Planar maps: bijections and applications

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Motivations for bijections

- efficient manipulation of maps (random generation algo.)

- key ingredient to study distances in random maps

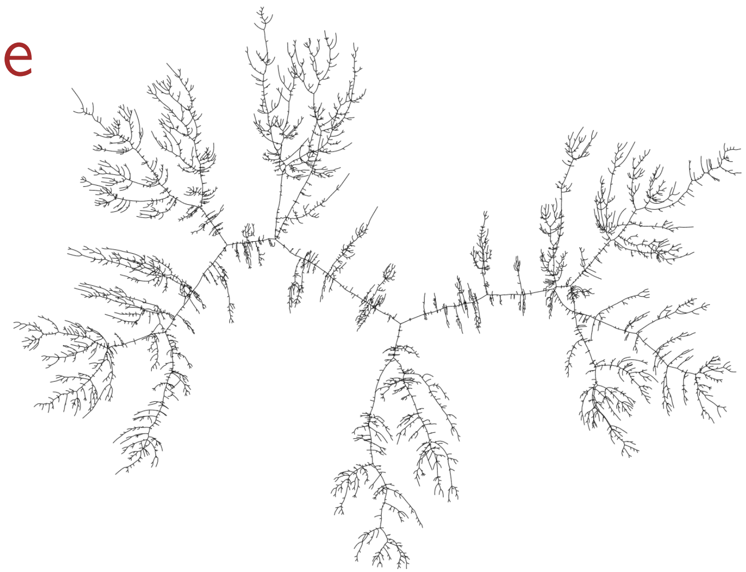
typical distances of order $n^{1/4}$ ($\neq n^{1/2}$ in random trees) today's topic!

Theo: [Le Gall, Miermont'13] $M_n :=$ random quadrangulation n faces

Upon rescaling distances by $n^{1/4}$, M_n converges to a continuum random metric space called the Brownian map

(**Rk:** for random trees, rescaling by $n^{1/2}$, convergence to CRT)

large tree



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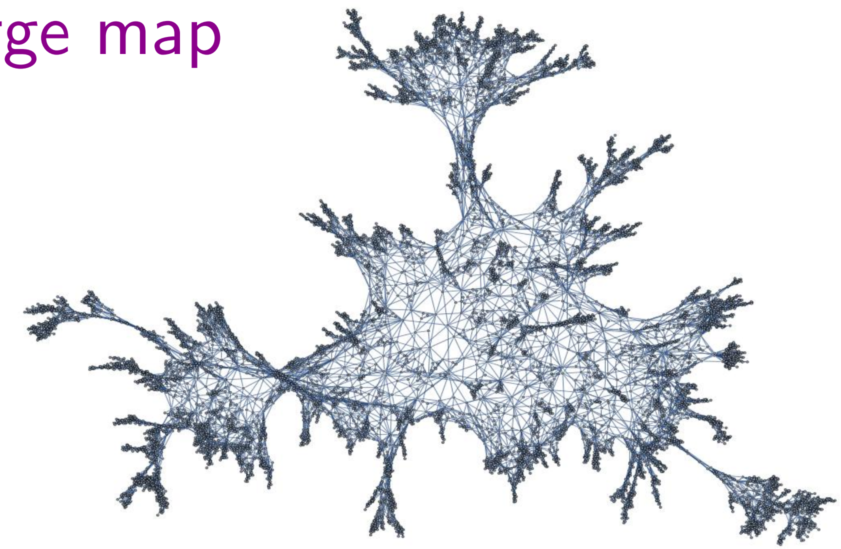
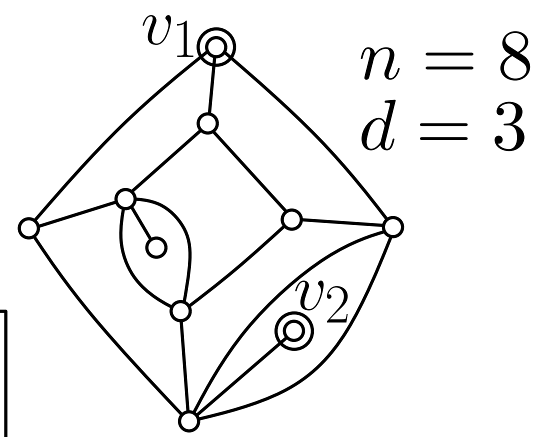


The 2-point function

- Let $\mathcal{G} = \cup_n \mathcal{G}[n]$ be a family of maps (or trees, or graphs) where n is a size-parameter (# faces, # edges, # vertices,...)

- Let $\mathcal{G}^{\circ\circ}$ = family of objects from $\mathcal{G}$ with 2 marked vertices v_1, v_2 (or one marked vertex and one rooted edge, etc.)

let $\mathcal{G}_d^{\circ\circ}$:= subfamily of $\mathcal{G}^{\circ\circ}$ where $\mathrm{dist}(v_2, v_2) = d$



The counting series $G_d \equiv G_d(t)$ of $\mathcal{G}_d^{\circ\circ}$ with respect to the size is called the **2-point function** of $\mathcal{G}$

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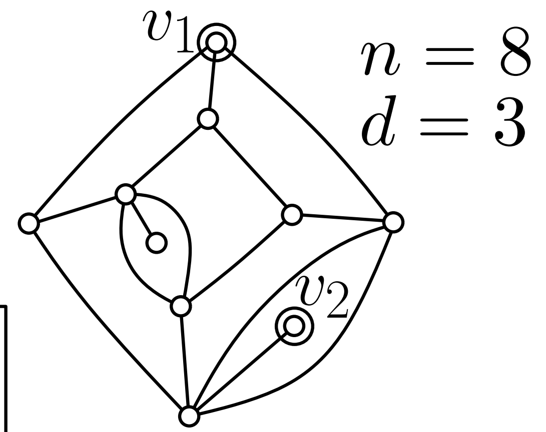

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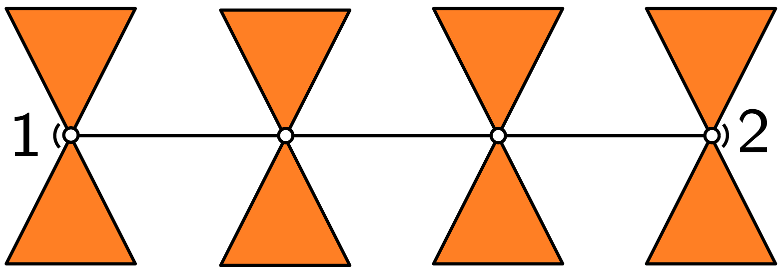
- Let $X_n := \mathrm{dist}(v_1, v_2)$ in a random object from $\mathcal{G}^{\circ\circ}[n]$

$$\text{Then } \mathbb{P}(X_n = d) = \frac{[t^n]G_d(t)}{[t^n]G^{\circ\circ}(t)}$$

The 2-point function of plane trees

Consider a random plane tree on n edges with two marked corners



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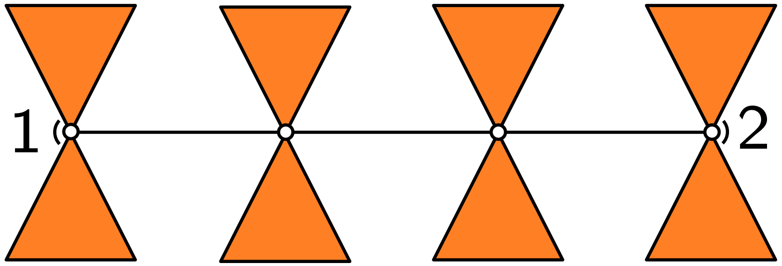

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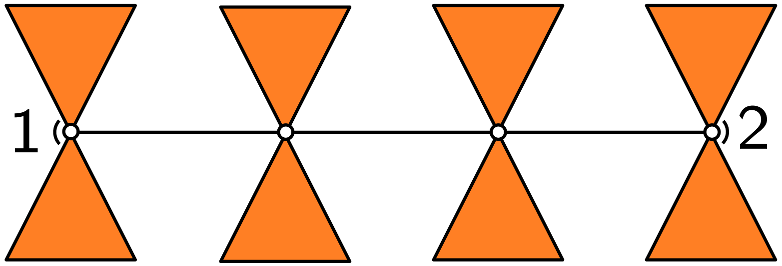

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$$\sim \frac{4^{n+1}}{n\sqrt{\pi}} x e^{-x^2} \quad \text{for } \frac{d}{\sqrt{n}} \to x$$

