A Hybrid of Darboux's method and Singularity Analysis in Combinatorial Asymptotics

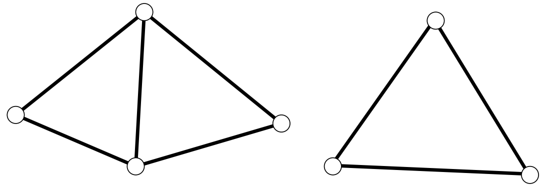
Philippe Flajolet, Éric Fusy, Xavier Gourdon, Daniel Panario, Nicolas Pouyanne

É.F.: Dept. Mathematics, Simon Fraser University (Vancouver)

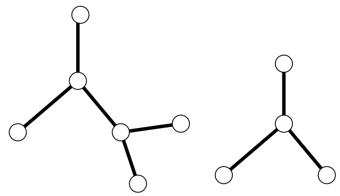
Motivations

- We consider classes of objects
  Each object has a size (e.g. $\#$(vertices))

Graphs



Forests



Permutations

1 3 5 2 4

- Given a class $\mathcal{C}$, let c_n be the number of objects of size n in $\mathcal{C}$
- **Our aim**: find automatic methods for estimating the coefficients c_n asymptotically.

Generating functions

- Let $\mathcal{C}$ be a class, with counting coefficients c_n.
  OGF (unlabelled class): $C(z) = \sum_n c_n z^n$

  EGF (labelled class): $C(z) = \sum_n c_n \frac{z^n}{n!}$

Generating functions

- Let $\mathcal{C}$ be a class, with counting coefficients c_n.
  OGF (unlabelled class): $C(z) = \sum_n c_n z^n$
  EGF (labelled class): $C(z) = \sum_n c_n \frac{z^n}{n!}$
- Dictionary for computing GF:

$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$C(z) = A(z) + B(z)$
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(z) = A(z) \cdot B(z)$
$\mathcal{C} = \mathrm{Set}(\mathcal{A})$	$C(z) = \exp(A(z))$
$\mathcal{C} = \mathrm{Cyc}(\mathcal{A})$	$C(z) = \log(1/(1 - A(z)))$

Generating functions

- Let $\mathcal{C}$ be a class, with counting coefficients c_n.
  OGF (unlabelled class): $C(z) = \sum_n c_n z^n$
  EGF (labelled class): $C(z) = \sum_n c_n \frac{z^n}{n!}$
- Dictionary for computing GF:

$\mathcal{C} = \mathcal{A} + \mathcal{B}$	$C(z) = A(z) + B(z)$
$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(z) = A(z) \cdot B(z)$
$\mathcal{C} = \mathrm{Set}(\mathcal{A})$	$C(z) = \exp(A(z))$
$\mathcal{C} = \mathrm{Cyc}(\mathcal{A})$	$C(z) = \log(1/(1 - A(z)))$

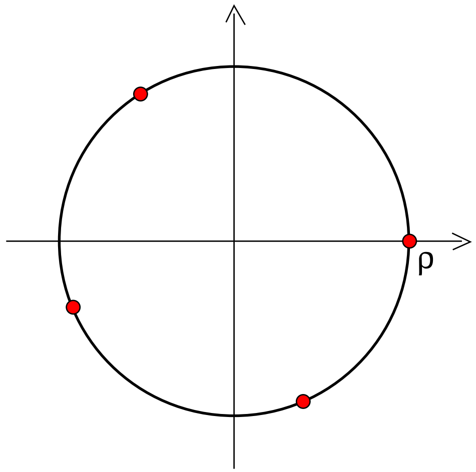
- **Example:** permutations with no fixed point:

$$\mathcal{C} = \mathrm{Set}(\mathrm{Cyc}_{\geq 2}(\mathcal{Z}))$$

$$\Rightarrow C(z) = \exp\left(\log\left(\frac{1}{1-z} - z\right)\right) = \frac{e^{-z}}{1-z}.$$

Complex analysis

- Generating function $C(z)$ as a complex function



$$\rho = \limsup \left([z^n]C(z)\right)^{1/n}$$

$C(z)$ is singular at ρ
(Pringsheim)

- Asymptotic methods:

singular behaviour of $C(z)$	$\Longrightarrow$ (i) Darboux (ii) Sing. analysis	asymptotic estimate of $[z^n]C(z)$

