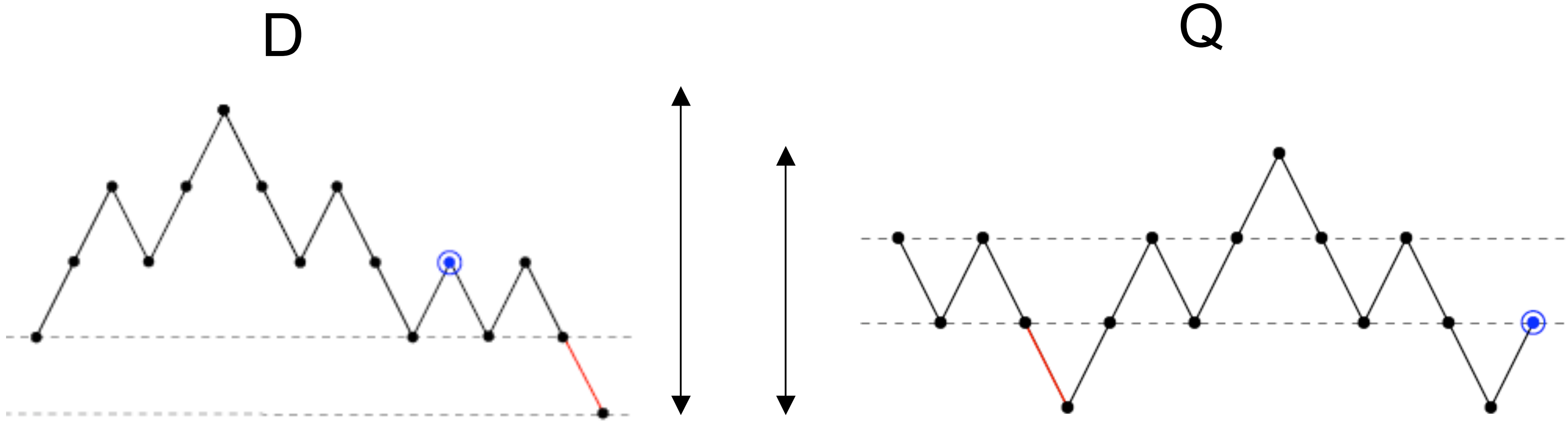


Vertical span and cyclic lemma



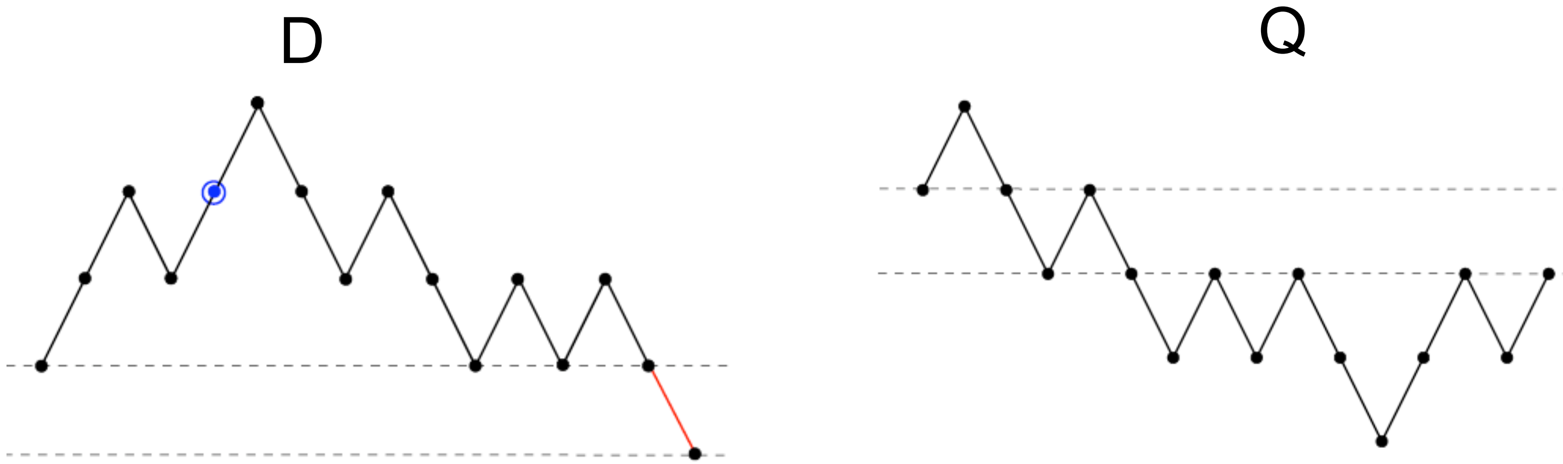
vs(D) = { vs(Q) if marked point before MaxOrdinate

Vertical span and cyclic lemma



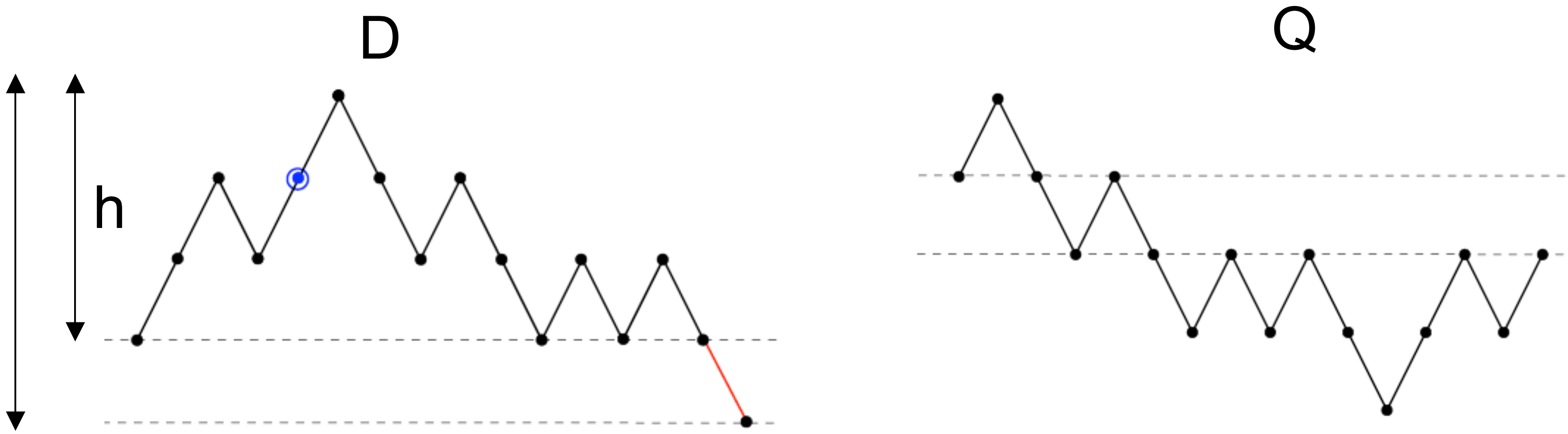
$$vs(D) = \begin{cases} vs(Q) \text{ if marked point before MaxOrdinate} \\ vs(Q) + 1 \text{ if marked point after MaxOrdinate} \end{cases}$$

Vertical span and cyclic lemma



$\Rightarrow$ vs(D) = vs(Q) + (0 or 1)

Vertical span and cyclic lemma



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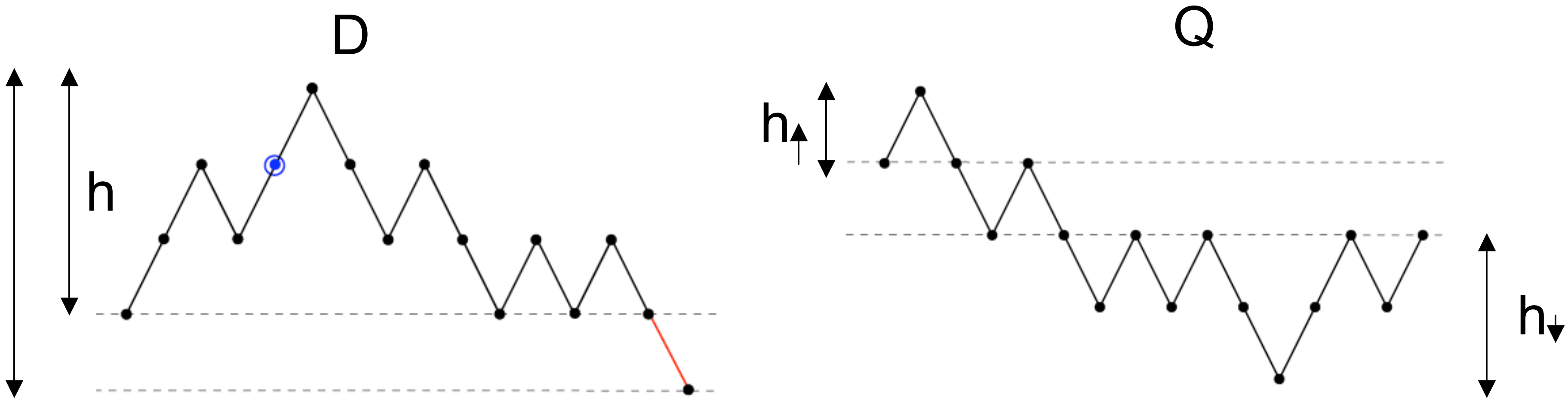
Also, vs(D) = h + 1

Vertical span and cyclic lemma



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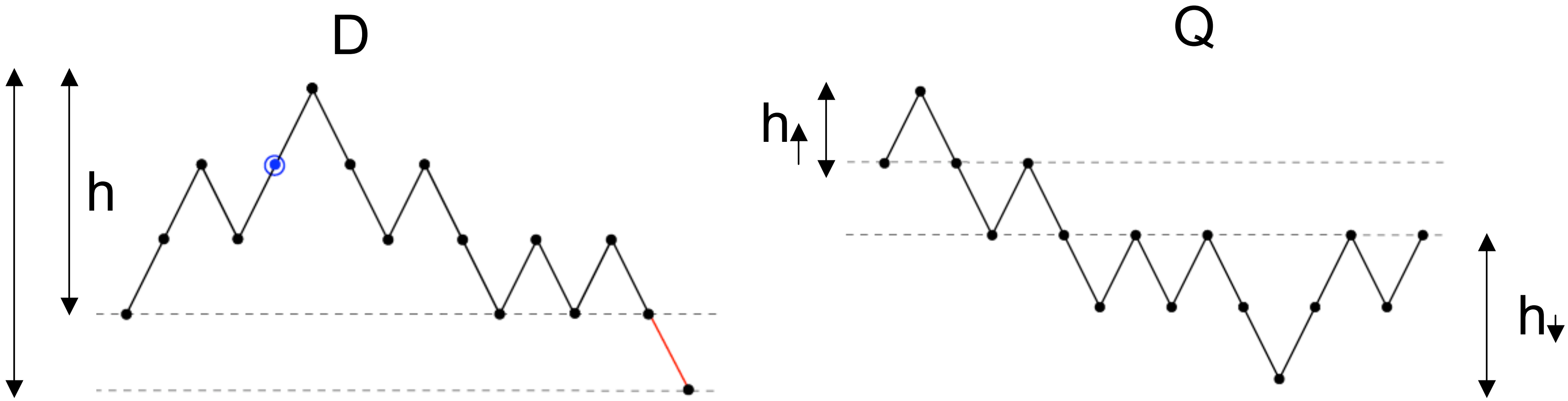
Also, vs(D) = h + 1

vs(Q) = $h_{\downarrow}$(Q) + $h_{\uparrow}$(Q) +1

Vertical span and cyclic lemma



$\Rightarrow$ vs(D) = vs(Q) + (0 or 1)

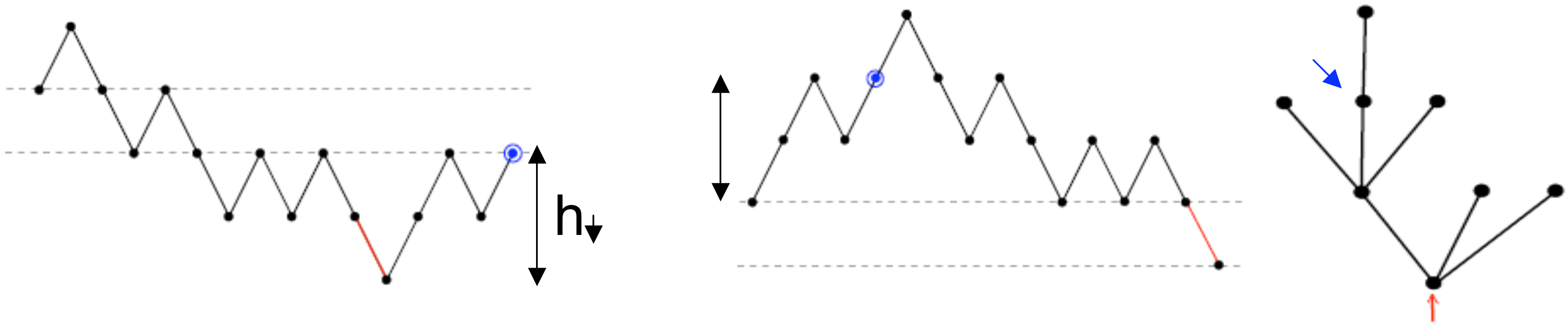
Also, vs(D) = h + 1

vs(Q) = $h_\downarrow$(Q) + $h_\uparrow$(Q) +1

Hence h(D) = $h_\downarrow$(Q) + $h_\uparrow$(Q) + (0 or -1)

Combinatorial interpretation of $h_{\downarrow}(Q)$

Q $\Leftrightarrow$ D + marked point $\Leftrightarrow$ T + marked corner



$h_{\downarrow}(Q)$ = distance L between the 2 marked corners

Combinatorial interpretation of $h_{\downarrow}(Q)$

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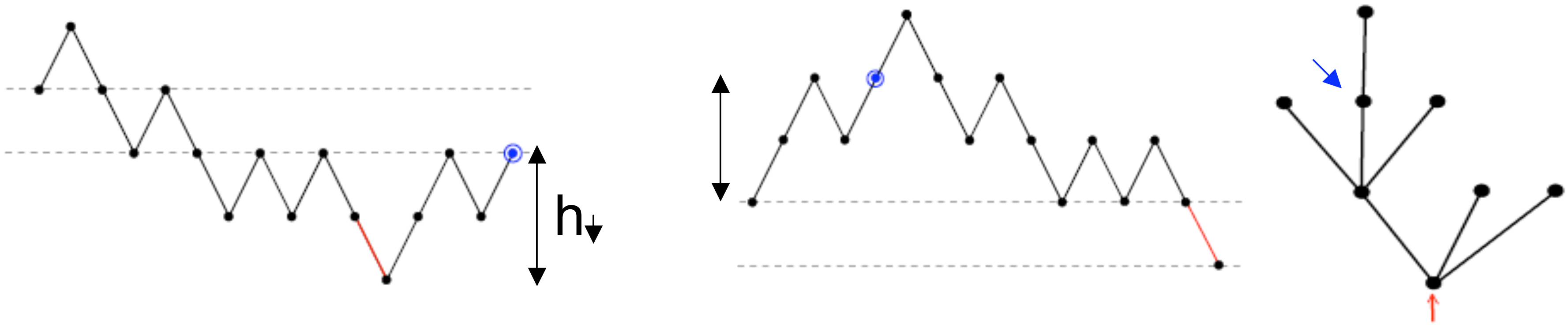


$h_{\downarrow}(Q)$ = distance L between the 2 marked corners

PATHS: $h(D) = h_{\downarrow}(Q) + h_{\uparrow}(Q) + (0 \text{ or } -1)$

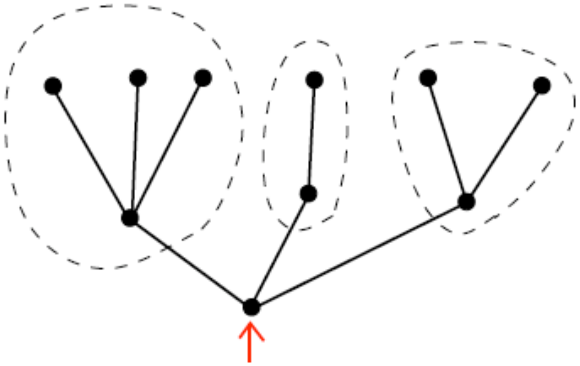
$\Downarrow$

TREES: $h(T) = L + L' + (0 \text{ or } -1)$

extremal typical same distribution as L

Distribution of L (Meir & Moon'78)

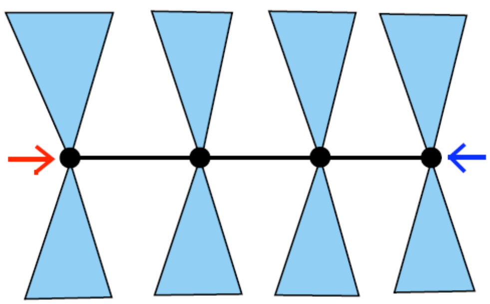
- Use generating functions (cf this morning)



$$T(z) = \frac{1}{1 - zT(z)}$$

- Two marked corners at distance k

L=3

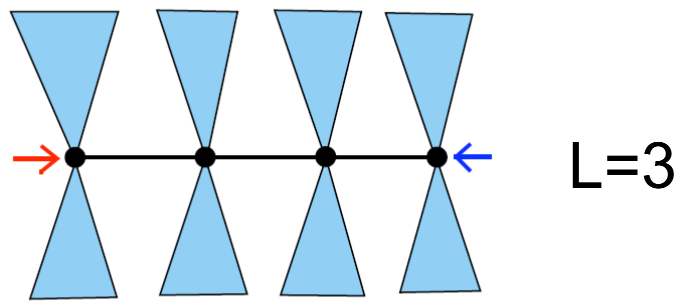


$$T_k(z) = z^k T(z)^{2k+2}$$

$$\mathbb{P}_n(L = k) = \frac{[z^n]T_k(z)}{(2n+1)[z^n]T(z)} = \frac{(2k+2)n!(n+1)!}{(n+k+2)!(n-k)!}$$