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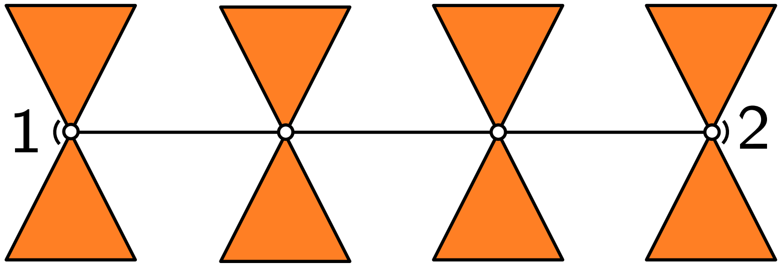

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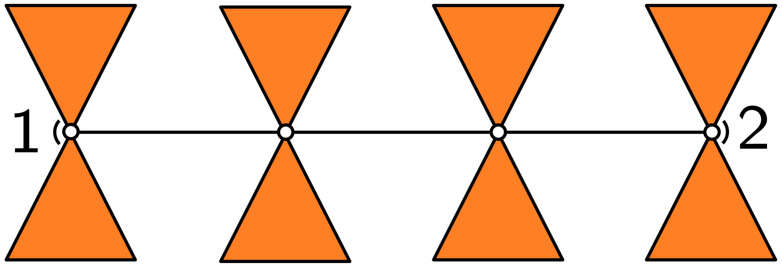

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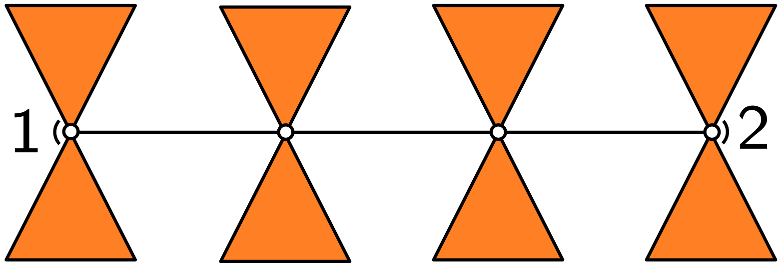

distance $d = 3$

$G_d(t) = t^d C(t)^{2d+2}$ where $C(t) = 1 + tC(t)^2$

$= C(t)^2 E(t)^d$ d th power of series $E(t) = C(t) - 1 = \frac{1-2t-\sqrt{1-4t}}{2t}$

with square-root singularity (also explains $\sqrt{n}$)

Lagrange inversion $\Rightarrow$ $[t^n]G_d(t) = \frac{d+1}{n}\binom{2n+2}{n+d}$

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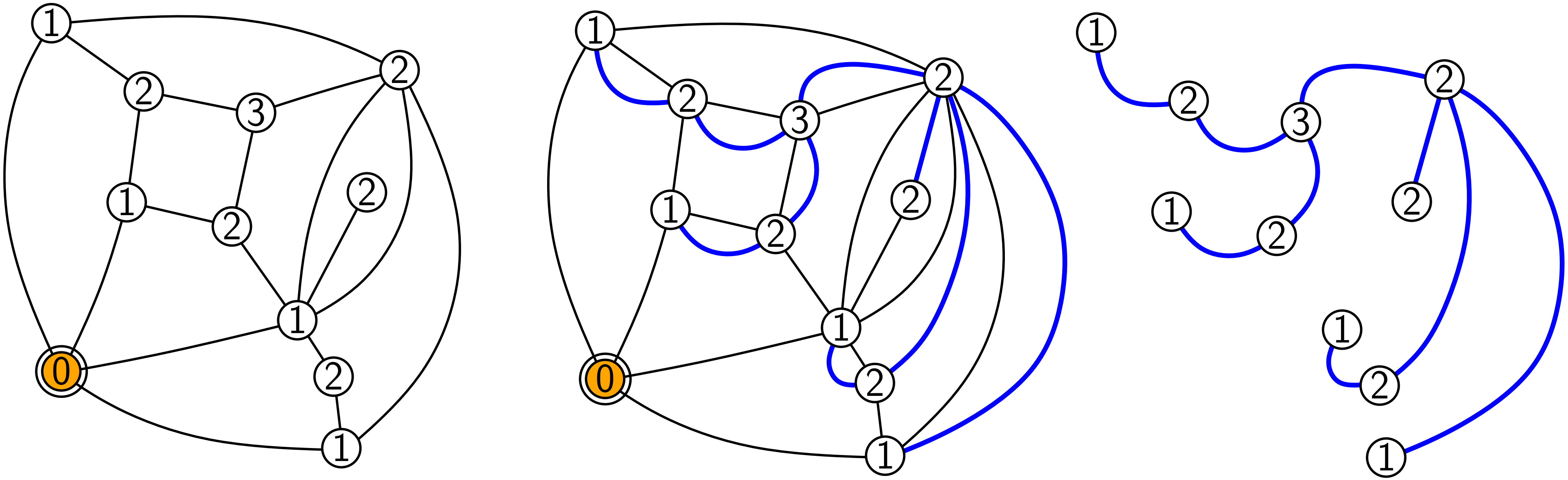
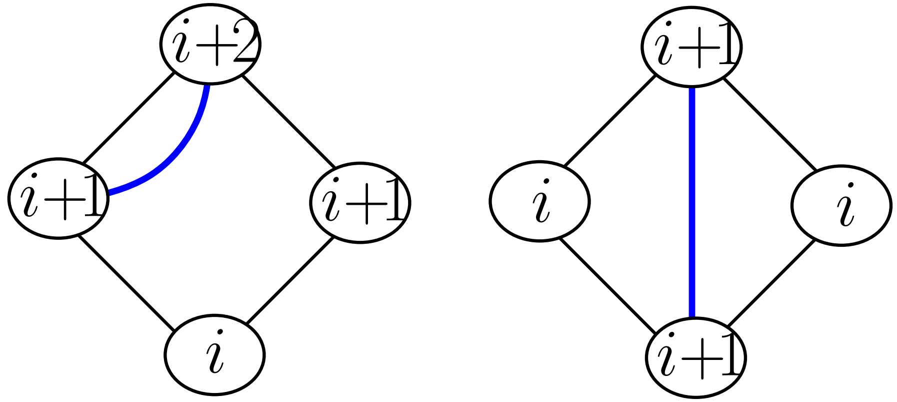
The Schaeffer bijection

bijection between: - vertex-pointed quadrangulations with n faces
- well-labelled trees with n edges and min-label$= 1$



Local rule in each face:



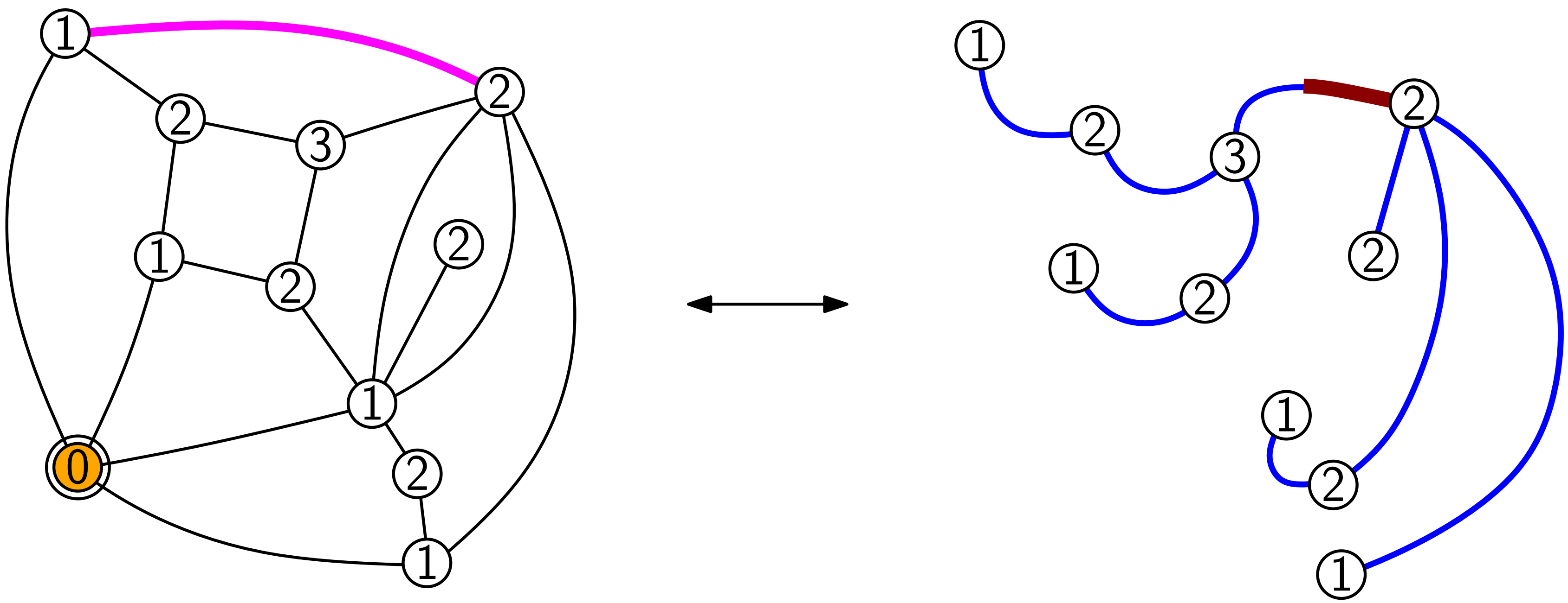
Crucial property: the label $\ell(v)$ of a vertex is its distance (in Q) from the pointed vertex