- **Remark:** we can assume $\rho = 1$ without loss of generality, using $[z^n]C(z) = \rho^{-n}[z^n]C(\rho \cdot z)$

Coefficients of basic functions

- A log-power function at 1 is a linear combination $\sigma(z)$ of functions of the form

$$(1-z)^{\alpha} \log^{k}\left(\frac{1}{1-z}\right), \quad \alpha \in \mathbb{R},\ k \in \mathbb{Z}_{\geq 0}$$

Coefficients of basic functions

- A log-power function at 1 is a linear combination $\sigma(z)$ of functions of the form

$$(1-z)^{\alpha}\log^{k}\left(\frac{1}{1-z}\right), \quad \alpha \in \mathbb{R},\ k \in \mathbb{Z}_{\geq 0}$$

- Coefficients have a full asymptotic expansion
Example:
$$[z^n]\frac{\log\left(\frac{1}{1-z}\right)}{\sqrt{1-z}} \sim \frac{\log n+\gamma+2\log 2}{\sqrt{\pi n}} - \frac{\log n+\gamma+2\log 2}{8\sqrt{\pi n^3}} + \cdots$$

Coefficients of basic functions

- A log-power function at 1 is a linear combination $\sigma(z)$ of functions of the form

$$(1-z)^{\alpha}\log^{k}\left(\frac{1}{1-z}\right), \quad \alpha\in\mathbb{R},\ k\in\mathbb{Z}_{\geq 0}$$

- Coefficients have a full asymptotic expansion
Example:

$$[z^n]\frac{\log\left(\frac{1}{1-z}\right)}{\sqrt{1-z}} \sim \frac{\log n+\gamma+2\log 2}{\sqrt{\pi n}} - \frac{\log n+\gamma+2\log 2}{8\sqrt{\pi n^3}} + \cdots$$

- Applies for log-power with finitely many singularities $\zeta_1,\ldots,\zeta_\ell$, using $[z^n]\sigma(z/\zeta_i)=\zeta_i^{-n}[z^n]\sigma(z)$

Darboux's method

- **Key-remark:** if $g(z)$ is $\mathcal{C}_s$-smooth on the closed unit disk, then
$$[z^n]g(z) = o(n^{-s}).$$
(from Cauchy's integral formula + int. by part)

- **Application**: given $C(z) = \sum_n c_n z^n$, decompose $C(z)$ as a sum
$$C(z) = \underbrace{\Sigma(z)}_{\text{log}-\text{power}} + \underbrace{g(z)}_{\mathcal{C}_s-\text{smooth}}.$$
Then $\boxed{[z^n]C(z) = [z^n]\Sigma(z) + o(n^{-s})}$.

Singularity analysis

- There holds the transfer rule (Flajolet-Odlyzko'90)

$$C(z) = \underset{z \to \rho}{O}(g(z)) \xrightarrow[\text{transfer}]{} [z^n]C(z) = O([z^n]g(z))$$

  ($+$ analytic continuation conditions to check)
- Applies for finitely many singularities $\zeta_1, \ldots, \zeta_\ell$

Singularity analysis

- There holds the transfer rule (Flajolet-Odlyzko'90)
$$C(z) = \underset{z \to \rho}{O}(g(z)) \underset{\text{transfer}}{\longrightarrow} [z^n]C(z) = O([z^n]g(z))$$
(+ analytic continuation conditions to check)
- Applies for finitely many singularities $\zeta_1, \ldots, \zeta_\ell$
- **Application**: given $C(z) = \sum_n c_n z^n$ with singularities $\zeta_1, \ldots, \zeta_\ell$, decompose $C(z)$ as a sum
$$\boxed{C(z) = \Sigma(z) + g(z)}$$
where $\Sigma(z)$ is a log-power and $g(z)$ is $O(z - \zeta_i)^\alpha$ at ζ_i.

 Then $\boxed{[z^n]C(z) = [z^n]\Sigma(z) + O(n^{-\alpha-1})}$.