(using the Lagrange inversion formula)

Distribution of L (Meir & Moon'78)



(i) $\mathbb{P}_n(L=k) = \dfrac{[z^n]T_k(z)}{(2n+1)[z^n]T(z)} = \dfrac{(2k+2)n!(n+1)!}{(n+k+2)!(n-k)!}$

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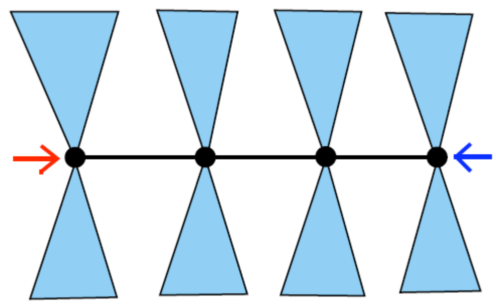

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$$\Downarrow$$

$$\forall x>0, \quad \mathbb{P}_n(L=x\sqrt{n}) \underset{n\to\infty}{\sim} \frac{1}{\sqrt{n}} 2x\exp(-x^2)$$

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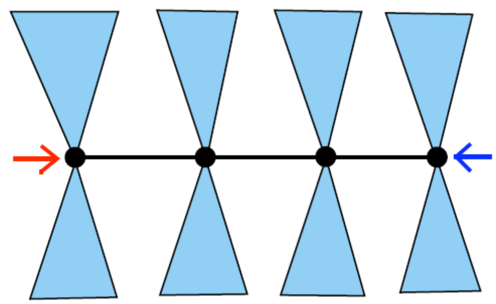

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$$L/\sqrt{n} \underset{n\to\infty}{\longrightarrow} \mathrm{d}x \cdot 2x\exp(-x^2) \qquad \text{Rayleigh law}$$

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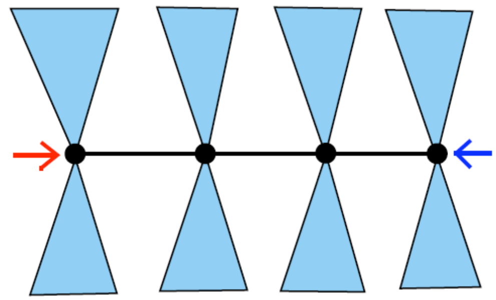

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$$L/\sqrt{n} \underset{n\to\infty}{\longrightarrow} \mathrm{d}x \cdot 2x \exp(-x^2) \qquad \text{Rayleigh law}$$

Rq: (i) implies uniform tail $\mathbb{P}_n(L/\sqrt{n} \geq x) \leq a\, e^{-cx} \; \forall n, x$

$\Rightarrow$ Moments of L / $n^{1/2}$ converge to moments of Rayleigh law

The Rayleigh law / stable laws

cf [Banderier, Flajolet, Schaeffer, Soria'01]

Case $\lambda = 1/2$

If $\mathbb{P}_n(X_n = k) \propto [z^n]T(u)^k$

with $T(u) = 1 - c(1-u)^{1/2} + \cdots$

then $\dfrac{X_n}{n^{1/2}} \to \textit{Rayleigh law}$

Rk: $T(u)^k = PGF\Big(\sum_{i=1}^{k} Z_i\Big), \quad \textit{with } Tail(Z_i) \sim k^{-3/2}$

$$\frac{1}{k^2}\sum_{i=1}^{k} Z_i \longrightarrow \textit{Stable law parameter } 1/2$$

The Rayleigh law / stable laws

cf [Banderier, Flajolet, Schaeffer, Soria'01]

General $\lambda \in (0,1)$

If $\mathbb{P}_n(X_n = k) \propto [z^n]T(u)^k$

with $T(u) = 1 - c(1-u)^\lambda + \cdots$

then $\dfrac{X_n}{n^\lambda} \to G_\lambda(u)\,\mathrm{d}u$

related to Stable_λ

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$$\frac{1}{k^{1/\lambda}}\sum_{i=1}^{k} Z_i \longrightarrow Stable\ law\ parameter\ \lambda$$

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$$\frac{1}{k^{1/\lambda}} \sum_{i=1}^{k} Z_i \longrightarrow Stable\ law\ parameter\ \lambda$$

Here $\lambda = 1/2$ **(for maps** $\lambda = 1/4$**)**

Expectation/tail for the height

$$h = L + L' + (0 \text{ or } -1)$$

Expectation/tail for the height

h = L + L' + (0 or -1)

Expectation: $\mathbb{E}_n(h) = 2\,\mathbb{E}_n(L) + \epsilon, \quad \text{with } \epsilon \in [-1, 0]$

$$\mathbb{E}_n(L) \sim \underbrace{\sqrt{\pi}/2}_{\mathbb{E}(\text{Rayleigh})} \cdot \sqrt{n}$$

$$\Rightarrow \boxed{\mathbb{E}_n(h) \sim \sqrt{\pi}\sqrt{n}}$$

[De Bruijn, Knuth, Rice'72]

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[De Bruijn, Knuth, Rice'72]

Exponential tail: $\mathbb{P}_n(h \geq k) \leq 2\,\mathbb{P}_n(L \geq k/2)$

$$\mathbb{P}_n(L/\sqrt{n} \geq x) \leq a\,e^{-cx} \ \forall n, x$$

$$\Rightarrow \boxed{\mathbb{P}_n(h/\sqrt{n} \geq x) \leq 2a\,e^{-cx}}$$

Limit distribution for the height

Two possible approaches:

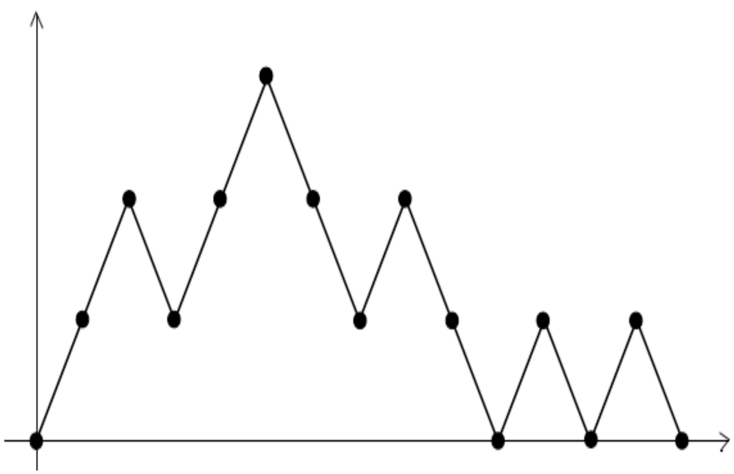
- Singularity analysis [Flajolet, Odlyzko'82], [Flajolet et al.'93]

$$\textbf{System } \; y_h(z) = 1/(1 - y_{h-1}(z)) \quad [\textbf{height} \leq h]$$

$$\textbf{Singular expansion of } y_h - y_{h-1} \textbf{ for } h = \lfloor x\sqrt{n} \rfloor$$

$$\Rightarrow \; \mathbb{P}\Big(\frac{height}{\sqrt{n}} \leq x\Big) \longrightarrow \sum_{k\in\mathbb{Z}} (2k^2x^2 - 1)e^{-k^2x^2}$$

- Continuous limit [Aldous]



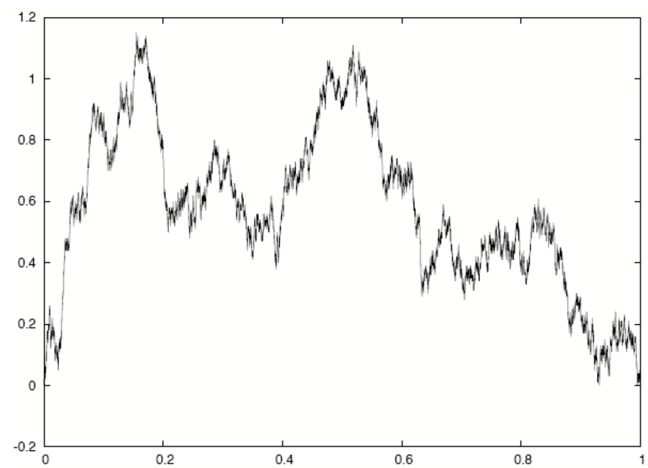


Image credit

J.F. Marckert

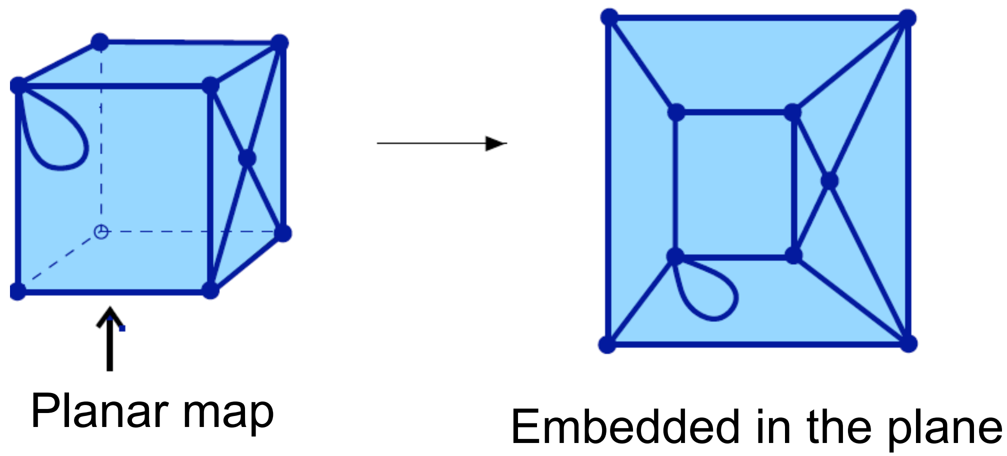
$$\textbf{If functional } F : \mathcal{C}[0,1] \to \mathbb{R} \textbf{ is continuous for } ||.||_\infty\textbf{, then}$$

$$F(D_n/\sqrt{n}) \longrightarrow F(brownian\ excursion)$$

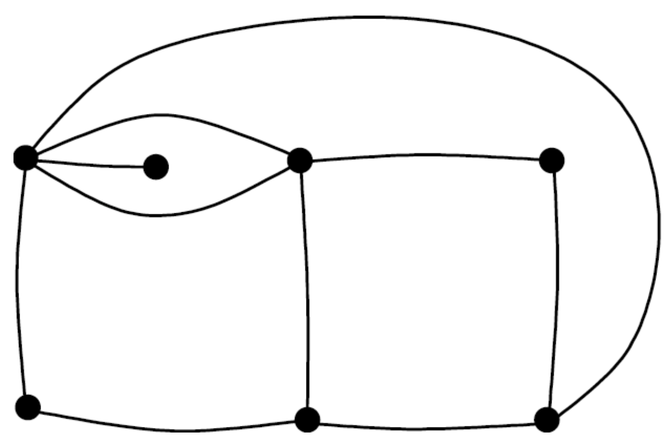
Part 2: distances in planar quadrangulations

Planar maps

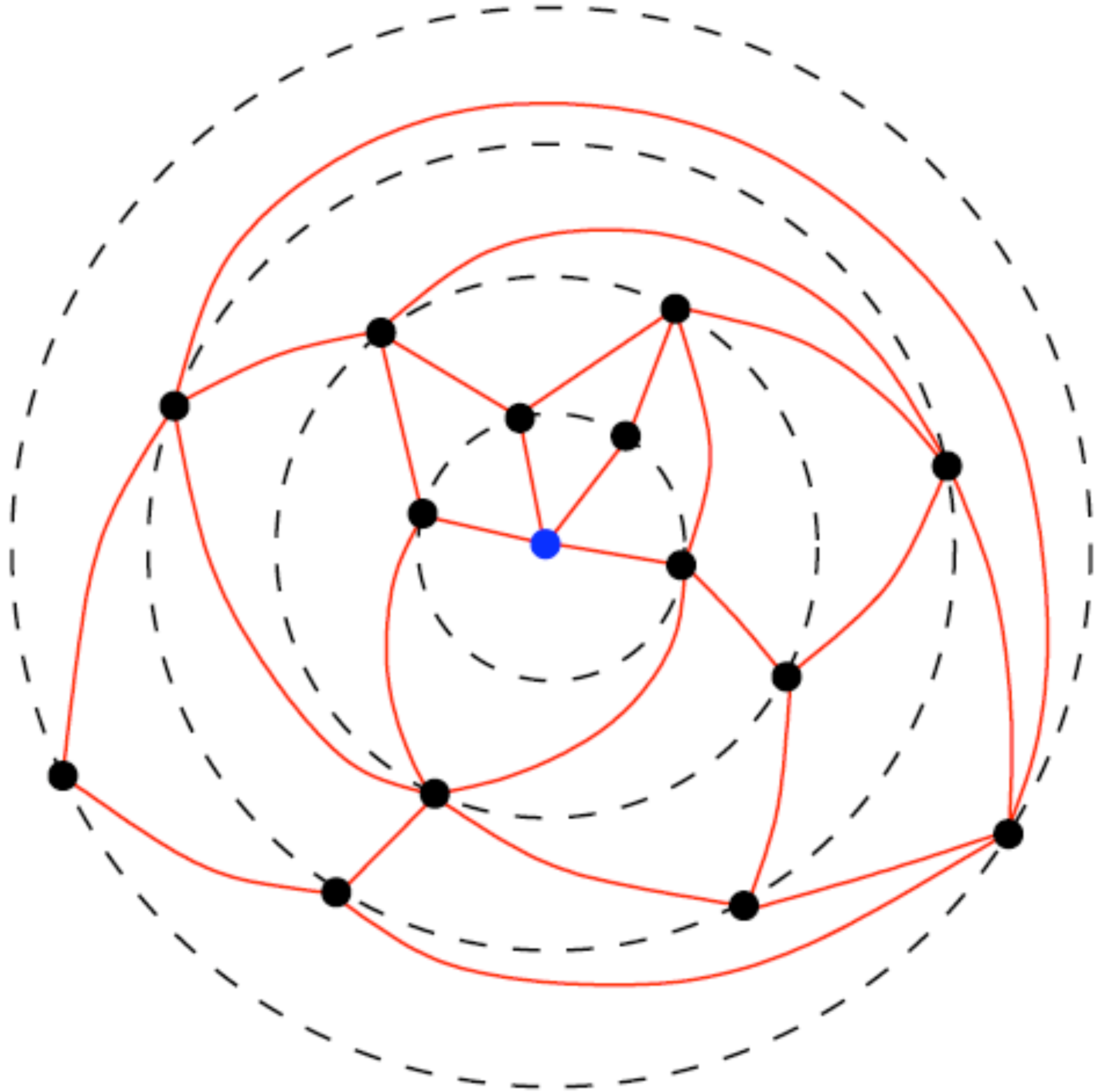
- Planar map = planar graph embedded on the sphere



- Quadrangulation = planar map with faces of degree 4



Profile of a pointed quadrangulation



Profile for vertices: (4,4,4,2)

Profile for edges: (4,8,8,6)

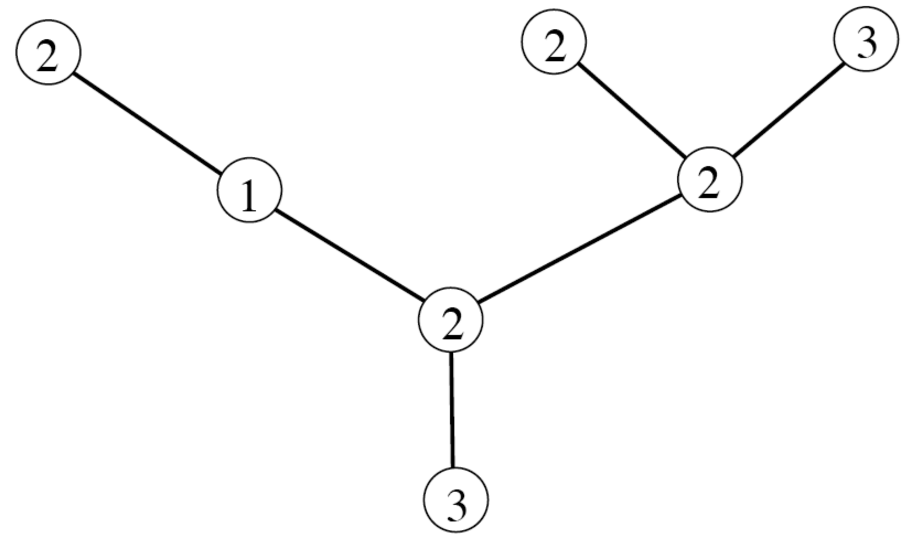
Well-labelled trees

- A well-labelled tree is a plane tree where:

- each vertex v has a non-negative label

- the labels at each edge (v,v') differ by at most 1

- at least one vertex has label 1



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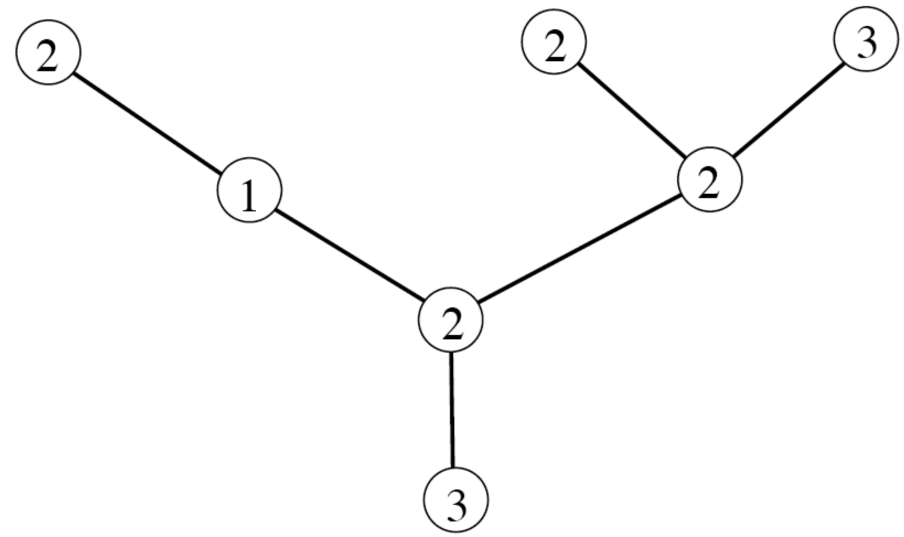
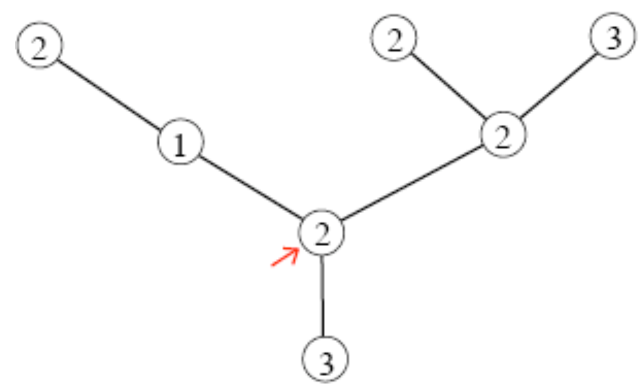


- Rooted well-labelled tree = well-labelled tree + marked corner



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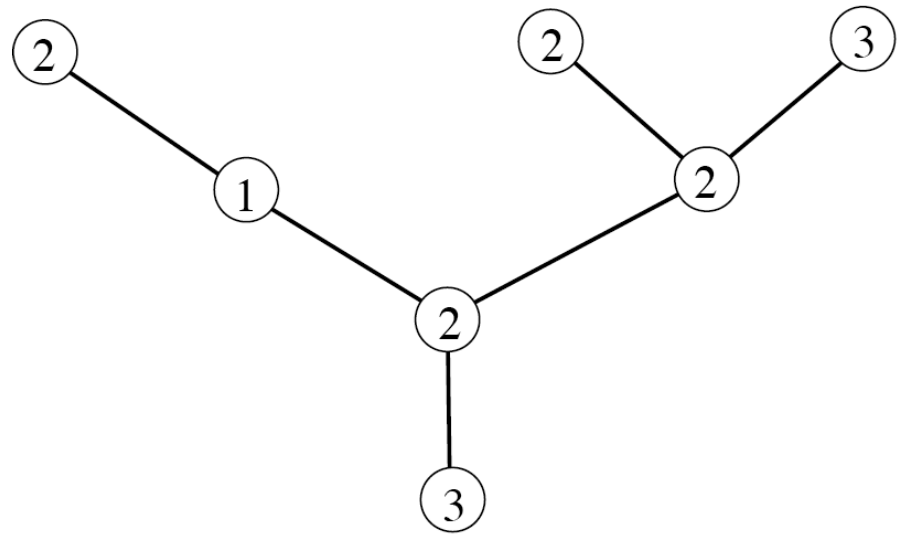

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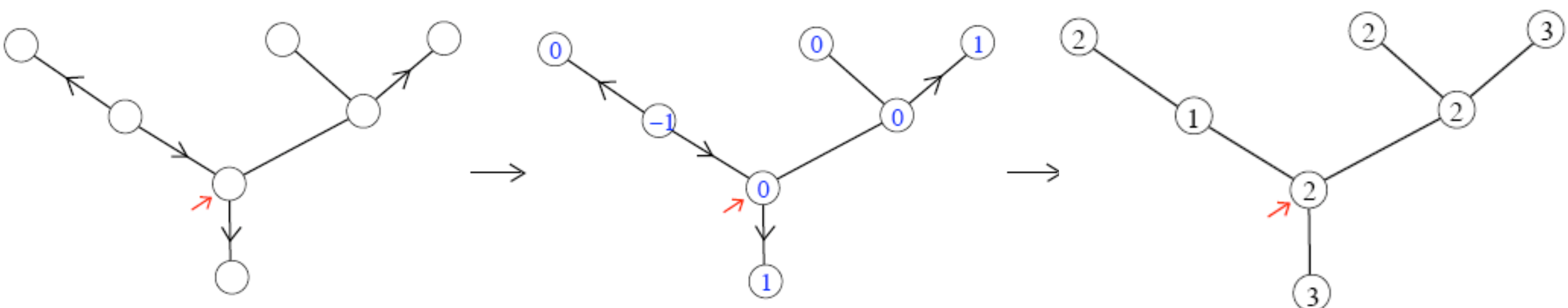
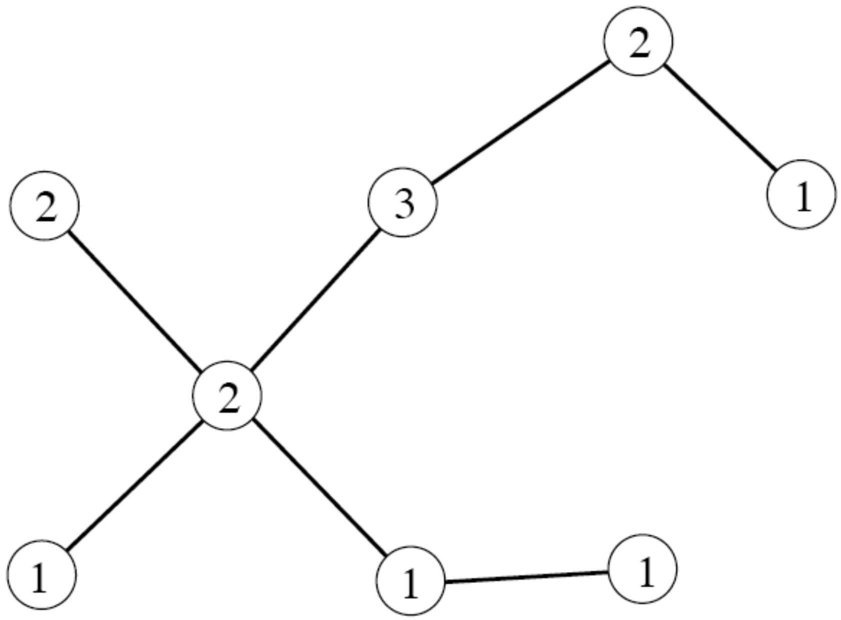


(there are $3^n \frac{(2n)!}{n!(n+1)!}$ such trees with n edges)

Well-labelled tree -> pointed quadrangulation

[Schaeffer'98], also [Cori&Vauquelin'81]



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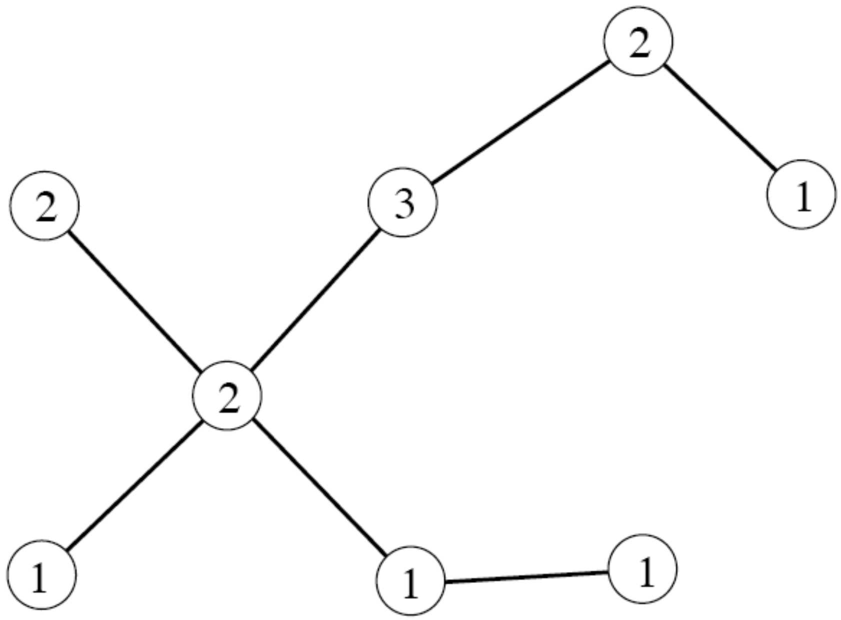

1) Place a red leg in each corner

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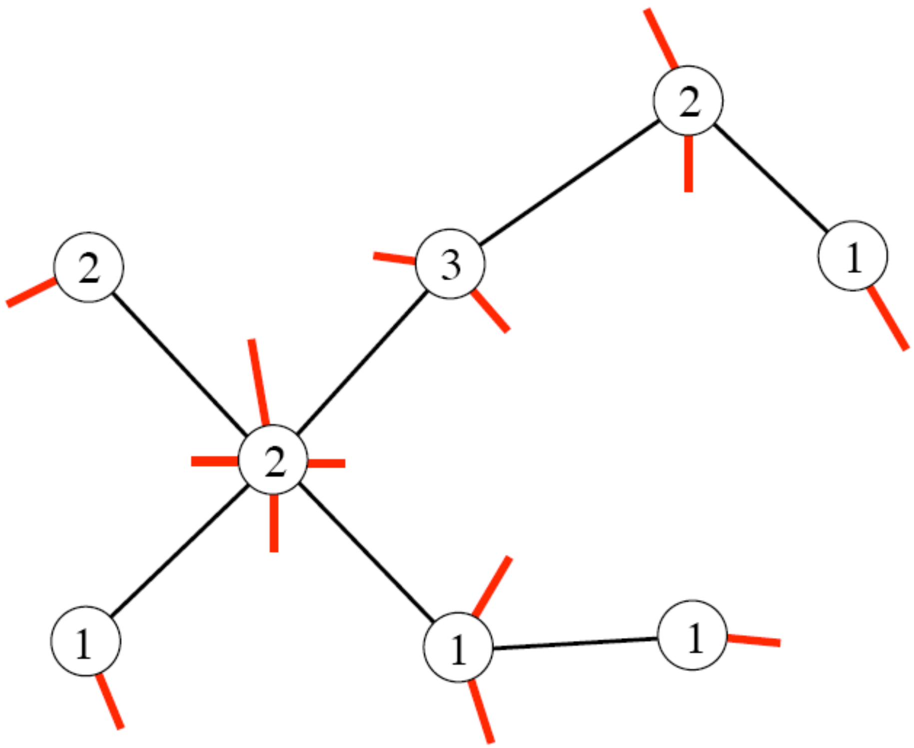

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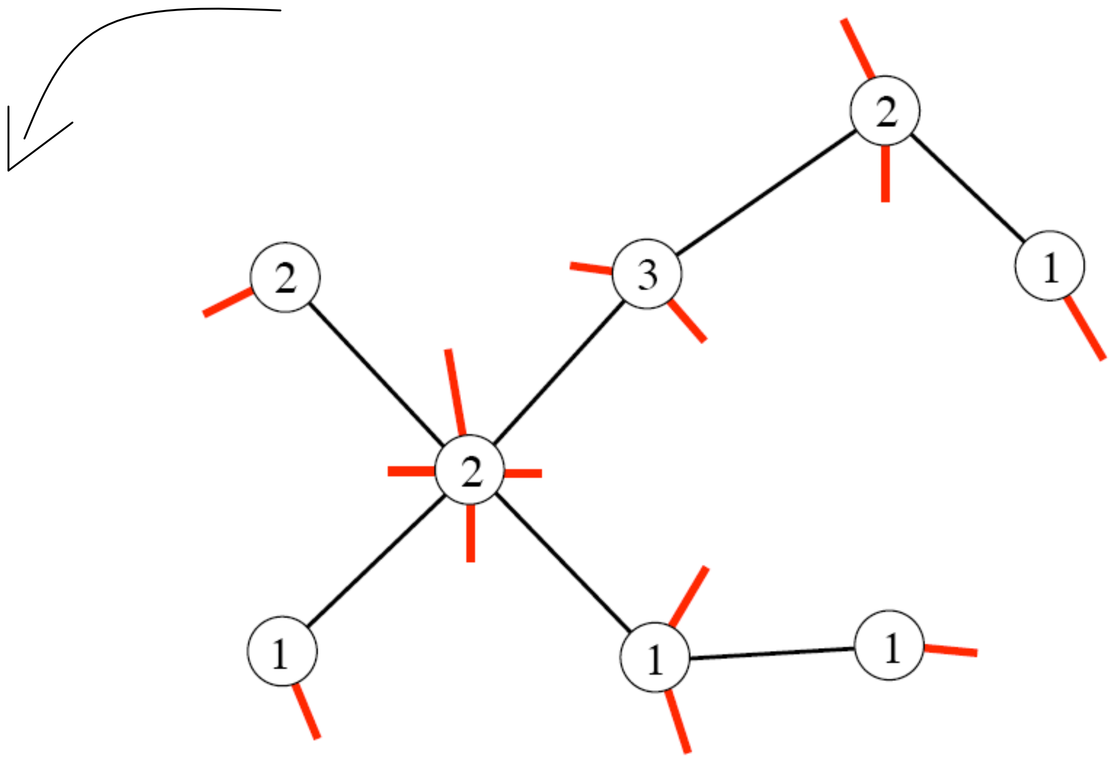

2) Repeat: - choose a leg of label i>1

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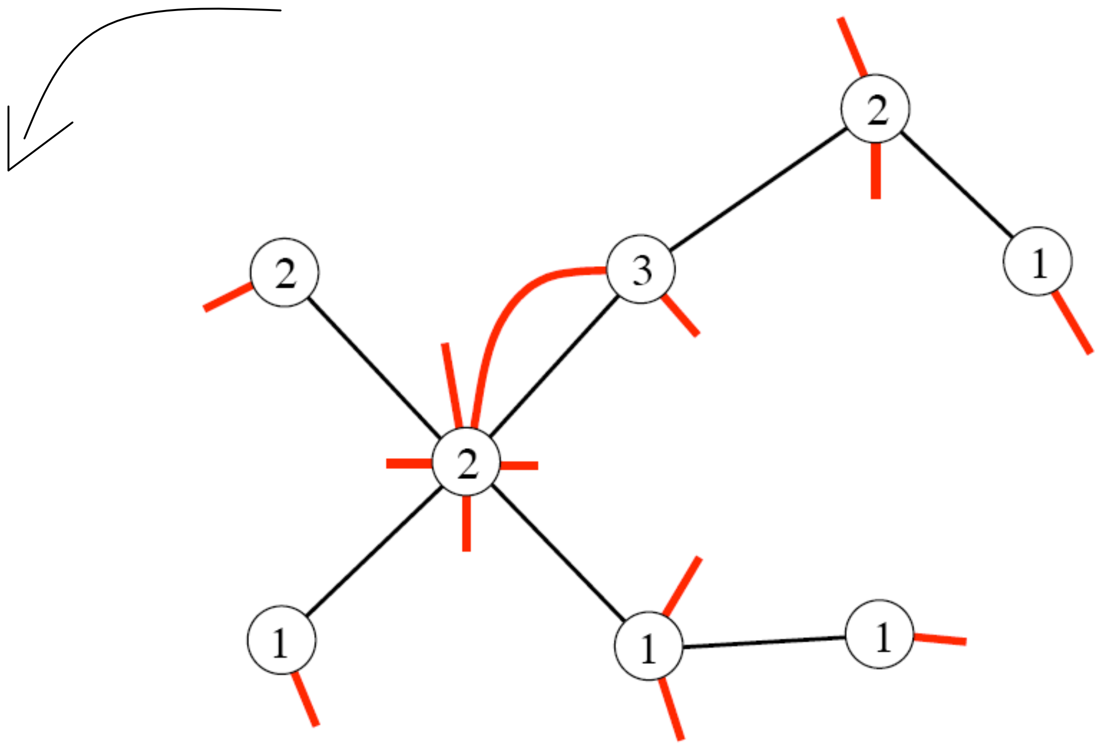