Reexpressing the 2-point function of quadrangulations

Let $\mathcal{G} = \cup_n \mathcal{G}[n]$ = family of quadrangulations, with $n = \#(\text{faces})$

Let $\mathcal{G}^{\circ\circ} = \{$ quadrangulations + marked vertex v + marked edge e $\}$

$\mathcal{G}_d^{\circ\circ} :=$ subfamily where $\mathrm{dist}(v, e) = d + 1$



Then $G_d(t)$= generating function (by edges) of rooted well-labelled trees with root-vertex label $= d$ and min-label $= 1$

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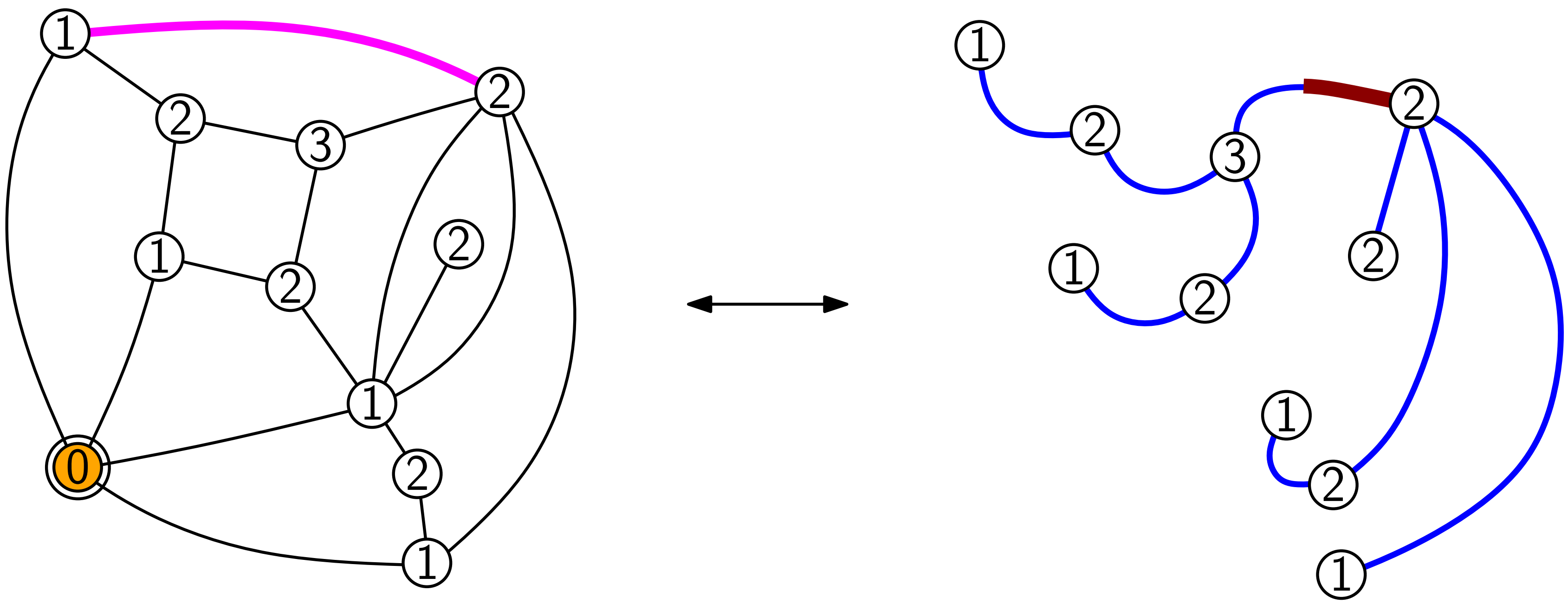


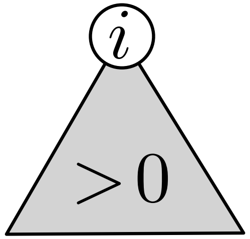
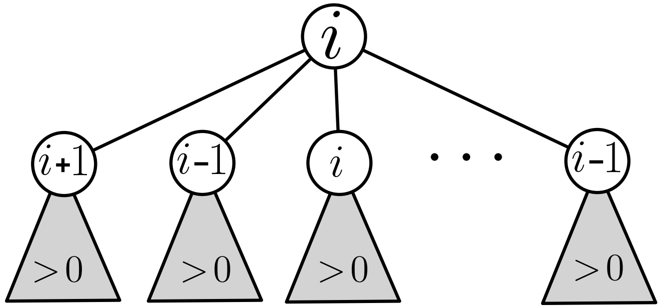
Then $G_d(t)$= generating function (by edges) of rooted well-labelled trees with root-vertex label $= d$ and min-label $= 1$

Rk: Let $R_i(t)$ = GF of rooted well-labelled trees where root-label $= i$ and min-label ≥ 1

Then $G_d(t) = R_d(t) - R_{d-1}(t)$

An equation system for the $R_i(t)$

[Bouttier, Di Francesco, Guitter'03]

$R_i(t)$ counts  $=$ 

$$= \frac{1}{1 - t(R_{i-1}(t) + R_i(t) + R_{i+1}(t))}$$

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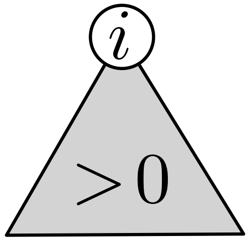

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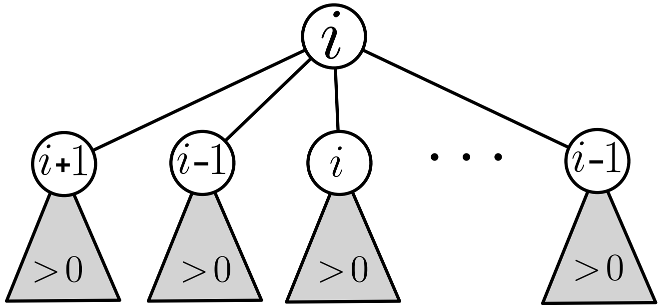

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Hence the $R_i(t)$ are specified by (infinite!) equation system:

$R_0 = 0, \qquad R_i(t) = 1 + tR_i(t) \cdot \big(R_{i-1}(t) + R_i(t) + R_{i+1}(t)\big)$ for $i \geq 1$

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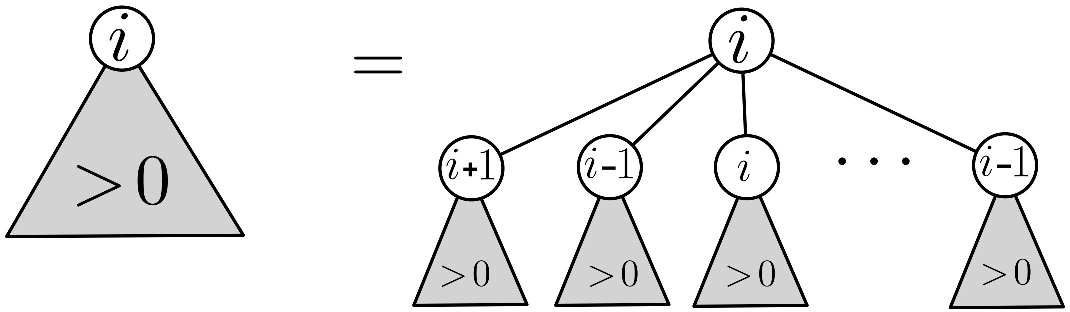
$R_i(t)$ counts 

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Rk: The series $R(t) = \lim_{i\to\infty} R_i(t)$ satisfies $R(t) = 1 + 3tR(t)^2$
$= \sum_{n\geq 0} 3^n \mathrm{Cat}_n t^n$

Computing the $R_i(t)$ iteratively

We have $R_1(t) = \sum_{n\geq 0} \frac{2\cdot 3^n (2n)!}{n!(n+2)!} t^n = R - tR^3$

and for $i \geq 1$ we have $R_i = 1 + tR_i \cdot (R_{i-1} + R_i + R_{i+1})$

$$\Downarrow$$

$$R_{i+1} = \frac{R_i - 1}{tR_i} - R_{i-1} - R_i$$

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each R_i has a rational expression in t and R