Singularity analysis

- There holds the transfer rule (Flajolet-Odlyzko'90)
$$C(z) = \underset{z\to\rho}{O}(g(z)) \underset{\text{transfer}}{\longrightarrow} [z^n]C(z) = O([z^n]g(z))$$
(+ analytic continuation conditions to check)
- Applies for finitely many singularities $\zeta_1, \dots, \zeta_\ell$
- **Application**: given $C(z) = \sum_n c_n z^n$ with singularities $\zeta_1, \dots, \zeta_\ell$, decompose $C(z)$ as a sum
$$\boxed{C(z) = \Sigma(z) + g(z)}$$
where $\Sigma(z)$ is a log-power and $g(z)$ is $O(z-\zeta_i)^\alpha$ at ζ_i.
Then $\boxed{[z^n]C(z) = [z^n]\Sigma(z) + O(n^{-\alpha-1})}$.
- **Remark:** $g(z)$ is $\mathcal{C}_{\lfloor\alpha\rfloor}$-smooth, as $[z^n]g(z) = O(n^{-\alpha-1})$.
By Darboux, this gives only $[z^n]g(z) = o(n^{-\lfloor\alpha\rfloor})$

Example

Let $\mathcal{C}$ be the class of labelled 2-regular graphs.

Example

Let $\mathcal{C}$ be the class of labelled 2-regular graphs.
2-regular graph = Set of undirected cycles of length ≥ 3

$$\Rightarrow C(z) = \exp\left(\frac{1}{2}\log(\frac{1}{1-z}) - \frac{z}{2} - \frac{z^2}{4}\right) = \frac{e^{-z/2-z^4/4}}{\sqrt{1-z}}$$

Example

Let $\mathcal{C}$ be the class of labelled 2-regular graphs.
2-regular graph $=$ Set of undirected cycles of length ≥ 3

$$\Rightarrow C(z) = \exp\left(\frac{1}{2}\log(\frac{1}{1-z}) - \frac{z}{2} - \frac{z^2}{4}\right) = \frac{e^{-z/2-z^4/4}}{\sqrt{1-z}}$$

- Singularity analysis:

$$C(z) \underset{z\to 1}{\sim} \frac{e^{-3/4}}{\sqrt{1-z}} \quad \Rightarrow \quad [z^n]C(z) \sim \frac{e^{-3/4}}{\sqrt{\pi n}}$$

Example

Let $\mathcal{C}$ be the class of labelled 2-regular graphs.
2-regular graph $=$ Set of undirected cycles of length ≥ 3

$$\Rightarrow C(z) = \exp\left(\frac{1}{2}\log(\frac{1}{1-z}) - \frac{z}{2} - \frac{z^2}{4}\right) = \frac{e^{-z/2-z^4/4}}{\sqrt{1-z}}$$

- Singularity analysis:

$$C(z) \underset{z\to 1}{\sim} \frac{e^{-3/4}}{\sqrt{1-z}} \quad \Rightarrow \quad [z^n]C(z) \sim \frac{e^{-3/4}}{\sqrt{\pi n}}$$

- Darboux's method:

$$C(z) = \frac{e^{-3/4}}{\sqrt{1-z}} + *\sqrt{1-z} + \underbrace{O((1-z)^{3/2})}_{\mathcal{C}_1-\text{smooth}}$$

$$\Rightarrow \quad [z^n]C(z) \sim \frac{e^{-3/4}}{\sqrt{\pi n}} + \frac{*}{n^{3/2}} + o(n^{-1}) \sim \frac{e^{-3/4}}{\sqrt{\pi n}}$$

The Hybrid method

- Classical Darboux's method: sum-decomposition of $C(z) = \sum_n c_n z^n$ as

$$C(z) = \underbrace{\Sigma(z)}_{\mathrm{log-power}} + \underbrace{g(z)}_{\mathcal{C}_s\mathrm{-smooth}} .$$

The Hybrid method

- Classical Darboux's method: sum-decomposition of $C(z) = \sum_n c_n z^n$ as

$$C(z) = \underbrace{\Sigma(z)}_{\mathrm{log-power}} + \underbrace{g(z)}_{\mathcal{C}_s-\mathrm{smooth}} .$$

- For many classes, $C(z)$ is an infinite product $\Rightarrow$ sum-decomposition does not easily apply
- **Example:** The EGF of permutations with distinct cycle lengths is

$$C(z) = \prod_{k \geq 1} \left(1 + \frac{z^k}{k}\right)$$

The Hybrid method

- Classical Darboux's method: sum-decomposition of $C(z) = \sum_n c_n z^n$ as

$$C(z) = \underbrace{\Sigma(z)}_{\mathrm{log-power}} + \underbrace{g(z)}_{\mathcal{C}_s\mathrm{-smooth}} \quad .$$

- For many classes, $C(z)$ is an infinite product $\Rightarrow$ sum-decomposition does not easily apply
- **Example:** The EGF of permutations with distinct cycle lengths is

$$C(z) = \prod_{k \geq 1} \left(1 + \frac{z^k}{k}\right)$$

- **Idea:** in such cases, decompose $C(z)$ as a product, where the first factor gathers the most salient singularities

Hybrid methodology

Given is a function $C(z) = \sum_n c_n z^n$ of finite order a, $C(z) = O(1 - |z|)^a$.