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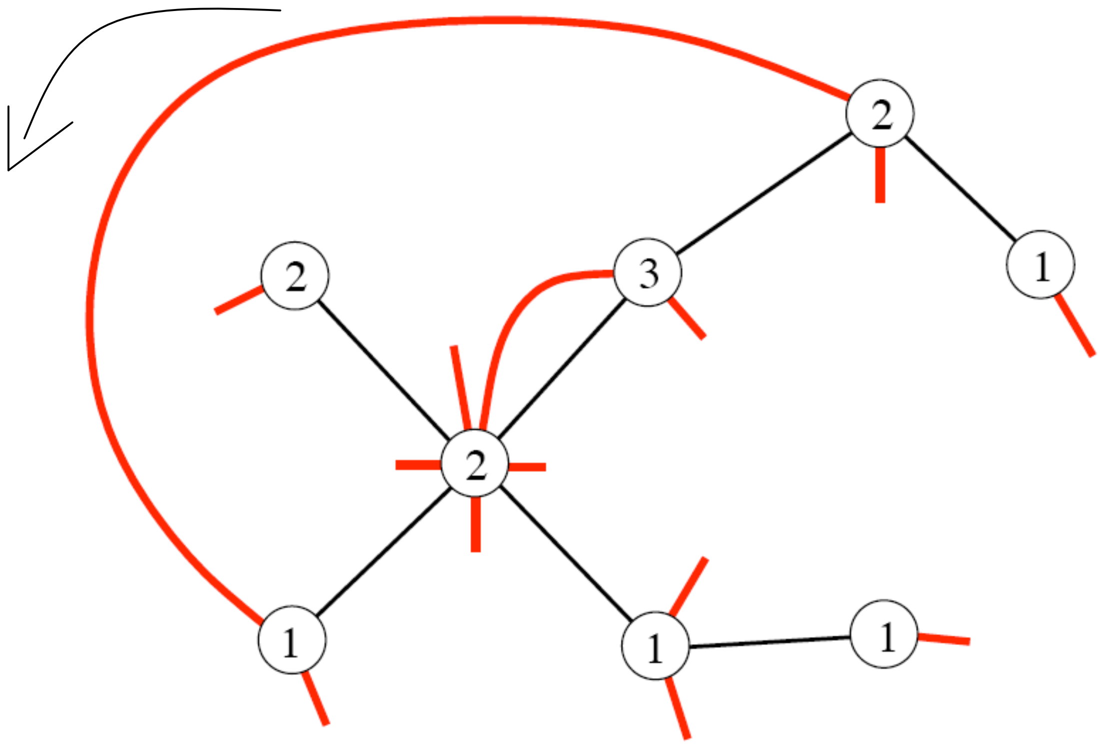

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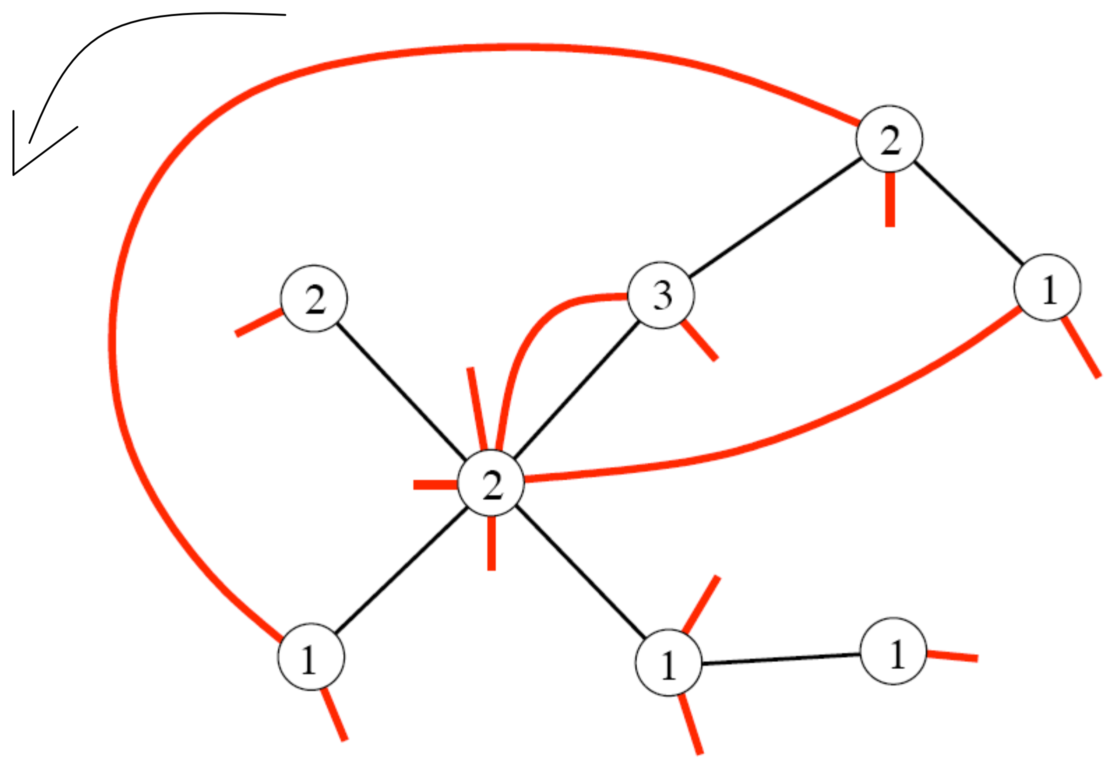

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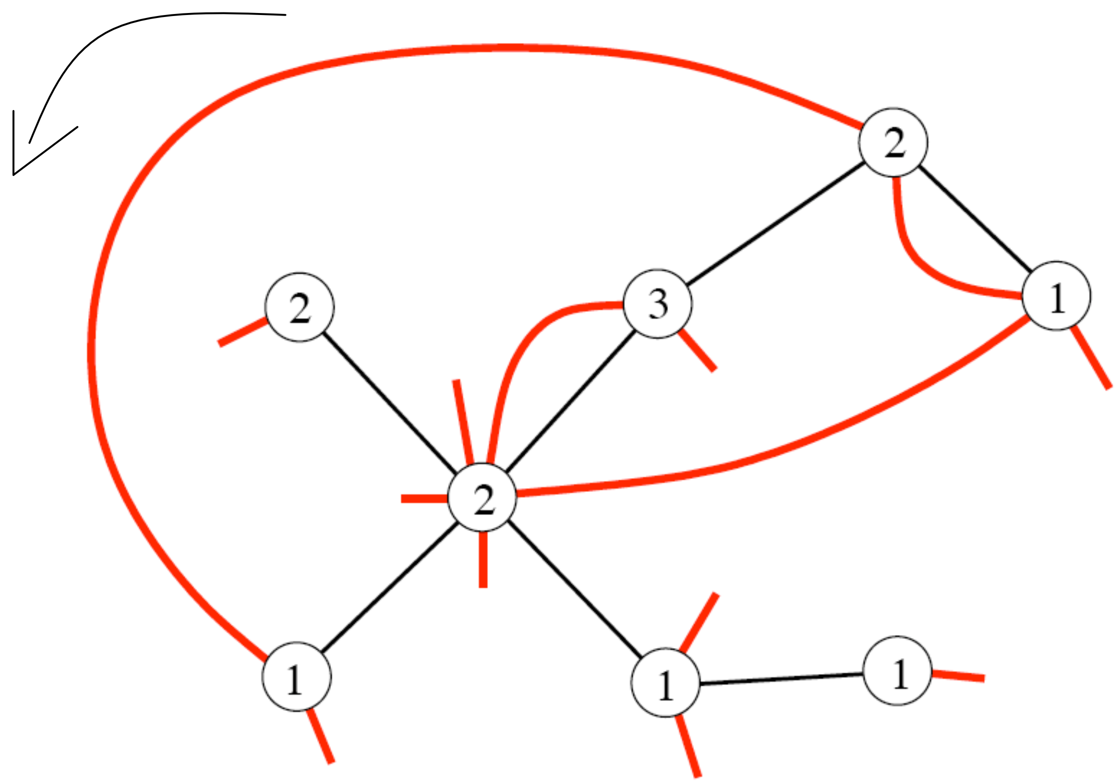

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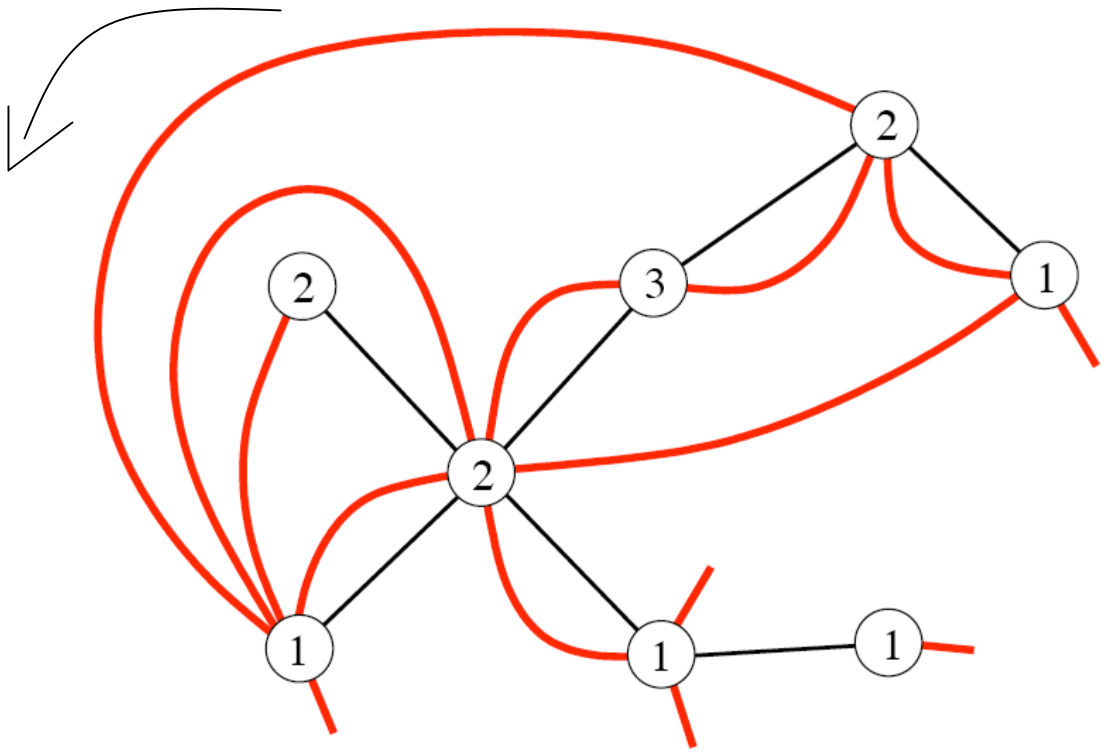

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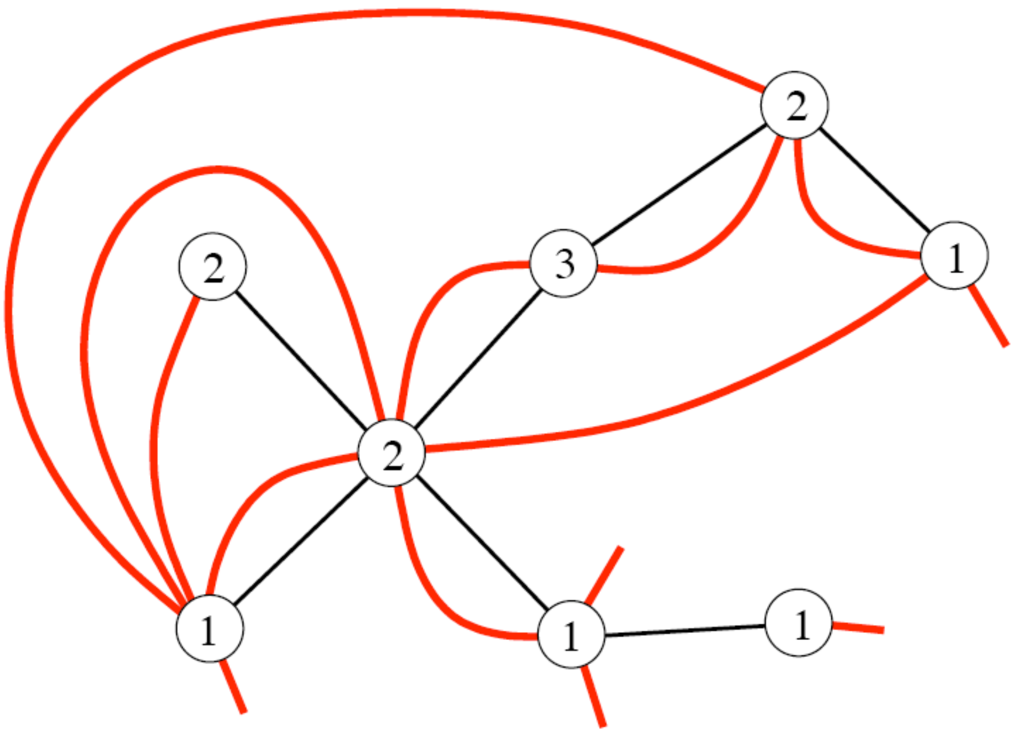

3) Create a new vertex labelled 0 in the outer face

Well-labelled tree -> pointed quadrangulation

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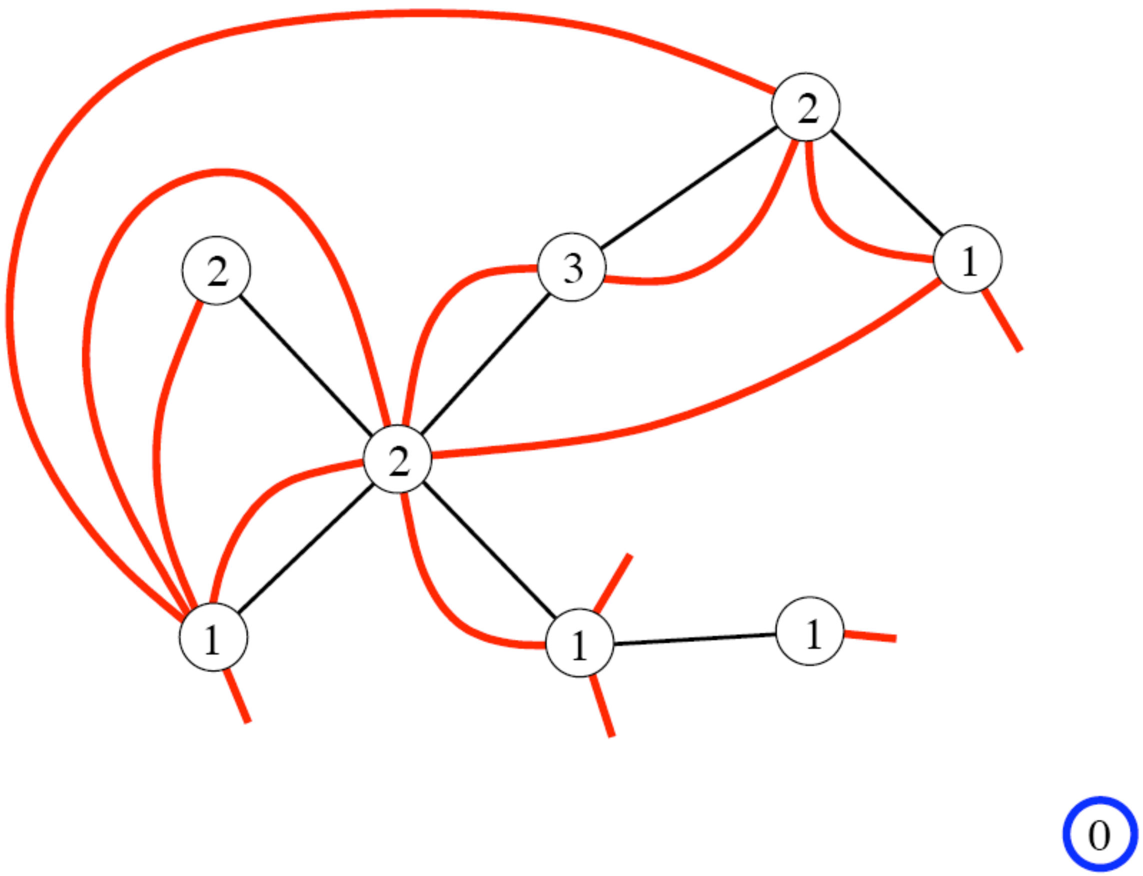