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this rational expression does not take a nice form by a simple inspection

Approach for finding a nice explicit expression

[Bouttier, Di Francesco, Guitter'03]

First step: ansatz $R_i(t) = R(t) \cdot \big(1 - c(t) \cdot x(t)^i + O(x^{2i})\big)$
with $x(t)$ to be determined

Rk: should have $x(t) = \Theta(t)$ as $t \to 0$ since $R(t) - R_i(t) = \Theta(t^i)$

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Inject into equation $R_i = 1 + tR_i \cdot (R_{i-1} + R_i + R_{i+1})$

$$R \cdot (1 - cx^i) = 1 + tR^2 \cdot (1 - cx^i)(3 - cx^{i-1} - cx^i - cx^{i+1}) + O(x^{2i})$$

$$\Updownarrow \epsilon = cx^i$$

$$R(1 - \epsilon) = 1 + tR^2 \cdot (1 - \epsilon)(3 - \epsilon(x^{-1} + 1 + x)) + O(\epsilon^2)$$

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extracting coefficient $[\epsilon]$ gives $\quad -R = tR^2 \cdot (-3 - x^{-1} - 1 - x)$

$$\Updownarrow$$

$$R - 3tR^2 = tR^2 \cdot (1 + x^{-1} + x)$$

$$\Updownarrow$$

$$\boxed{1 + x + x^{-1} = \frac{1}{tR^2}}$$

Expressing the $R_i(t)$ in terms of $x(t)$

[Bouttier, Di Francesco, Guitter'03]

- We have $1 + x + x^{-1} = \dfrac{1}{tR^2} = \dfrac{3}{R-1}$

hence $R(t)$ is rational in terms of $x(t)$, we find $R = \dfrac{x^2+4x+1}{x^2+x+1}$

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- We then substitute in the expressions of $R_1, R_2, R_3, \ldots$ in terms of R

  and recognize the explicit expression $$R_i = R\frac{(1-x^i)(1-x^{i+3})}{(1-x^{i+1})(1-x^{i+2})}$$

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  we have to check that this gives a power series for each $i \geq 0$,

  (true since $x(t) = tR(t)^2 \cdot (1 + x(t) + x(t)^2)$)

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  We let $F(x,y)=R(x)\frac{(1-y)(1-yx^3)}{(1-yx)(1-yx^2)}$ (y plays the role of x^i)

and check $F(t,x)=1+t(x)F(x,y)\cdot(F(x,yx^{-1})+F(x,y)+F(x,yx))$

Exact expression

[Bouttier, Di Francesco, Guitter'03]

The generating functions $R_i \equiv R_i(t)$ are expressed as

$$R_i = R\frac{(1-x^i)(1-x^{i+3})}{(1-x^{i+1})(1-x^{i+2})}$$

with $R \equiv R(t)$ given by $R = 1 + 3tR^2$
and $x \equiv x(t)$ given by $x = tR^2(1 + x + x^2)$

References:

- first derivation in BDG'03: 'Geodesic distances in planar graphs'
- combinatorial derivations in

  [Bouttier, Guitter'12]: 'planar maps and continued fractions'
  (+ general determinant expressions for maps with bounded face-degrees)

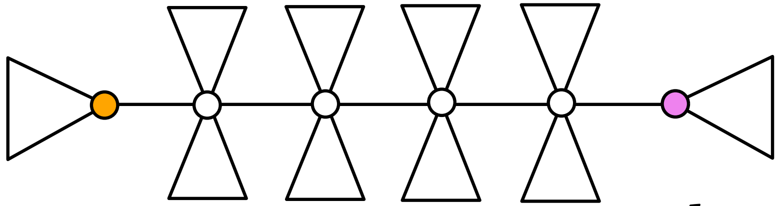
  [Guitter'17]: 'The distance-dependent two-point function of quadrangulations: a new derivation by direct recursion'

Asymptotic considerations

- Two-point function of (plane) trees:

$$\boxed{G_d(t) = (tR^2)^d}$$

with $R = 1 + tR^2 = \frac{1-\sqrt{1-4t}}{2t}$



$d = 5$

G_d is the d th power of a series having a square-root singularity

$\Rightarrow d/n^{1/2}$ converges in law (Rayleigh law, density $u \exp(-u^2/2)$)

- Two-point function of quadrangulations:

$$\boxed{G_d(g) \sim_{d\to\infty} a_1 x^d + a_2 x^{2d} + \cdots}$$

where $x = x(t)$ has a quartic singularity

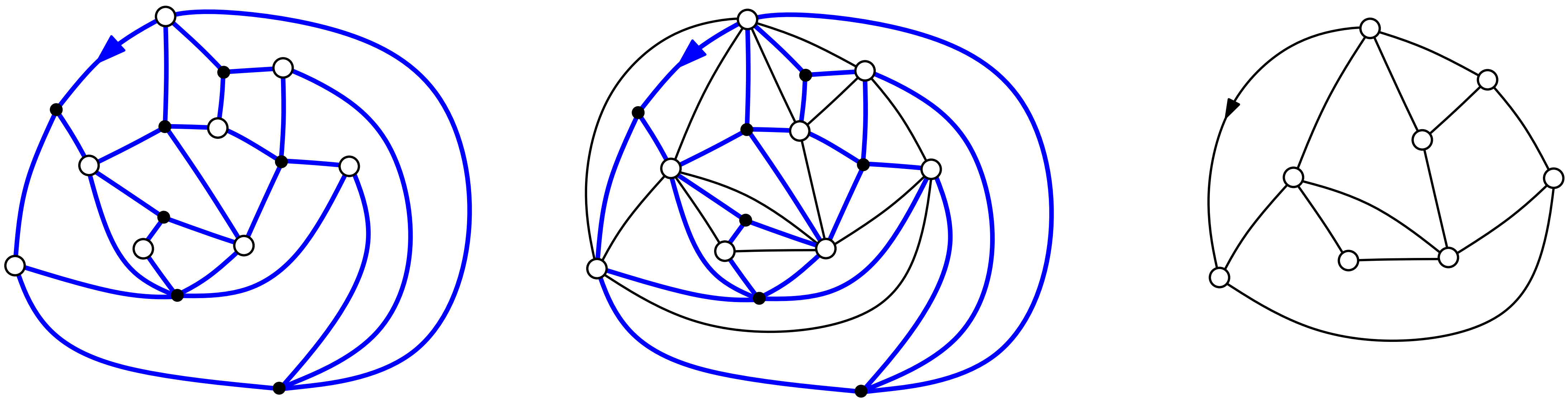
$\Rightarrow d/n^{1/4}$ converges to an explicit law **[BDG'03]**

Convergence in the two cases "follows" from (proof by Hankel contour) **[Banderier, Flajolet, Louchard, Schaeffer'03]**: for $0 < \alpha < 1$,

$$x(t) \underset{t\to 1}{\sim} 1 - (1-t)^\alpha \Rightarrow [t^n]x^{un^\alpha} \sim \frac{1}{2\pi n}\int_0^\infty e^{-s}\mathrm{Im}(\exp(-us^\alpha e^{i\pi\alpha}))\mathrm{d}s$$