Hybrid methodology

Given is a function $C(z) = \sum_n c_n z^n$ of finite order a, $C(z) = O(1 - |z|)^a$.

1. Product-decomposition:

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1,\ldots,\zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}$$

Hybrid methodology

Given is a function $C(z) = \sum_n c_n z^n$ of finite order a, $C(z) = O(1-|z|)^a$.

1. Product-decomposition:

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1,\ldots,\zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}$$

2. Sum-decompositions:

$$P(z) = \underbrace{\Sigma(z)}_{\text{log}-\text{power}} + \underbrace{R(z)}_{\mathcal{C}_t-\text{smooth}} \quad || \quad Q(z) = \underbrace{\overline{Q}(z)}_{\substack{\text{interpolation}\\ \text{polynomial}\\ \text{of order } c}} + \underbrace{S(z)}_{\substack{\text{high contact}\\ (\text{order } c)\\ \text{at } \zeta_1,\ldots,\zeta_\ell}}$$

Hybrid methodology

Given is a function $C(z) = \sum_n c_n z^n$ of finite order a, $C(z) = O(1-|z|)^a$.

1. Product-decomposition:

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1,\ldots,\zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}$$

2. Sum-decompositions:

$$P(z) = \underbrace{\Sigma(z)}_{\log-\text{power}} + \underbrace{R(z)}_{\mathcal{C}_t-\text{smooth}} \quad || \quad Q(z) = \underbrace{\overline{Q}(z)}_{\substack{\text{interpolation}\\ \text{polynomial}\\ \text{of order } c}} + \underbrace{S(z)}_{\substack{\text{high contact}\\ (\text{order } c)\\ \text{at } \zeta_1,\ldots,\zeta_\ell}}$$

3. Combine the terms:

$$C(z) = \underbrace{\Sigma(z)\overline{Q}(z)}_{\log-\text{power}} + \underbrace{\Sigma(z)S(z)}_{\substack{\mathcal{C}_u-\text{smooth}\\ u=\min(\lfloor c+a \rfloor, s-c)}} + \underbrace{R(z)\overline{Q}(z)}_{\mathcal{C}_s-\text{smooth}} + \underbrace{R(z)S(z)}_{\mathcal{C}_{\min(s,t)}-\text{smooth}}$$

Hybrid methodology

Given is a function $C(z)=\sum_n c_n z^n$ of finite order a, $C(z)=O(1-|z|)^a$.

1. Product-decomposition:

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1,\ldots,\zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}$$

2. Sum-decompositions:

$$P(z) = \underbrace{\Sigma(z)}_{\text{log}-\text{power}} + \underbrace{R(z)}_{\mathcal{C}_t-\text{smooth}} \quad || \quad Q(z) = \underbrace{\overline{Q}(z)}_{\substack{\text{interpolation}\\ \text{polynomial}\\ \text{of order } c}} + \underbrace{S(z)}_{\substack{\text{high contact}\\ \text{(order } c)\\ \text{at } \zeta_1,\ldots,\zeta_\ell}}$$

3. Combine the terms:

$$C(z) = \underbrace{\Sigma(z)\overline{Q}(z)}_{\text{log}-\text{power}} + \underbrace{\Sigma(z)S(z)}_{\substack{\mathcal{C}_u-\text{smooth}\\ u=\min(\lfloor c+a\rfloor, s-c)}} + \underbrace{R(z)\overline{Q}(z)}_{\mathcal{C}_s-\text{smooth}} + \underbrace{R(z)S(z)}_{\mathcal{C}_{\min(s,t)}-\text{smooth}}$$

4. Adjust the parameters: with $t=\lfloor\frac{s+\lfloor a\rfloor}{2}\rfloor$, $c=\lfloor\frac{s-\lfloor a\rfloor}{2}\rfloor$,

we obtain $\boxed{[z^n]C(z) = [z^n]\Sigma(z)\overline{Q}(z) + o(n^{-t})}$

Hybrid in practice

Given is a function $C(z) = \sum_n c_n z^n$ of order a, $C(z) = O(1 - |z|)^a$.