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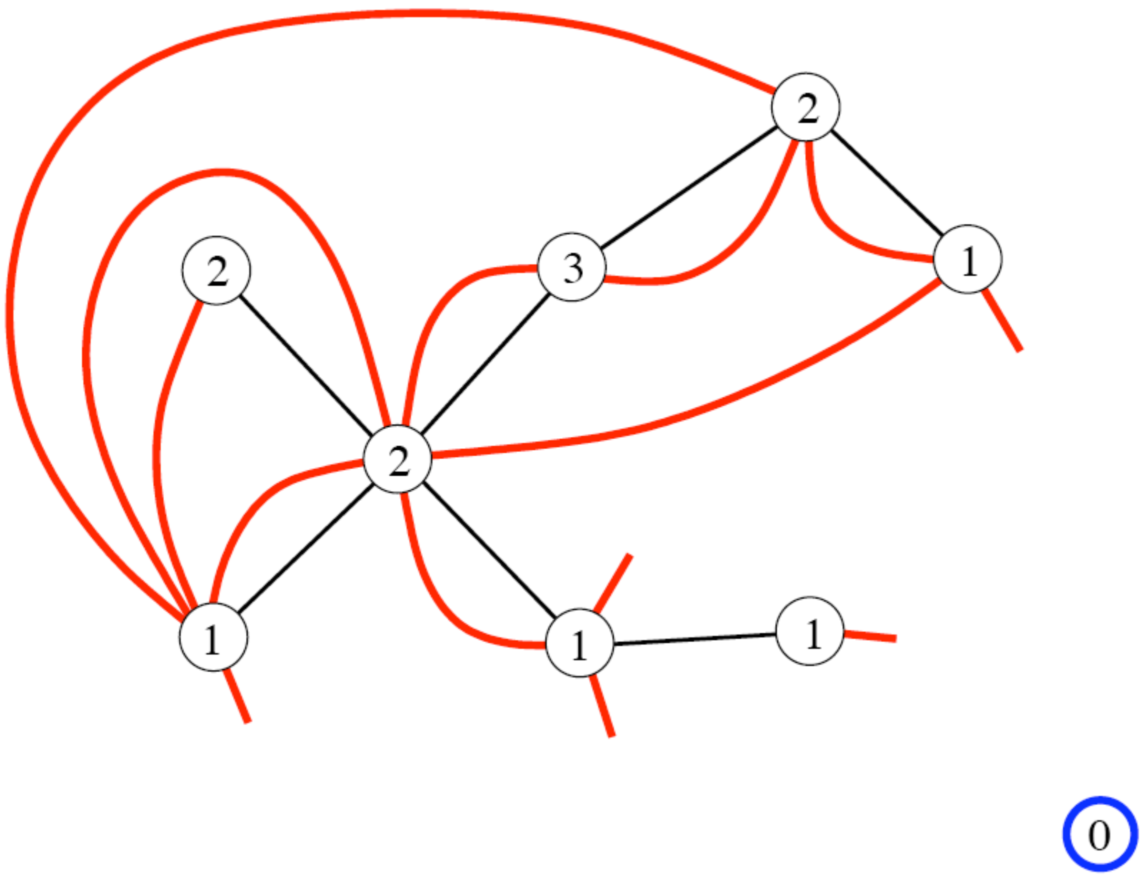

4) Connect all remaining legs (label 1) to the new vertex

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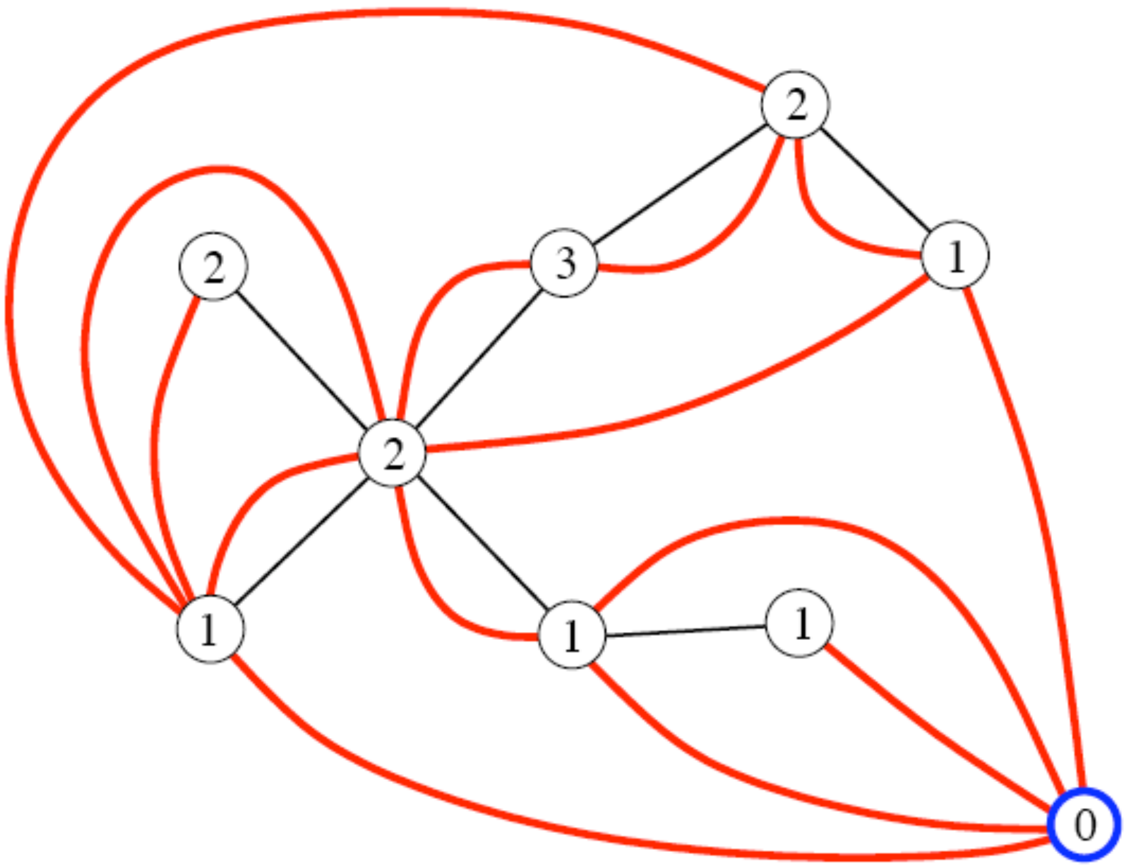

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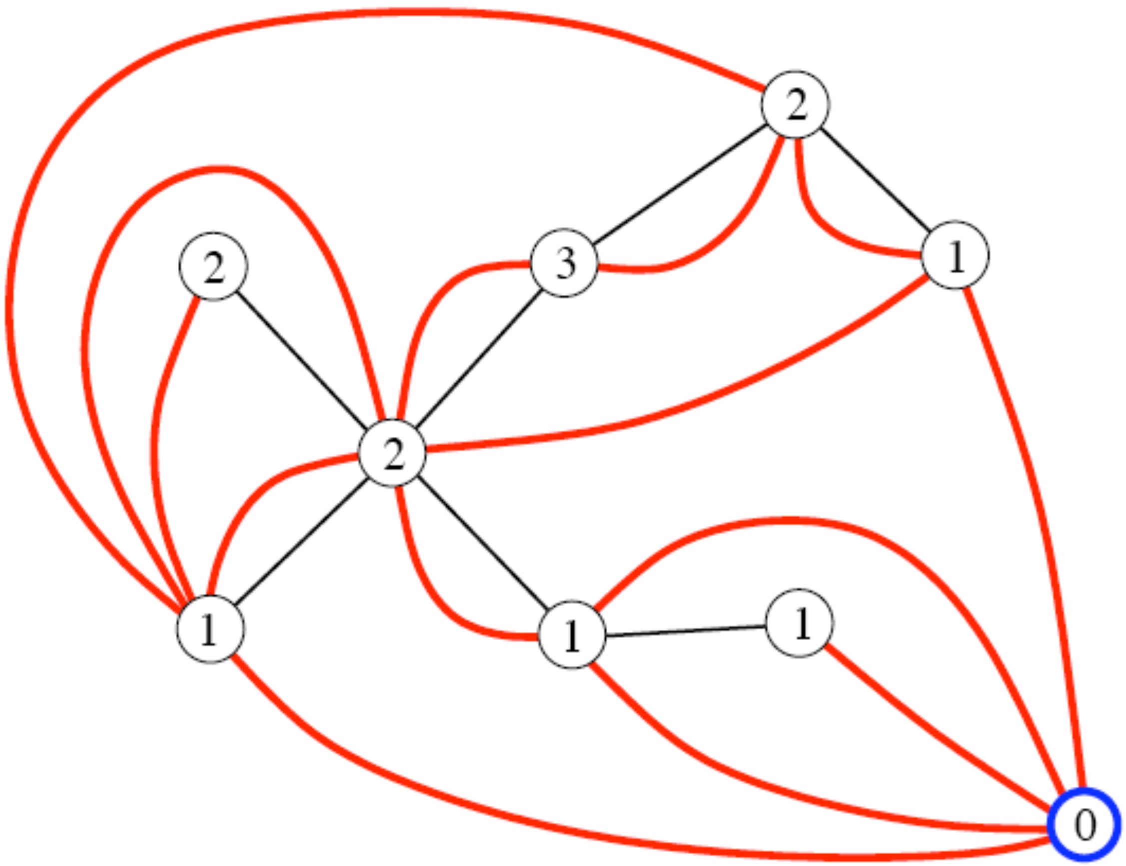

5) Delete the black edges

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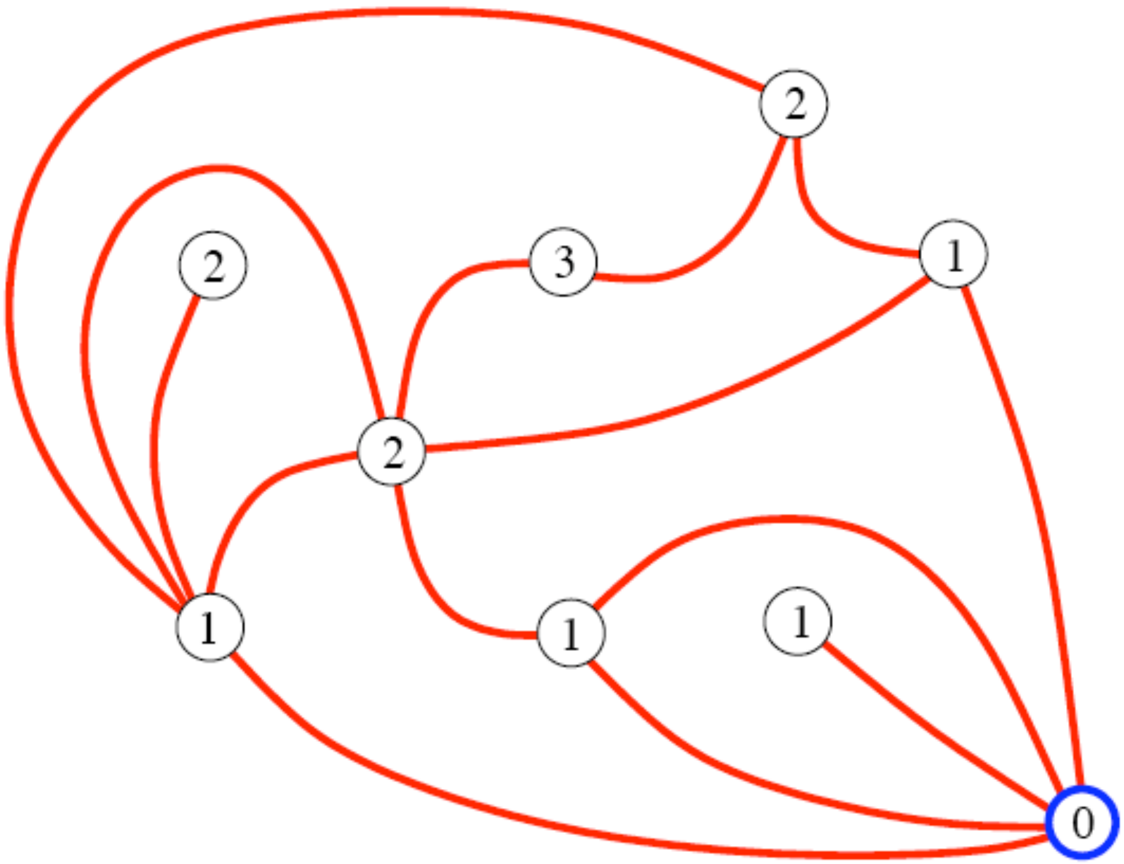

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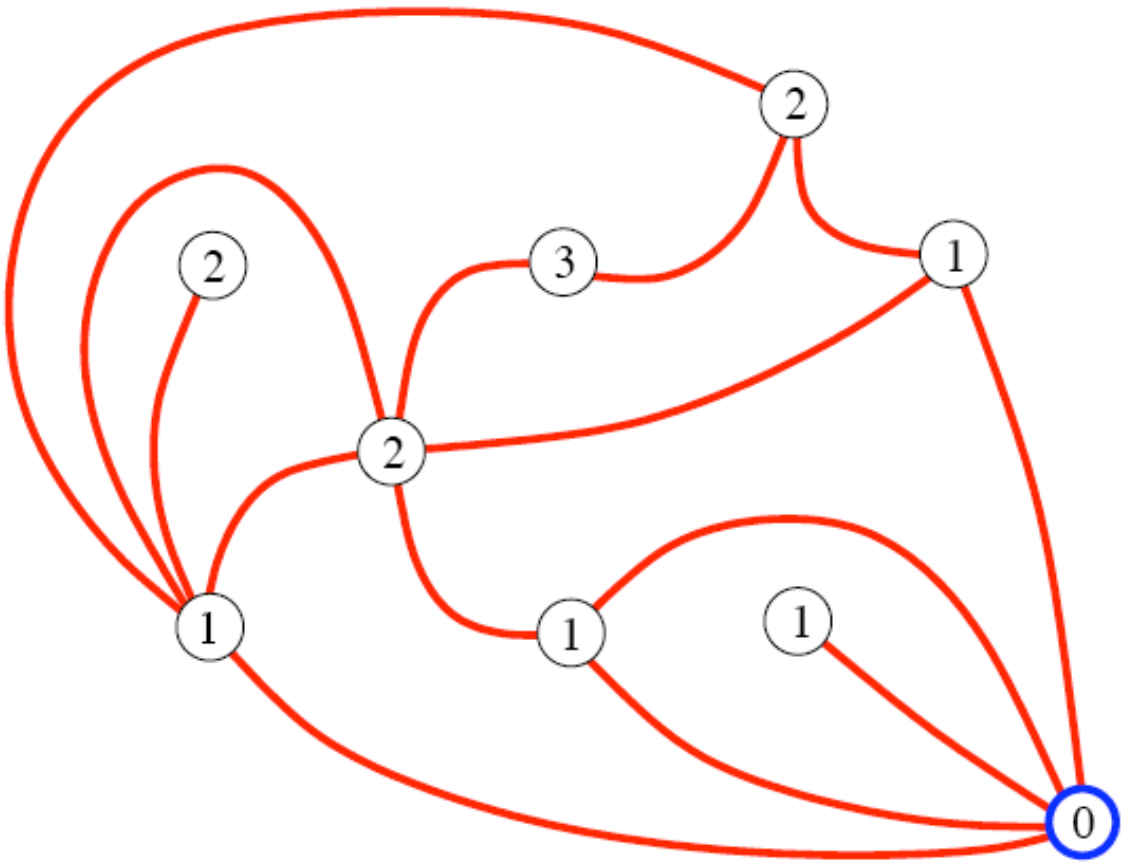

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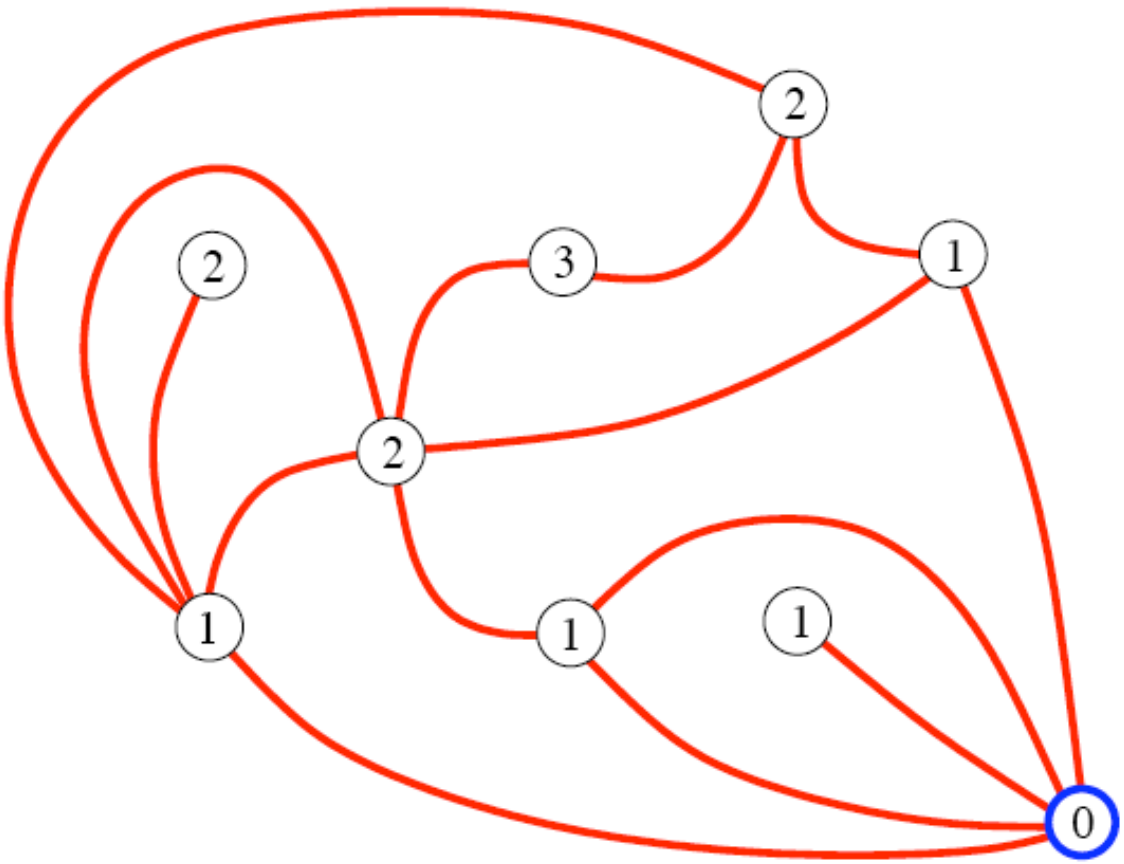