The 2-point function of planar maps

Recall the classical bijection from (rooted) quadrangulations to maps



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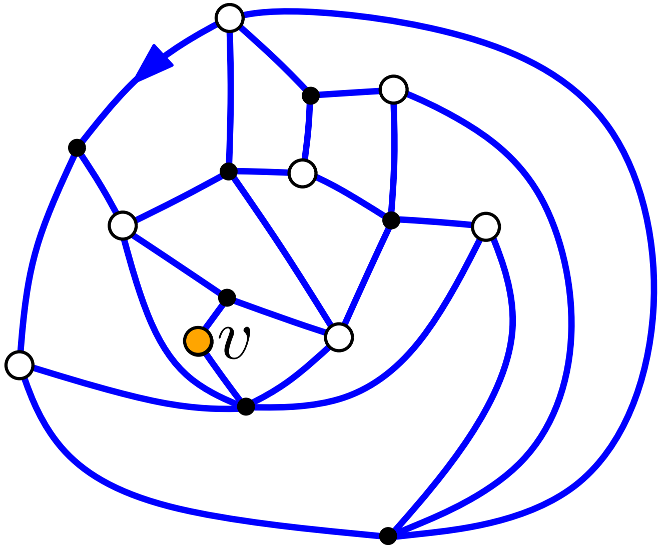
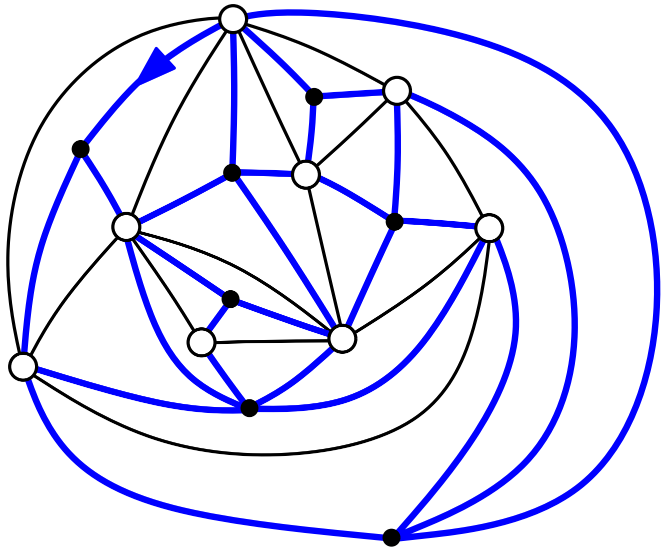
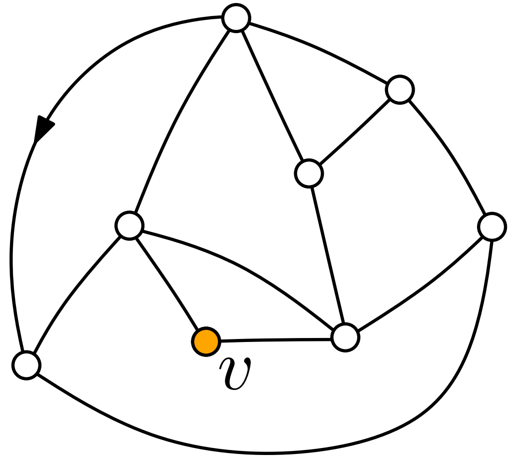


vertex v at edge-distance $2k$
from the root-vertex

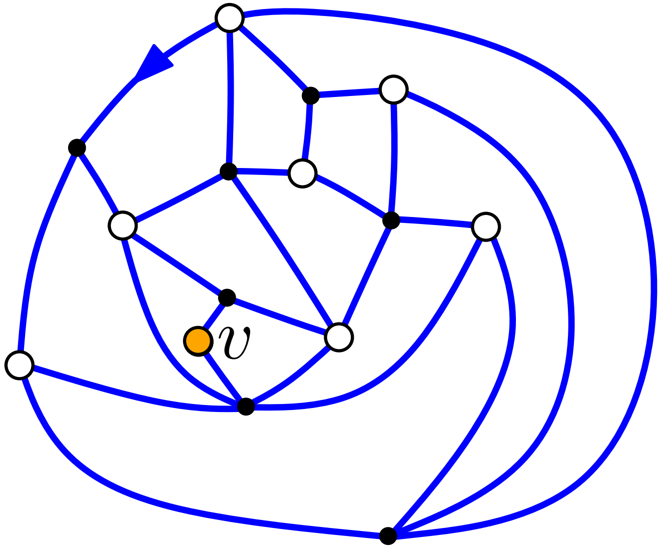


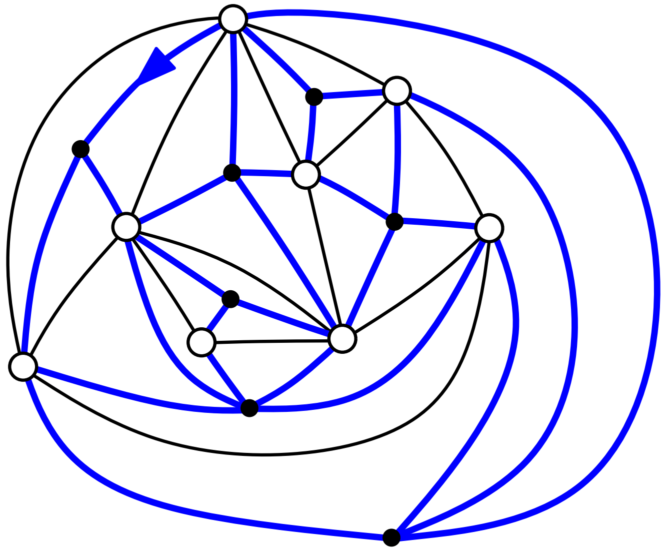


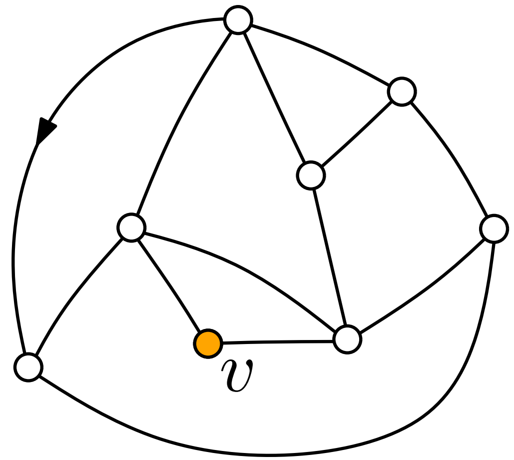
vertex v at face-distance k
from the root-vertex

The 2-point function of planar maps

Recall the classical bijection from (rooted) quadrangulations to maps







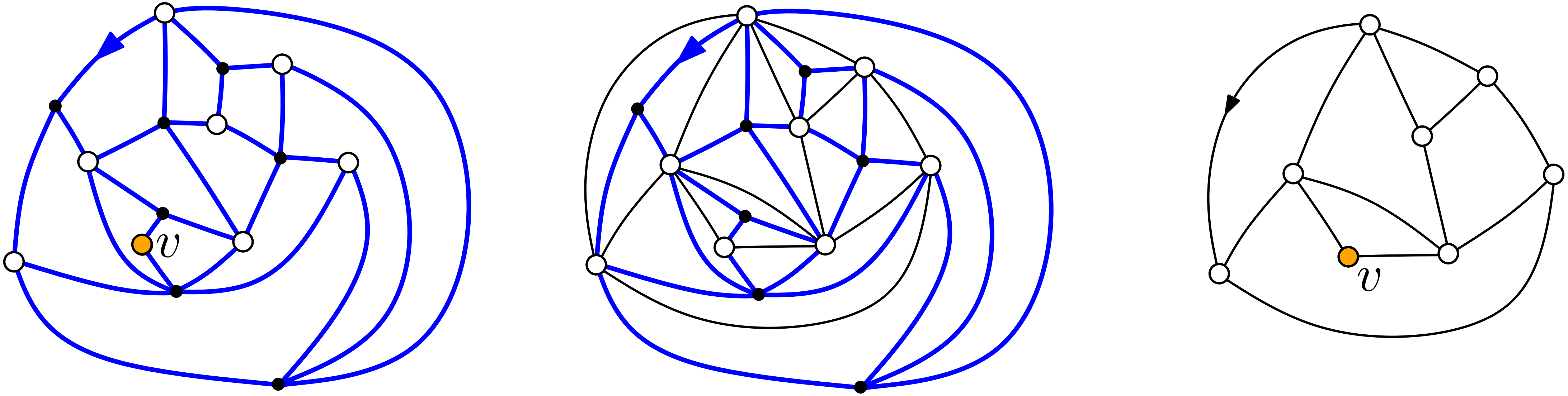
vertex v at edge-distance $2k$ from the root-vertex

vertex v at face-distance k from the root-vertex

Hence $R_{2d+1}(t) =$ GF (by edges) of rooted maps + marked vertex v such that v is at face-distance $\leq d$ from root-vertex