Hybrid in practice

Given is a function $C(z) = \sum_n c_n z^n$ of order a, $C(z) = O(1 - |z|)^a$.

1. Check that there exists a product-decomposition

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1,\ldots,\zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}$$

Hybrid in practice

Given is a function $C(z) = \sum_n c_n z^n$ of order a,
$C(z) = O(1 - |z|)^a$.

1. Check that there exists a product-decomposition

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1,\ldots,\zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}$$

(then, with $t = \lfloor \frac{s+\lfloor a \rfloor}{2} \rfloor, \; c = \lfloor \frac{s-\lfloor a \rfloor}{2} \rfloor$,

$$[z^n]C(z) = [z^n] \underbrace{\Sigma(z)}_{\substack{\text{log}-\text{power} \\ \text{expansion of } P \\ \text{of order } t}} \cdot \underbrace{\overline{Q}(z)}_{\substack{\text{interpolation} \\ \text{polynomial of } Q \\ \text{of order } c}} + o(n^{-t}))$$

Hybrid in practice

Given is a function $C(z) = \sum_n c_n z^n$ of order a,
$C(z) = O(1-|z|)^a$.

1. Check that there exists a product-decomposition

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1,\ldots,\zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}$$

(then, with $t = \lfloor \frac{s+\lfloor a \rfloor}{2} \rfloor$, $c = \lfloor \frac{s-\lfloor a \rfloor}{2} \rfloor$,

$$[z^n]C(z) = [z^n] \underbrace{\Sigma(z)}_{\substack{\text{log}-\text{power} \\ \text{expansion of } P \\ \text{of order } t}} \cdot \underbrace{\overline{Q}(z)}_{\substack{\text{interpolation} \\ \text{polynomial of } Q \\ \text{of order } c}} + o(n^{-t}))$$

Theorem: $\Sigma(z)\overline{Q}(z)$ is the sum of the radial singular expansions of order t of $C(z)$ at $\zeta_1, \ldots, \zeta_\ell$.

Hybrid in practice

Given is a function $C(z) = \sum_n c_n z^n$ of order a, $C(z) = O(1-|z|)^a$.

1. Check that there exists a product-decomposition

$$C(z) = \underbrace{P(z)}_{\text{sing. } \zeta_1,\ldots,\zeta_\ell} \cdot \underbrace{Q(z)}_{\mathcal{C}_s-\text{smooth}}$$

   (then, with $t = \lfloor \frac{s+\lfloor a \rfloor}{2} \rfloor,\ c = \lfloor \frac{s-\lfloor a \rfloor}{2} \rfloor$,

$$[z^n]C(z) = [z^n] \underbrace{\Sigma(z)}_{\substack{\text{log}-\text{power} \\ \text{expansion of } P \\ \text{of order } t}} \cdot \underbrace{\overline{Q}(z)}_{\substack{\text{interpolation} \\ \text{polynomial of } Q \\ \text{of order } c}} + o(n^{-t}))$$

   Theorem: $\Sigma(z)\overline{Q}(z)$ is the sum of the radial singular expansions of order t of $C(z)$ at $\zeta_1, \ldots, \zeta_\ell$.

2. Compute the radial expansions of $C(z)$ of order t (all computations within the disk of convergence)

Example (computation steps)

$C(z)$ is the EGF of permutations with dist. cycle lengths

$$C(z) = \prod_{k \geq 1} \left(1 + \frac{z^k}{k}\right)$$

Example (computation steps)

$C(z)$ is the EGF of permutations with dist. cycle lengths

$$C(z) = \prod_{k \geq 1} \left(1 + \frac{z^k}{k}\right)$$

1. exp-log transformation gives

$$C(z) = (1+z)\exp(\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell}(A_\ell(z))),$$

where $A_\ell(z) = \mathrm{Li}_\ell(z^\ell) - z^\ell,\ \mathrm{Li}_\ell(t) = \sum_{n\geq 1} \frac{t^n}{n^\ell}$ (polylog.)