- faces are of degree 4

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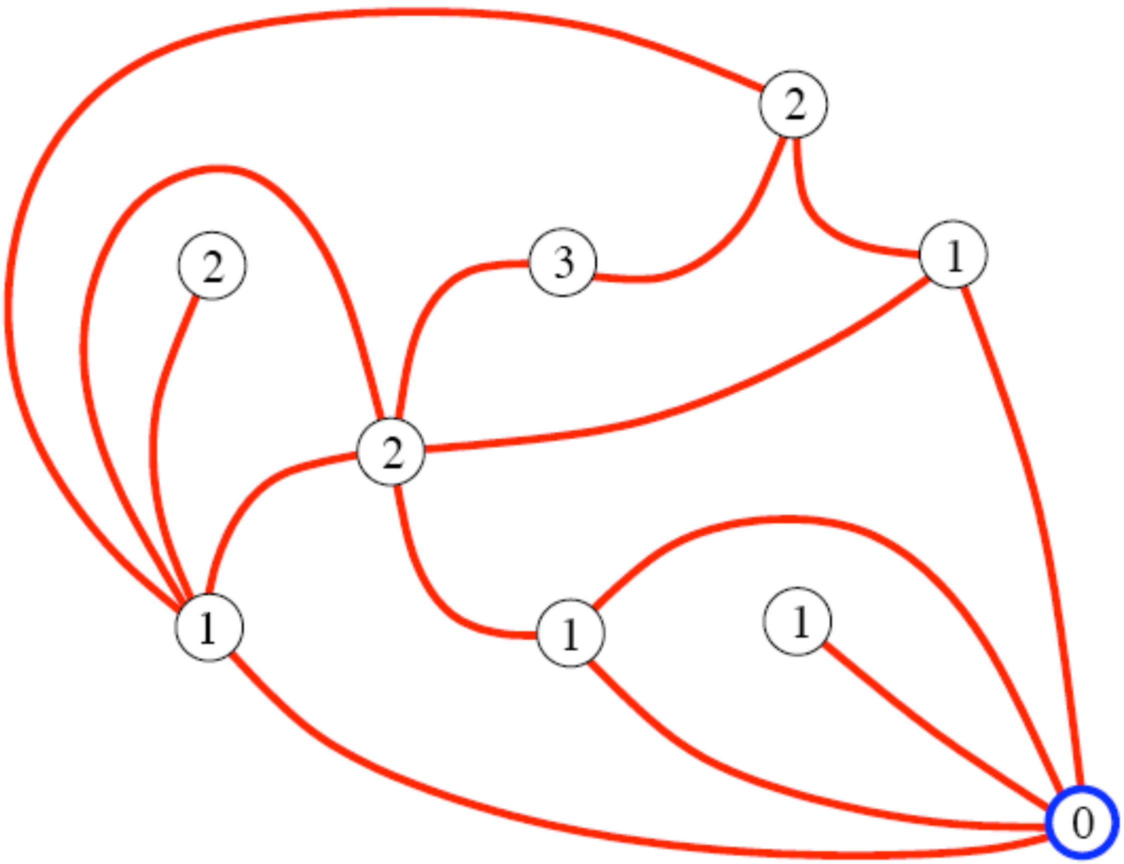

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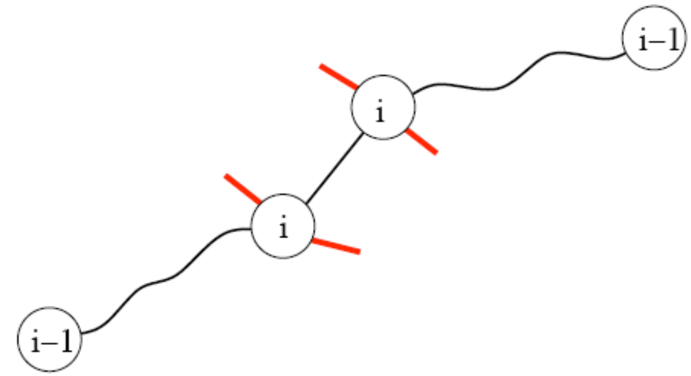

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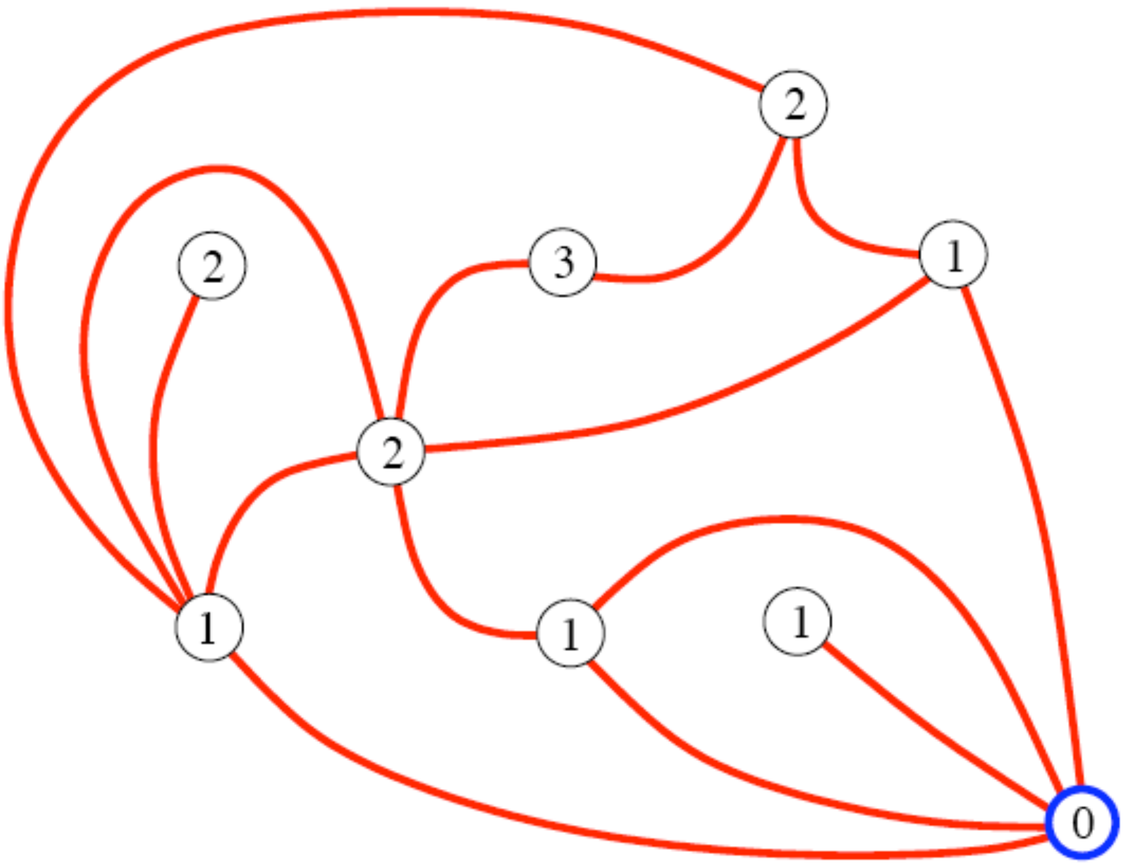

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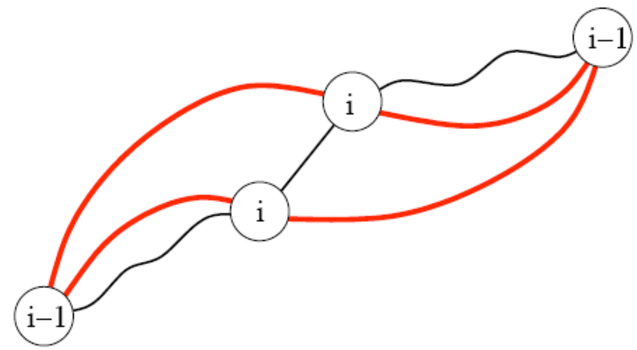

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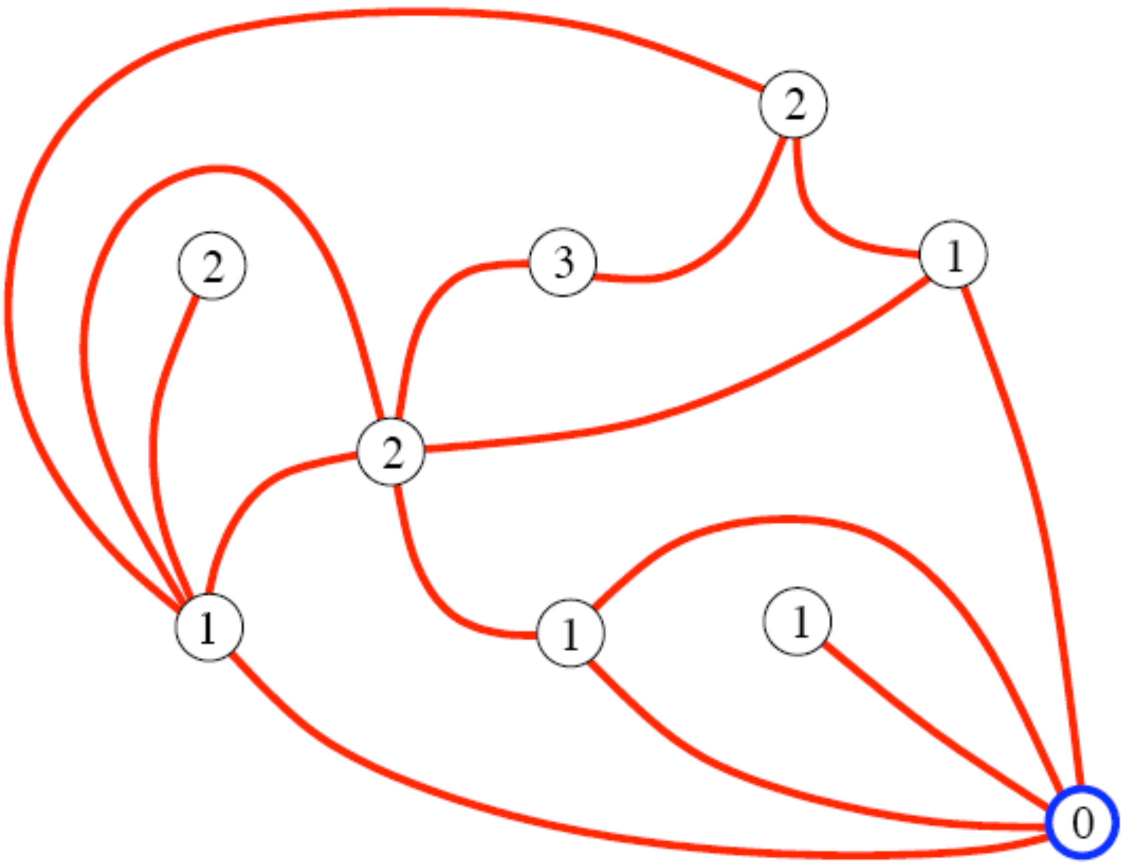

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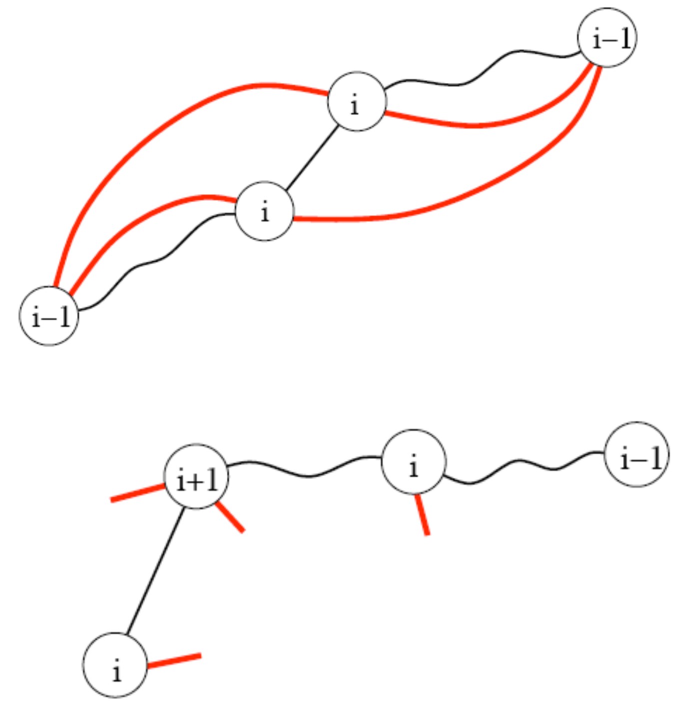

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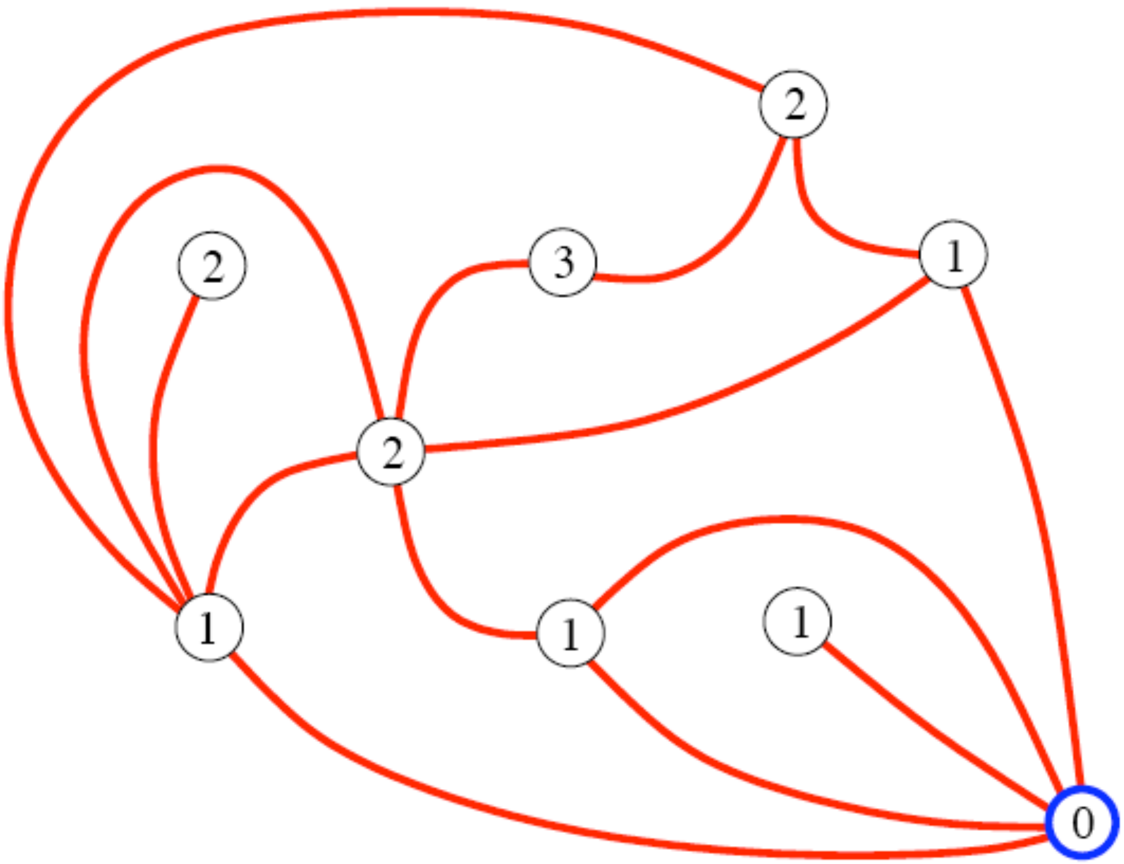

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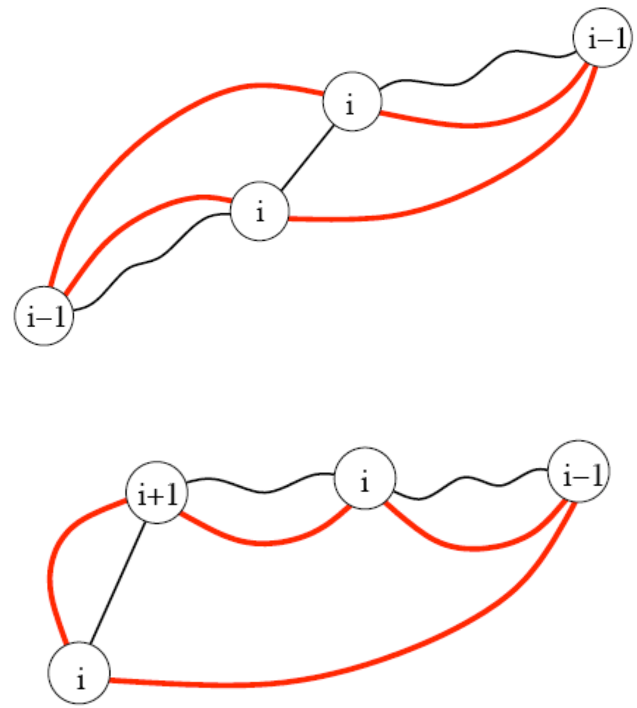