The 2-point function of planar maps

Recall the classical bijection from (rooted) quadrangulations to maps



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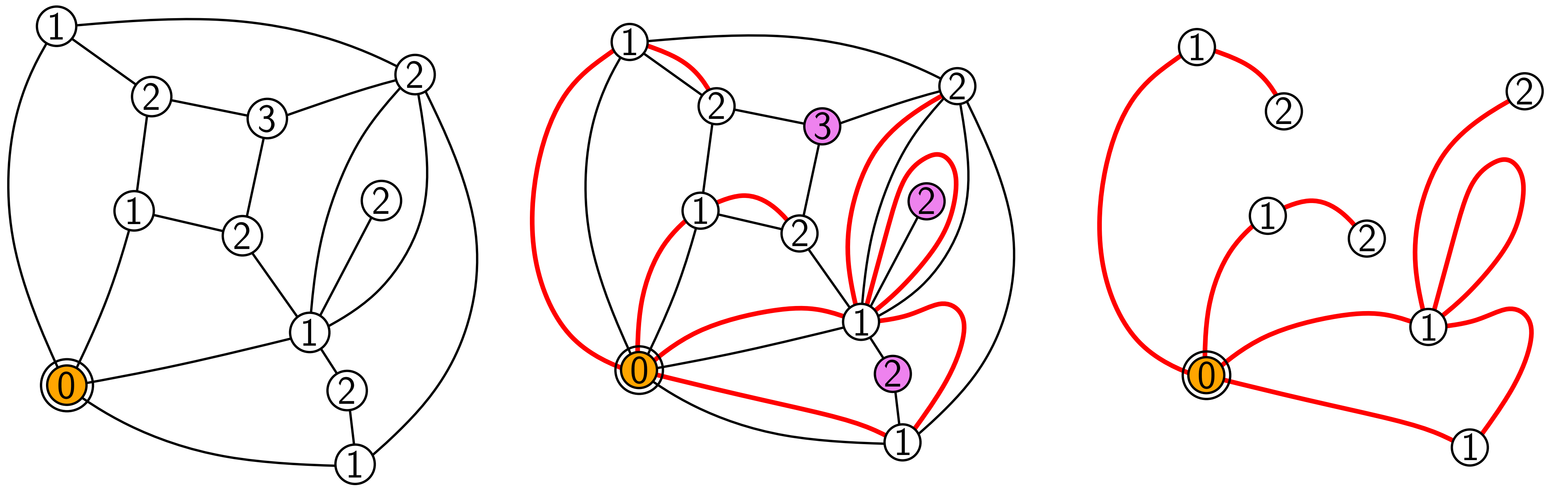
Hence $R_{2d+1}(t) =$ GF (by edges) of rooted maps + marked vertex v such that v is at face-distance $\leq d$ from root-vertex

What about the 2-point function of maps for the edge-distance?

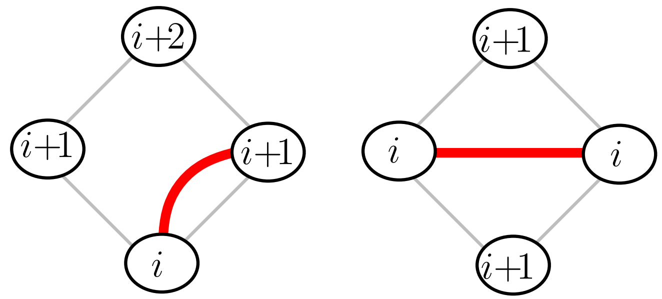
The Ambjørn-Budd bijection

[Ambjørn-Budd'13]

a different bijection between quadrangulations and maps



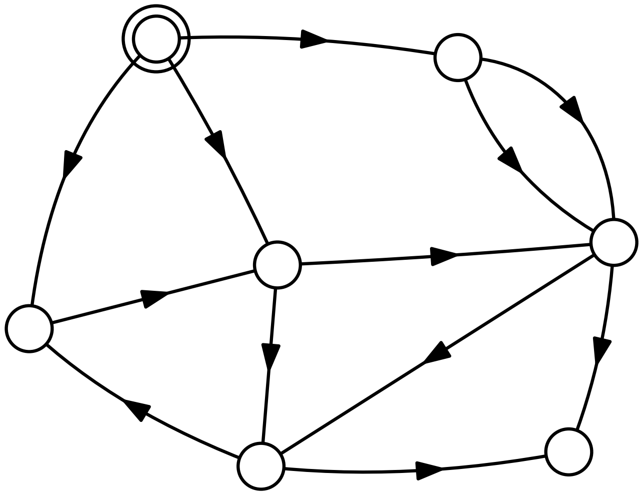
'opposite' Schaeffer rules



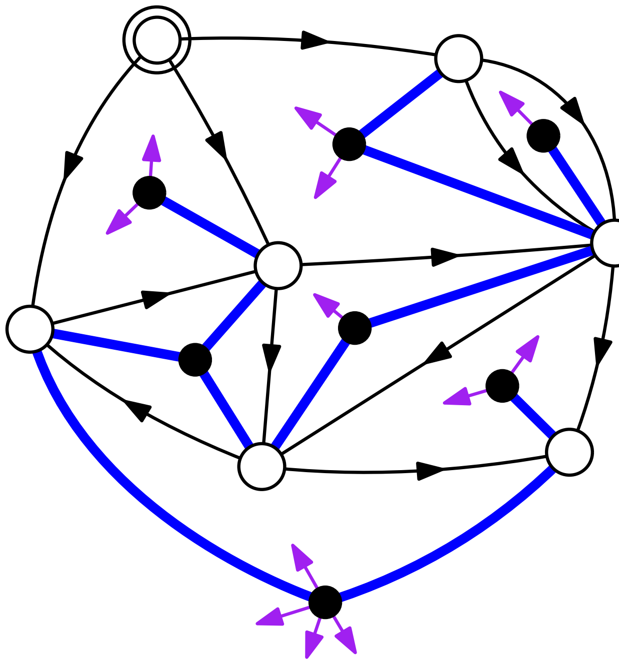
Hence $R_d(t) =$ GF (by edges) of rooted maps + marked vertex v such that v is at edge-distance $\leq d-1$ from root-vertex

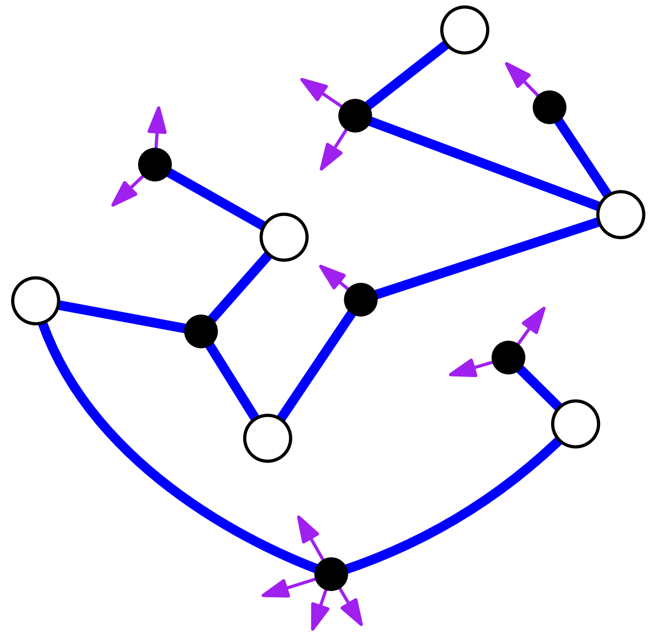
Distances from the meta-bijection Φ ?

Example for a 0-gonal source (pointed vertex v_0)

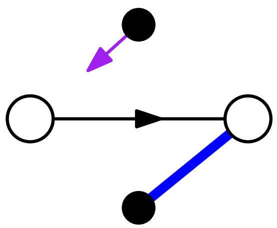


accessible from v_0
no ccw cycle





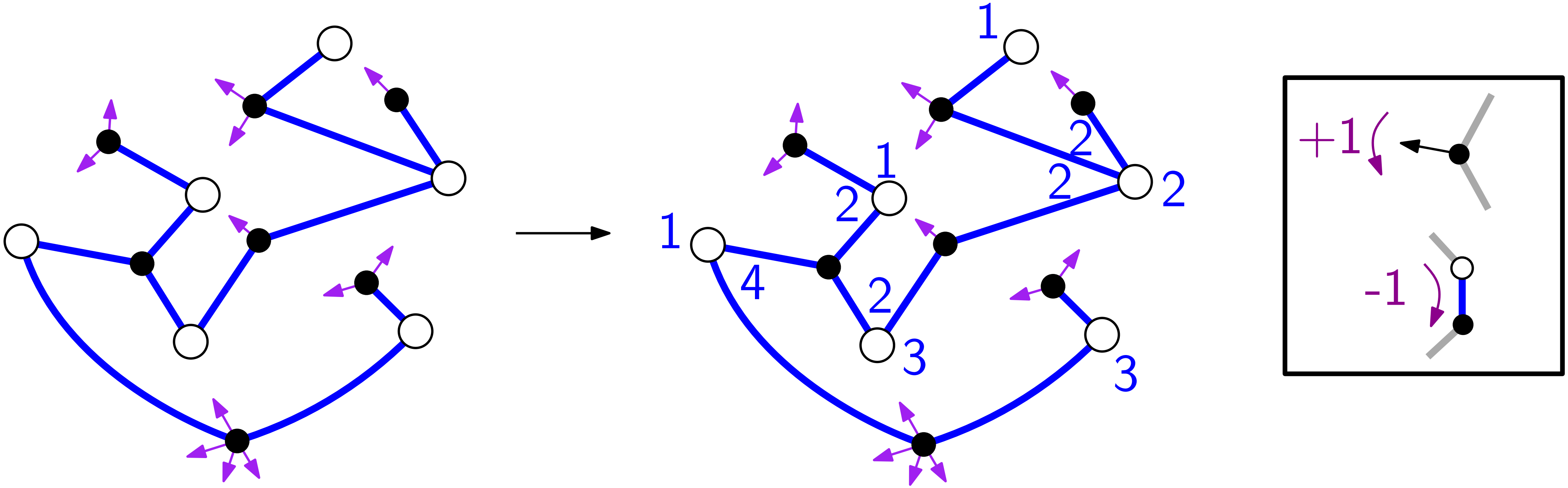
local rule



Distances from the meta-bijection Φ ?