Example (computation steps)

$C(z)$ is the EGF of permutations with dist. cycle lengths

$$C(z) = \prod_{k\geq 1}\left(1+\frac{z^k}{k}\right)$$

1. exp-log transformation gives

$$C(z) = (1+z)\exp(\sum_{\ell\geq 1}\frac{(-1)^{\ell-1}}{\ell}(A_\ell(z))),$$

where $A_\ell(z) = \mathrm{Li}_\ell(z^\ell) - z^\ell$, $\mathrm{Li}_\ell(t) = \sum_{n\geq 1}\frac{t^n}{n^\ell}$ (polylog.)

2. $\mathrm{Li}_\ell(t)$ is $\mathcal{C}_s$-smooth for $\ell \geq s+2$, so prod.-decomp. is

$$C(z) = \underbrace{P(z)}_{(1+z)\exp(\sum_{\ell\leq s+1} A_\ell(z))} \cdot \underbrace{Q(z)}_{\exp(\sum_{\ell\geq s+2} A_\ell(z))}$$

Example (computation steps)

$C(z)$ is the EGF of permutations with dist. cycle lengths

$$C(z) = \prod_{k\geq 1}\left(1+\frac{z^k}{k}\right)$$

1. exp-log transformation gives

$$C(z) = (1+z)\exp\Big(\sum_{\ell\geq 1}\frac{(-1)^{\ell-1}}{\ell}(A_\ell(z))\Big),$$

where $A_\ell(z) = \mathrm{Li}_\ell(z^\ell) - z^\ell$, $\mathrm{Li}_\ell(t) = \sum_{n\geq 1}\frac{t^n}{n^\ell}$ (polylog.)

2. $\mathrm{Li}_\ell(t)$ is $\mathcal{C}_s$-smooth for $\ell \geq s+2$, so prod.-decomp. is

$$C(z) = \underbrace{P(z)}_{(1+z)\exp(\sum_{\ell\leq s+1} A_\ell(z))} \cdot \underbrace{Q(z)}_{\exp(\sum_{\ell\geq s+2} A_\ell(z))}$$

3. Singularities of $P(z)$ are roots of unity till order $s+1$,

$\Rightarrow$ compute the radial expansions of $C(z)$ at these roots,

using the singular expansions of polylogarithms

(Zagier-Cohen'91, Flajolet'99)

Example (results)

Proposition: The probability that a permutation is made of cycles of distinct lengths admits a full asymptotic expansion of the form

$$f_n \sim e^{-\gamma} + \frac{e^{-\gamma}}{n} + \frac{e^{-\gamma}}{n^2}(-\log n - 1 - \gamma + \log 2)$$
$$+\frac{1}{n^3}\left[e^{-\gamma}\left(\log^2 n + c_{3,1}\log n + c_{3,0}\right) + 2(-1)^n + \Re\left(\frac{18\Gamma(\frac{2}{3})\omega^{-n}}{\Gamma(\frac{1}{6}+\frac{i\sqrt{3}}{6})\Gamma(\frac{1}{2}-\frac{i\sqrt{3}}{6})}\right)\right]$$
$$+\sum_{r\geq 4}\frac{P_r(n)}{n^r},$$

where $c_{3,1}$ and $c_{3,0}$ are explicit constants. Each $P_r(n)$ is a polynomial of degree $r-1$ in $\log n$ with coefficients that are periodic functions of n with period $D(r) = \mathrm{lcm}(2, 3, \ldots, r)$.

Scope of applications

- The hybrid method typically applies for a function $C(z)$ of the form

$$C(z) = \prod_{n\geq 1}(1 + c_n z^n), \text{ with } c_n \sim n^{-\alpha}, \ \alpha \geq 1$$

$Q_{-1} = \prod_{n\geq 1}\left(1 + \frac{z^n}{n}\right)$	Perm. dist. cycle lengths	$\propto 1$
$Q_{-3/2} = \prod_{n\geq 1}\left(1 + \frac{z^n}{n^{3/2}}\right)$	Forest dist. comp. sizes	$\propto n^{-3/2}$
$Q_{-2} = \prod_{n\geq 1}\left(1 + \frac{z^n}{n^2}\right)$	Perm. same cycle type	$\propto n^{-2}$

- Outside of scope:
  - Partitions dist. summands: $\prod_{n\geq 1}(1 + z^n)$
  - Set partitions dist. sizes. $\prod_{n\geq 1}\left(1 + \frac{z^n}{n!}\right)$