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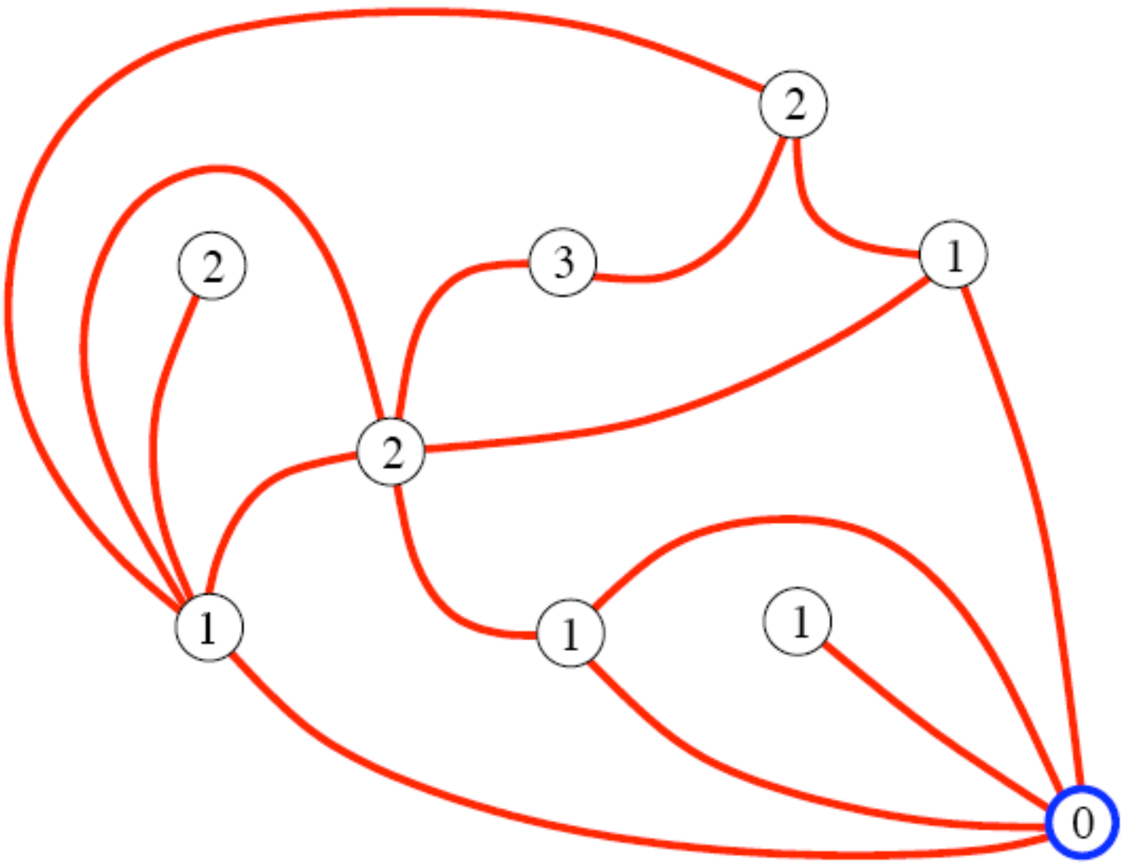

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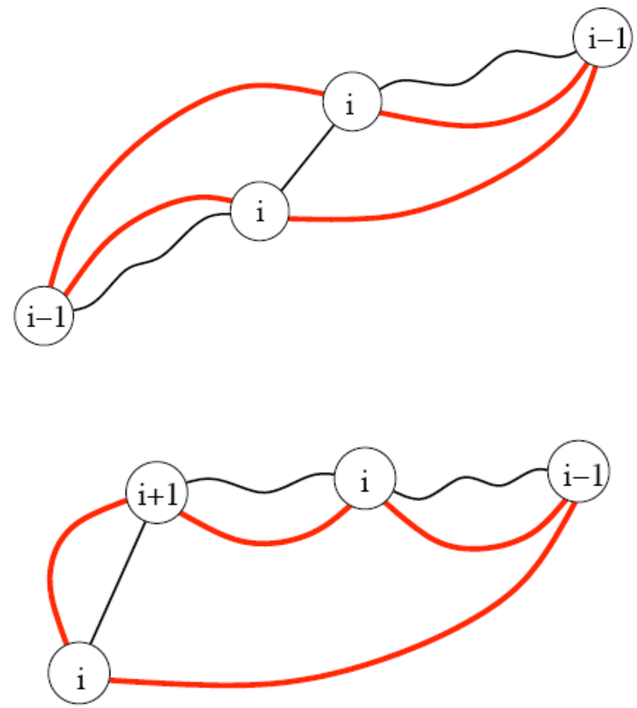
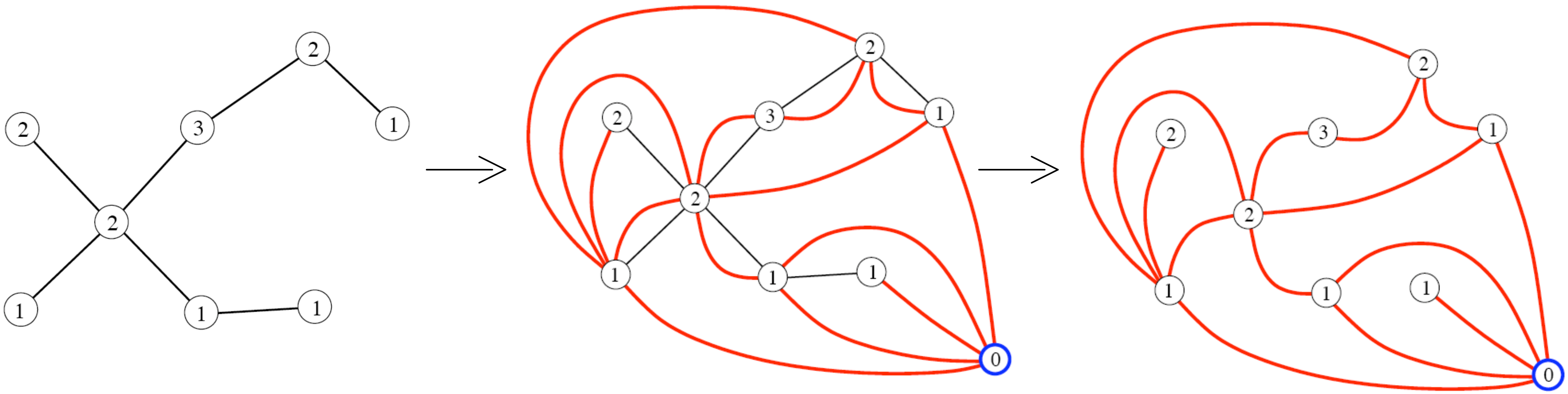


- labels = distances from pointed vertex

The mapping is a bijection

Theorem [Schaeffer'98]: The mapping is a bijection from well-labelled trees to pointed quadrangulations



vertex label i ↔ vertex at distance i

corner label i ↔ edge at level i

edge ↔ face

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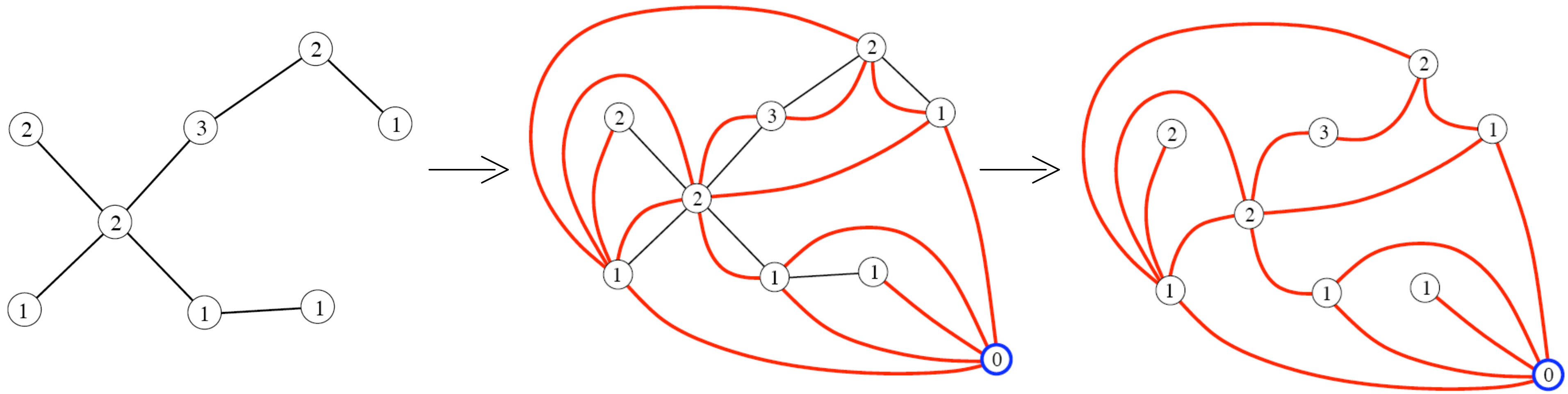

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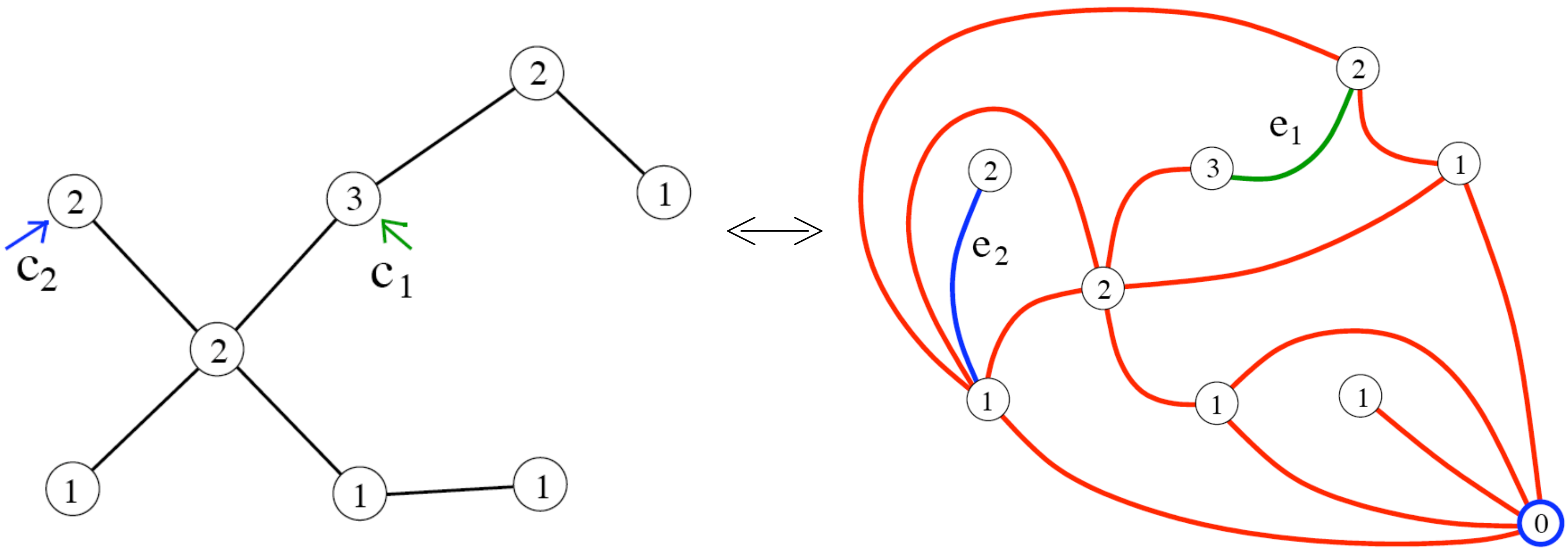
Corollary: there are $3^n \frac{(2n)!}{n!(n+1)!}$ quadrangulations with n faces, a marked vertex, and a marked edge

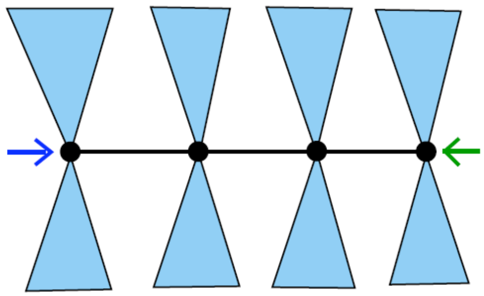
Relative levels

$$T + 2\,\textbf{marked corners}\ c_1, c_2 \leftrightarrow (Q, v) + 2\,\textbf{marked edges}\ e_1, e_2$$

$$\ell(c_2) - \ell(c_1) \ = \ \textbf{level}(e_2) - \textbf{level}(e_1)$$



Relative levels are in the scale $n^{1/4}$



L **is of order** $n^{1/2}$

$\Delta := \ell(c_2) - \ell(c_1)$ **is of order** $\sqrt{L}$**, i.e.,** $n^{1/4}$

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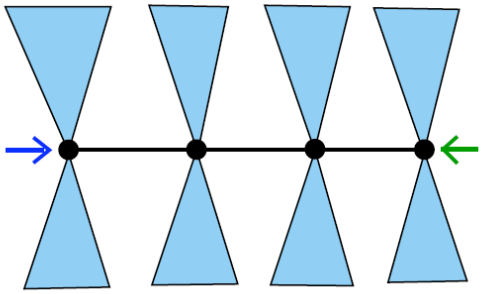

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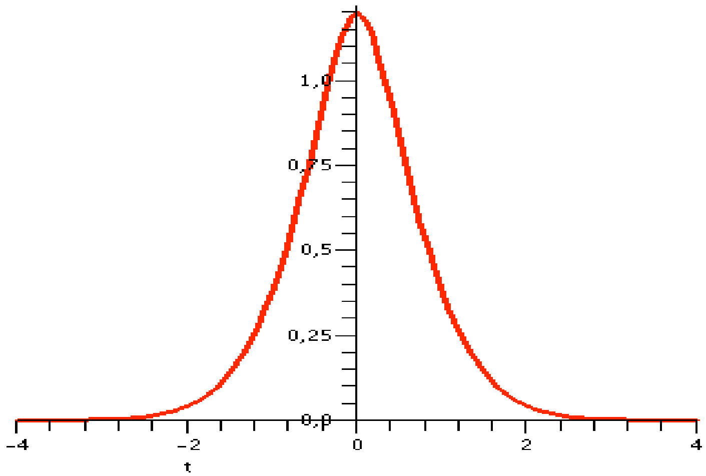
Precisely $$\frac{\Delta}{n^{1/4}} \xrightarrow[n\to\infty]{} \mathrm{d}t\, g(t)$$

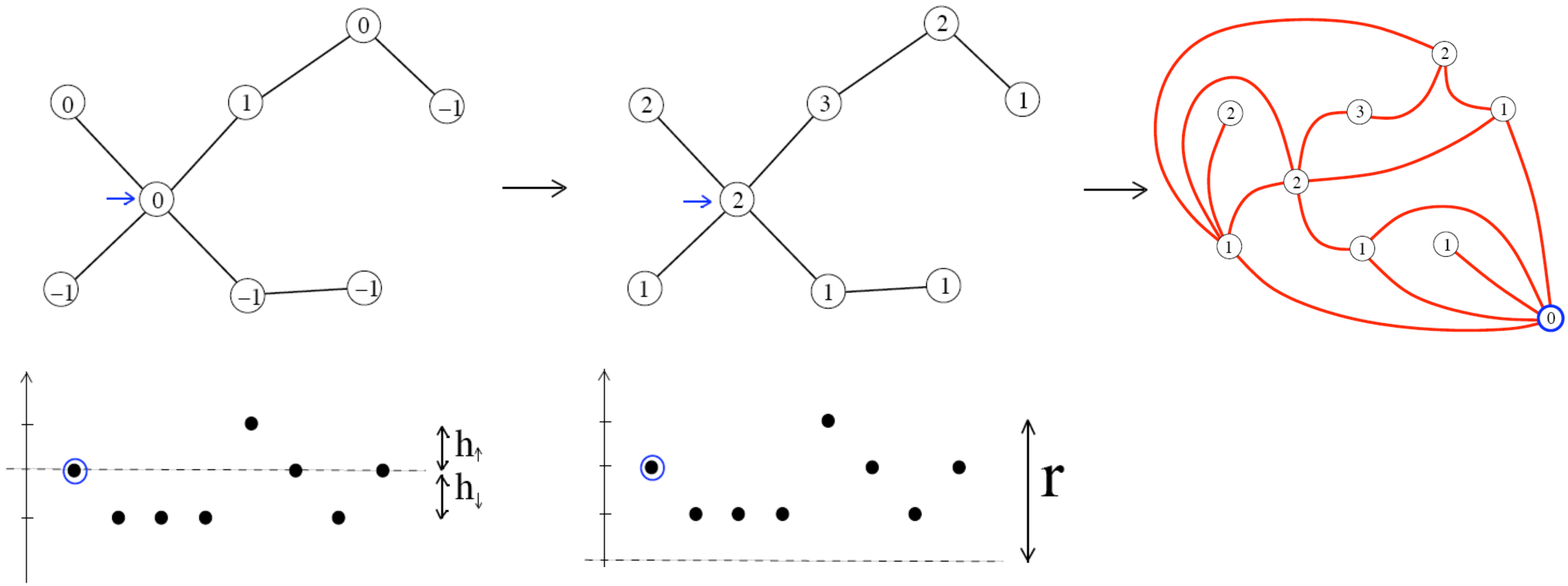
where $$g(t) := 2\sqrt{\frac{3}{\pi}} \int_0^{+\infty} e^{-3t^2/4x} \sqrt{x}\, e^{-x^2} \mathrm{d}x$$