inverse bijection can be done via growing a cactus from the mobile

other way of doing the inverse bijection by labelling the white corners



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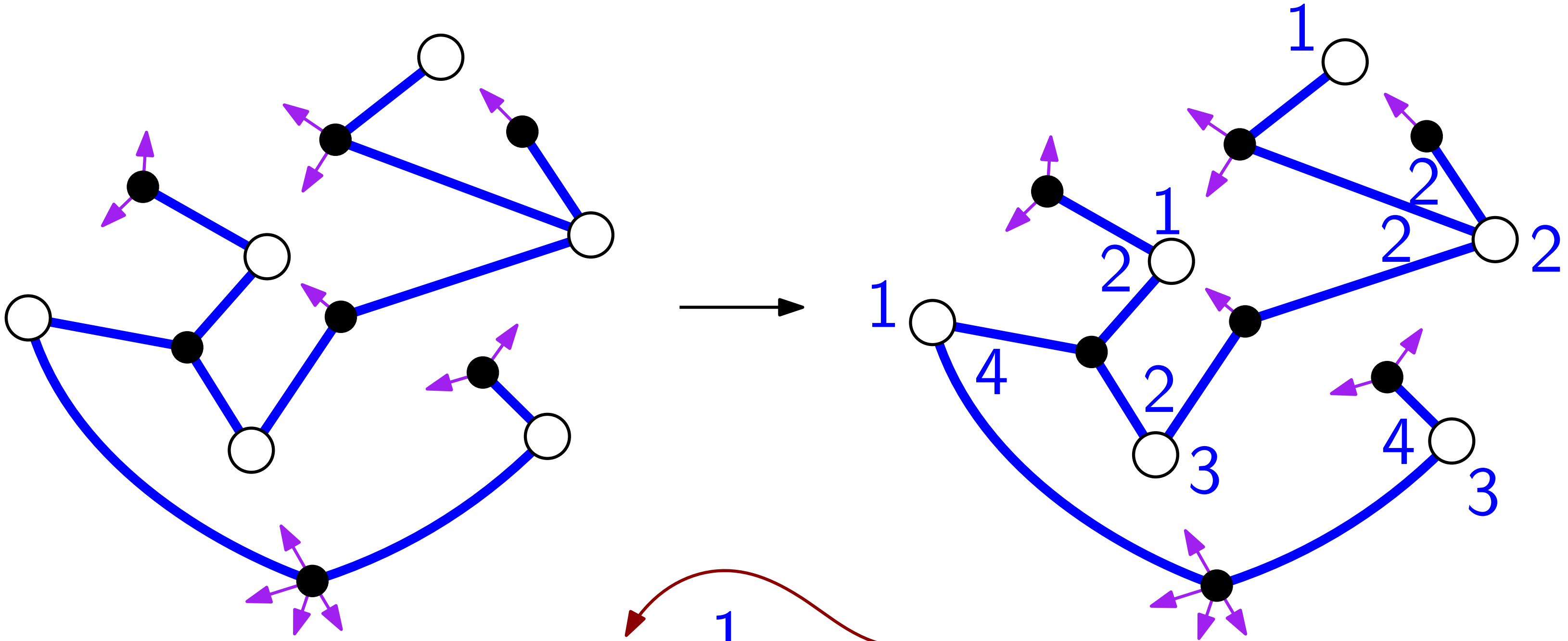

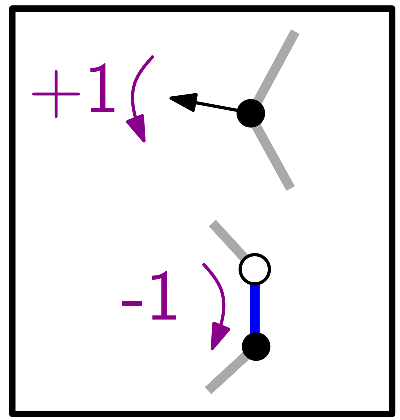

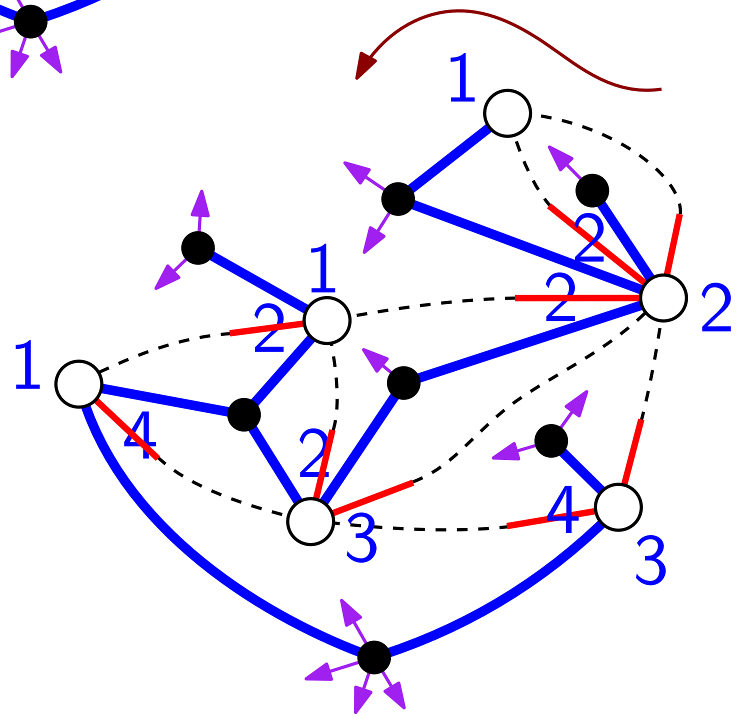

- every white corner of label $i \geq 2$ throws an edge to next corner $i-1$ in a ccw walk around the tree

Distances from the meta-bijection Φ ?