$$g(t) = \Theta(t^{1/3} e^{-ct^{4/3}})$$

$$c := 3^{2/3} \frac{5}{8}$$

Relation typical level / radius



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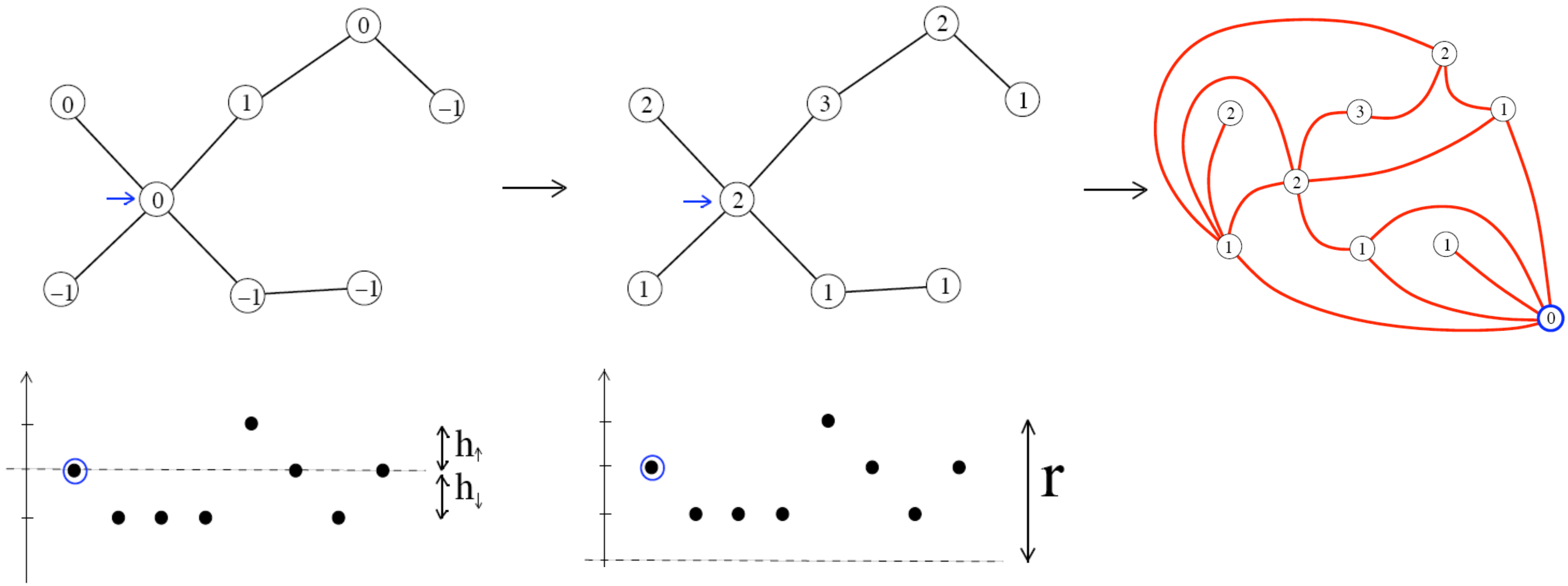

$h_\downarrow + 1$ = Level(random edge)   L := $h_\downarrow + 1/2$ = Level - 1/2

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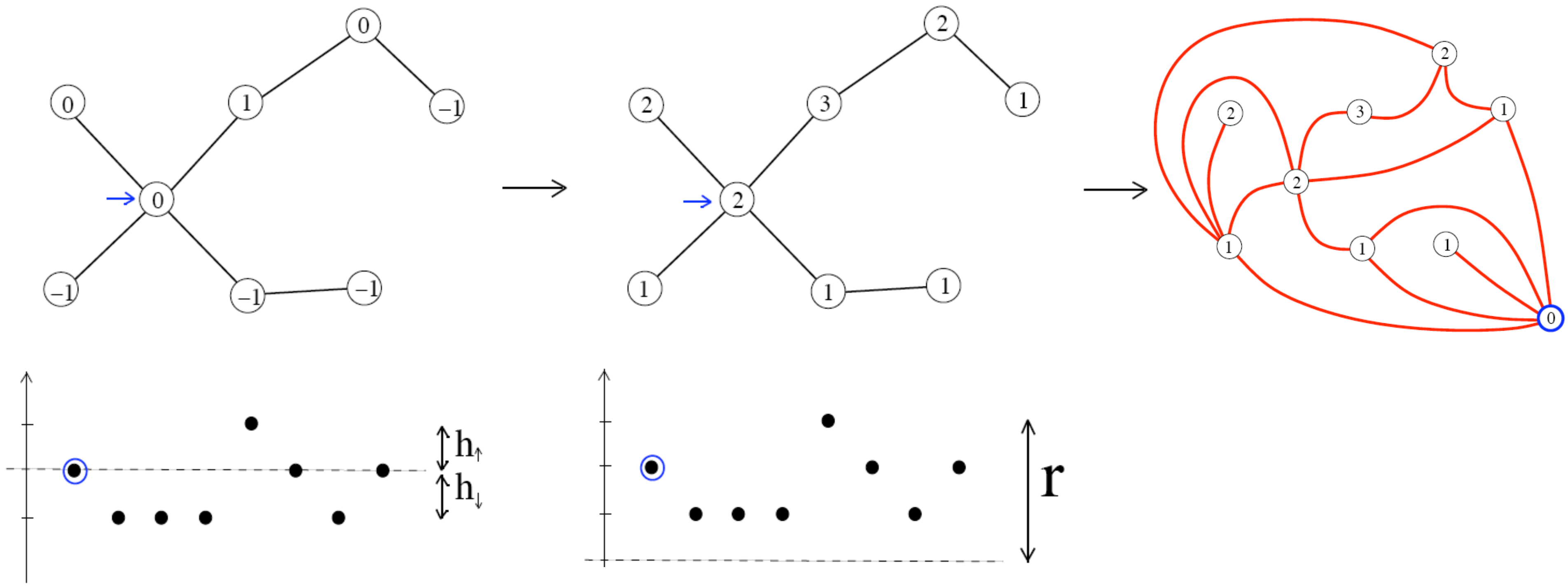


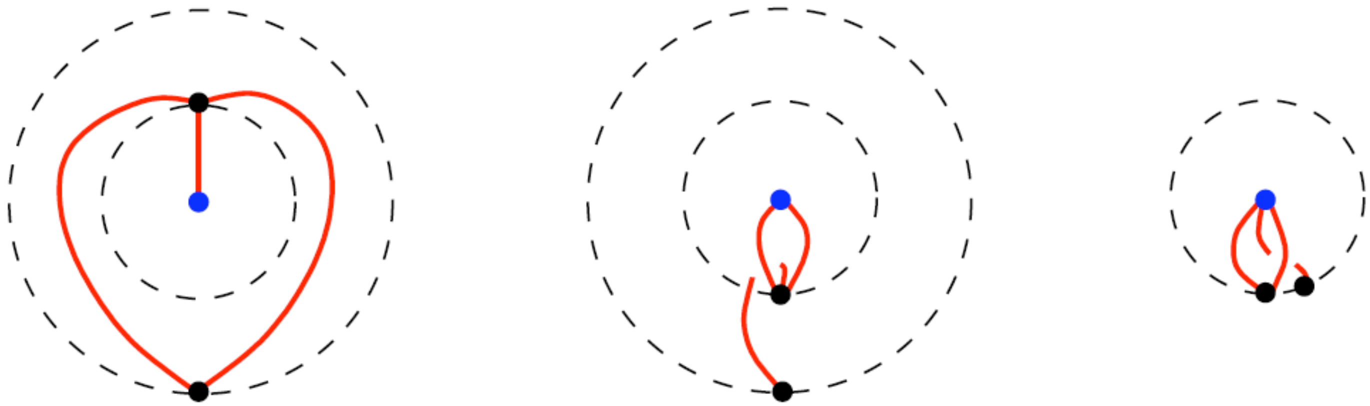
$h_\downarrow + 1 = \text{Level(random edge)}$ $\quad\quad$ $L := h_\downarrow + 1/2 = \text{Level} - 1/2$

$$r = L + L'$$

extremal (r), typical (L), same distribution as L (L')

Illustration

- For pointed quadrangulations with 2 faces



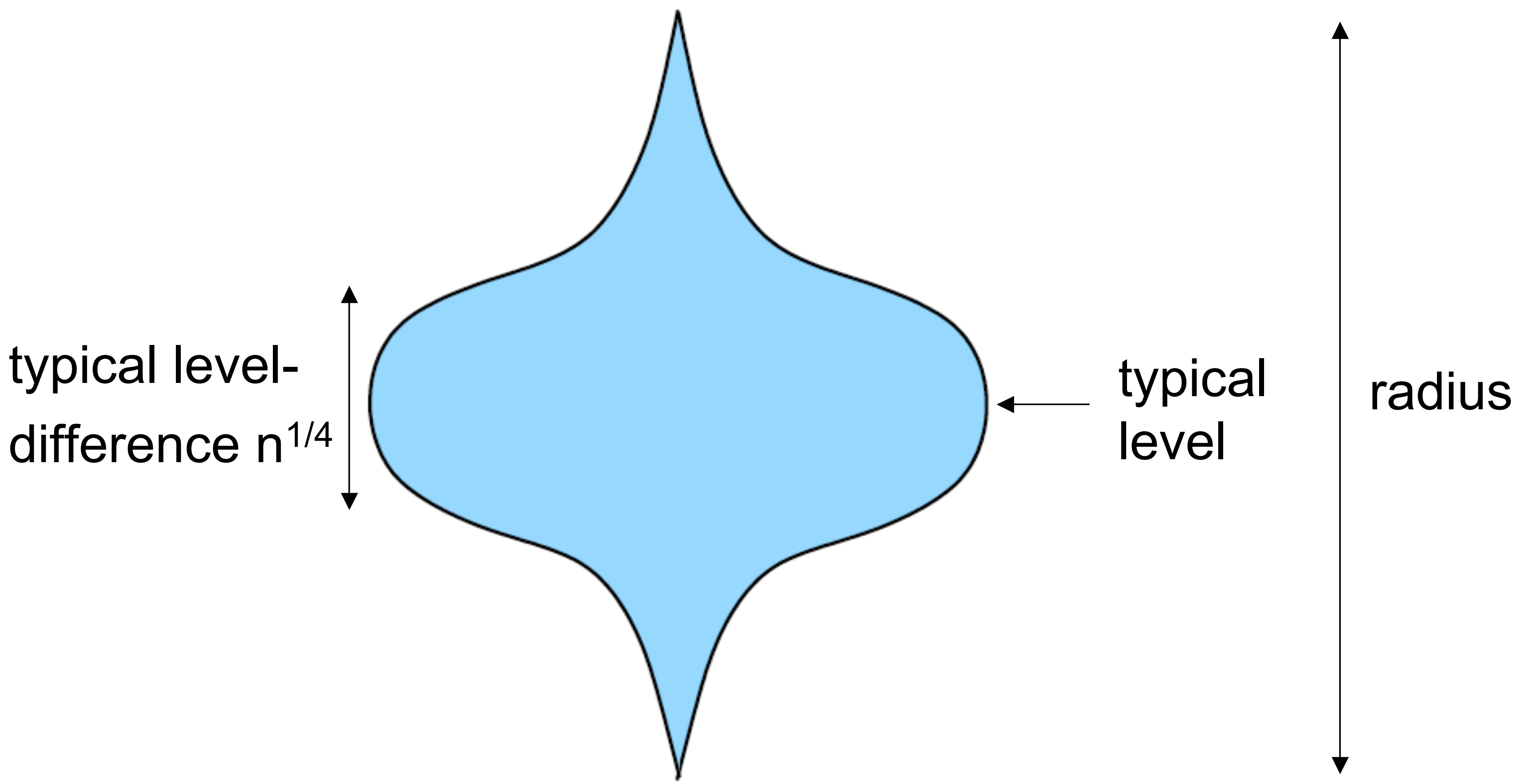
(1/2, 3/2, 3/2)	(1/2, 1/2, 3/2)	(1/2, 1/2, 1/2)	distance L
2	2	1	radius r

E(r) = (2+2+1)/3 = 5/3

E(L) = (7/2 + 5/2 + 3/2)/9 = 5/6

E(r) = 2 **E**(L) in each fixed size

Consequence on the profile



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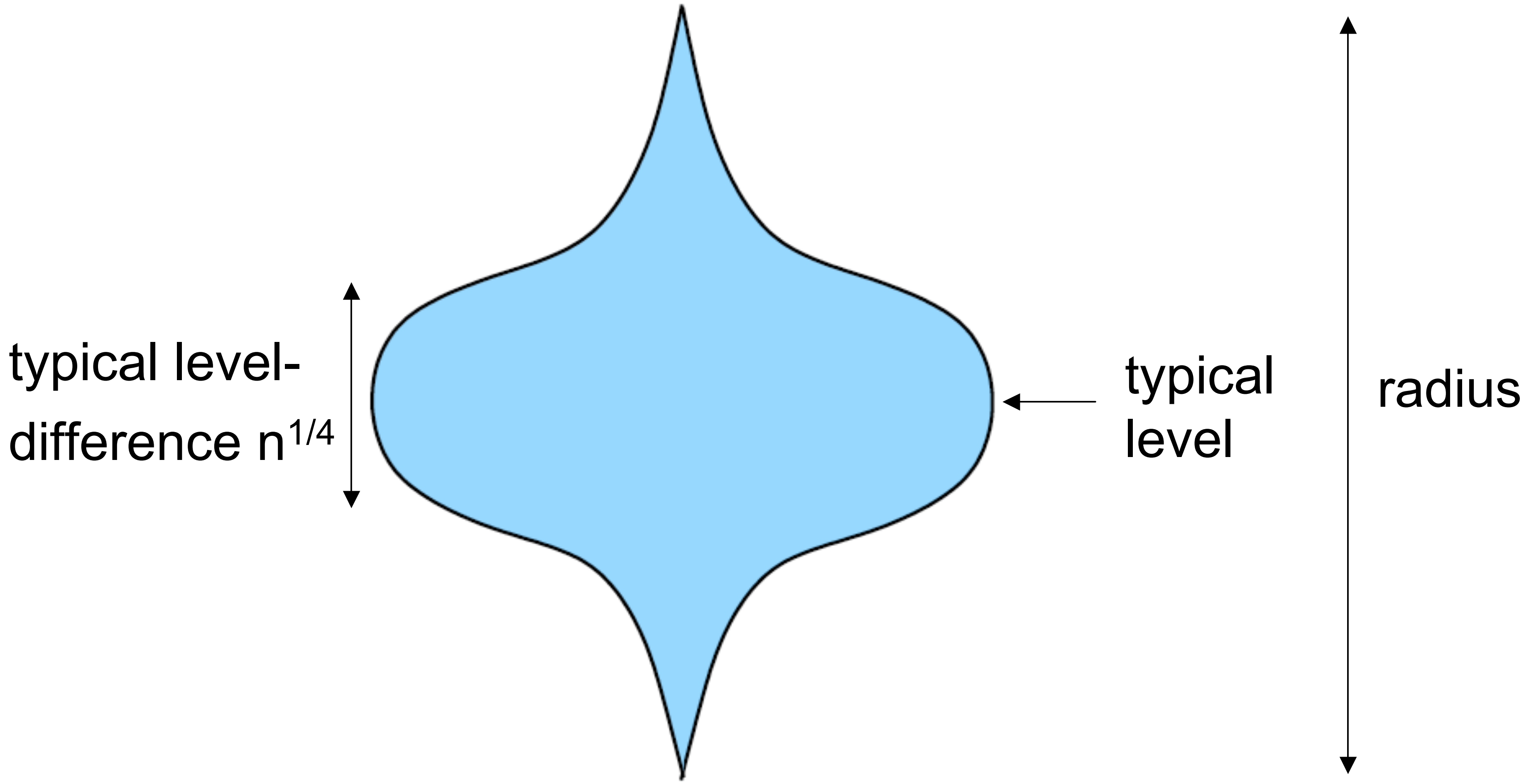

Typical level (& radius) also of order $n^{1/4}$:

- Chassaing-Schaeffer'04: continuous limit (brownian snake)
- Bouttier-Di Francesco-Guitter'03: exact GF expressions

Exact GF expression

[Bouttier, Di Francesco, Guitter'03]

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Equation: $R_k(z) = \dfrac{1}{1 - z(R_{k-1}(z) + R_k(z) + R_{k+1}(z))}$

$R = \lim_k R_k$ **satisfies** $R = \dfrac{1}{1 - 3zR}$

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Exact solution: $R_k = R\dfrac{(1 - x^k)(1 - x^{k+3})}{(1 - x^{k+1})(1 - x^{k+2})}$

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Rk: $x = 1 - c(1 - z/\rho)^{1/4} + \cdots$

$$\Rightarrow \frac{Level}{n^{1/4}} \longrightarrow \mathrm{du}\ g(u)$$

related to $\text{Stable}_{1/4}$