inverse bijection can be done via growing a cactus from the mobile

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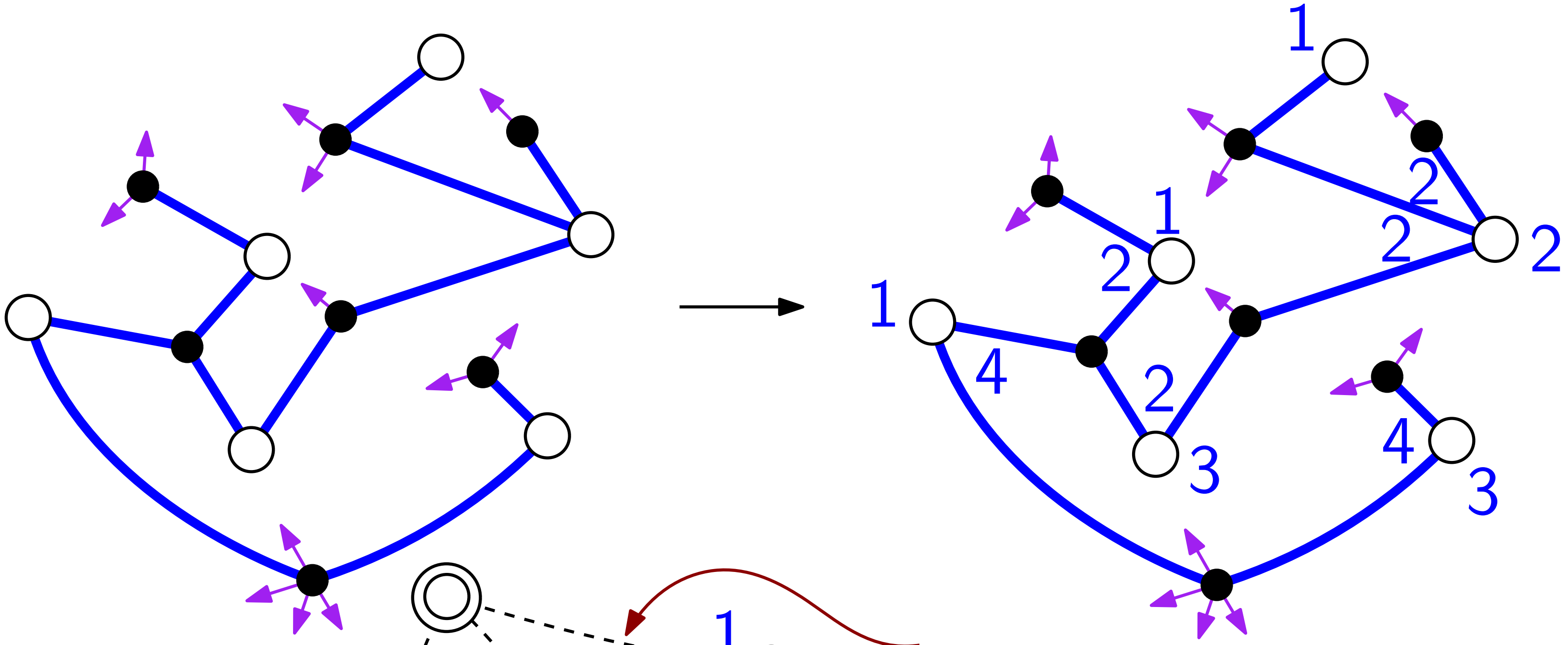
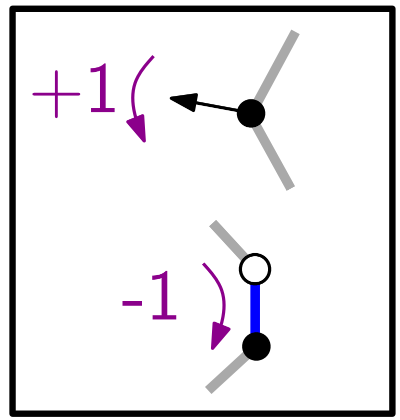
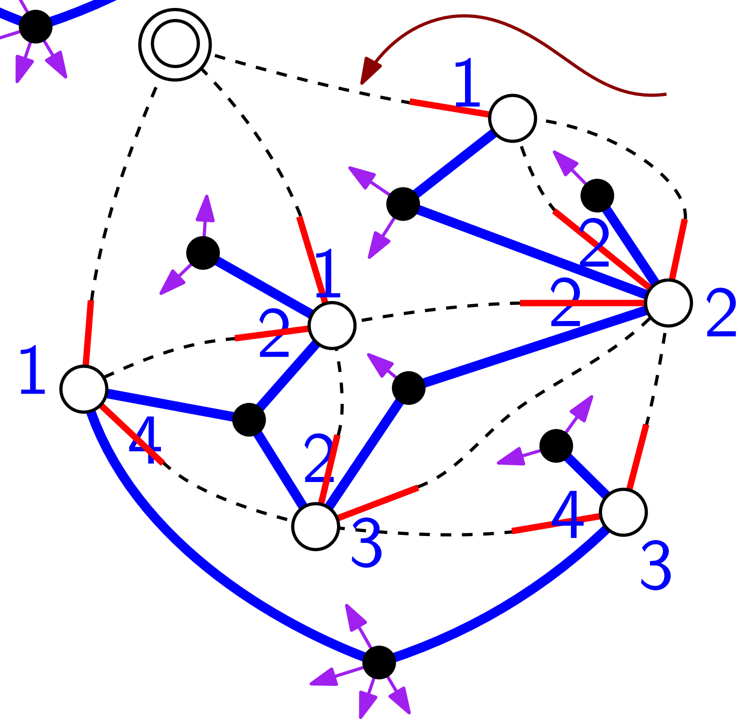


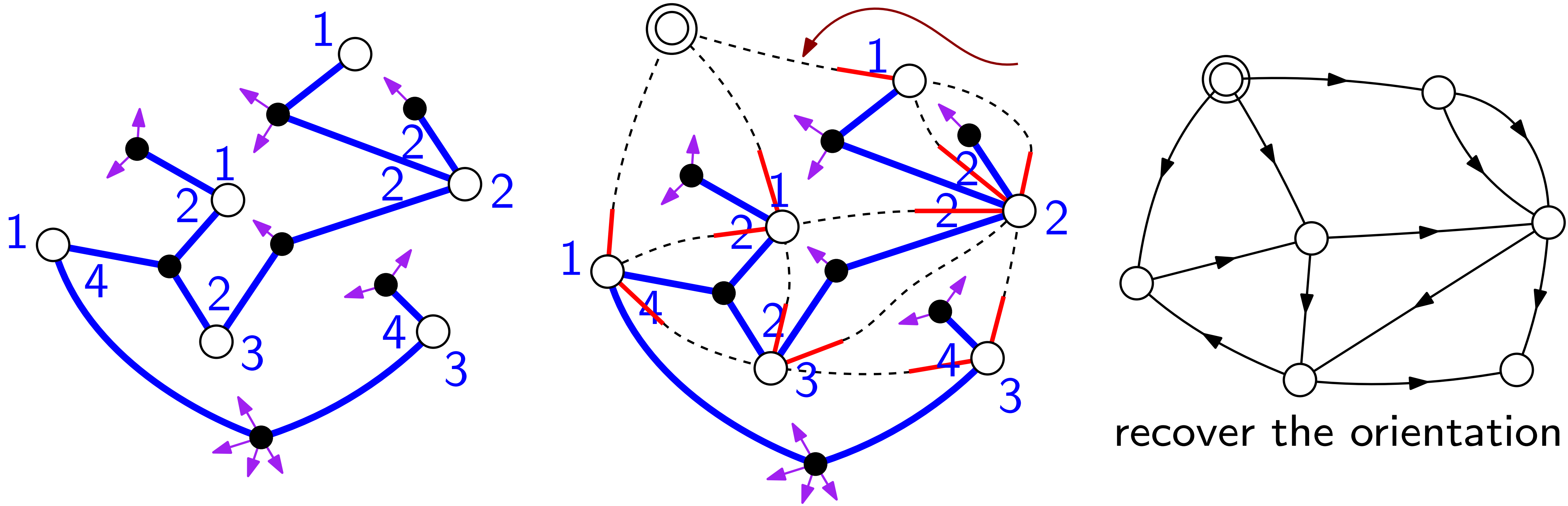




- every white corner of label $i \geq 2$ throws an edge to next corner $i-1$ in a ccw walk around the tree
- then every white corner of label 1 throws an edge to new created vertex

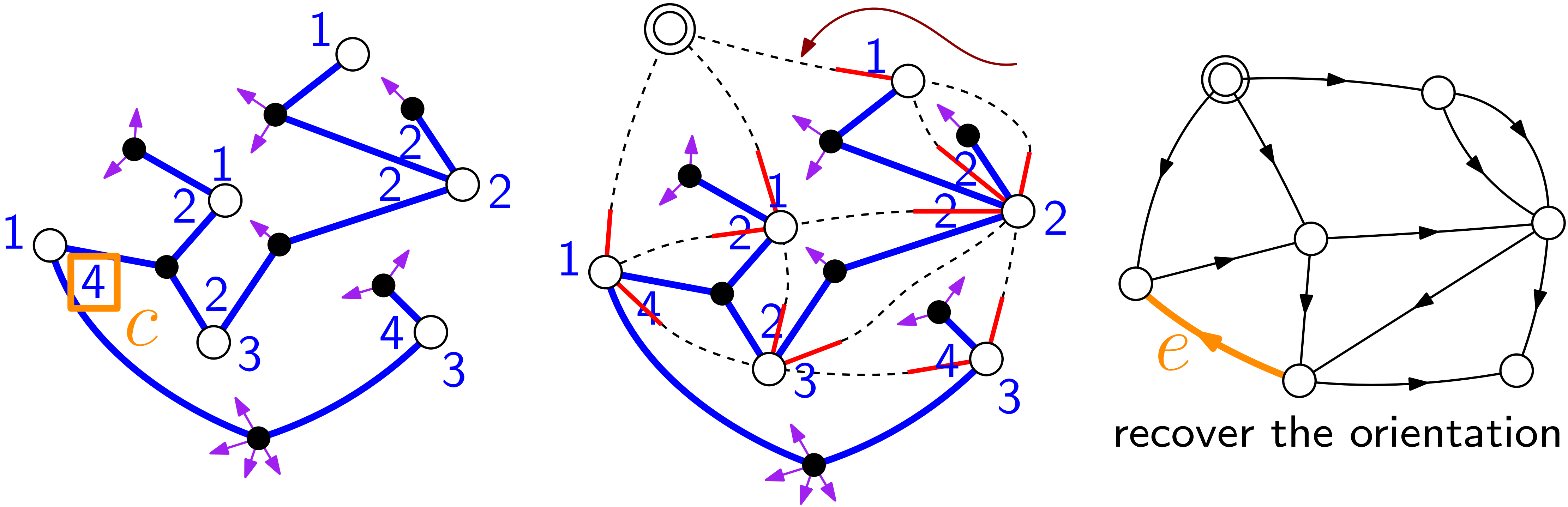
cf Schaeffer's bijection

Distances from the meta-bijection Φ ?



white corner c $\longleftrightarrow$ edge e

Distances from the meta-bijection Φ ?

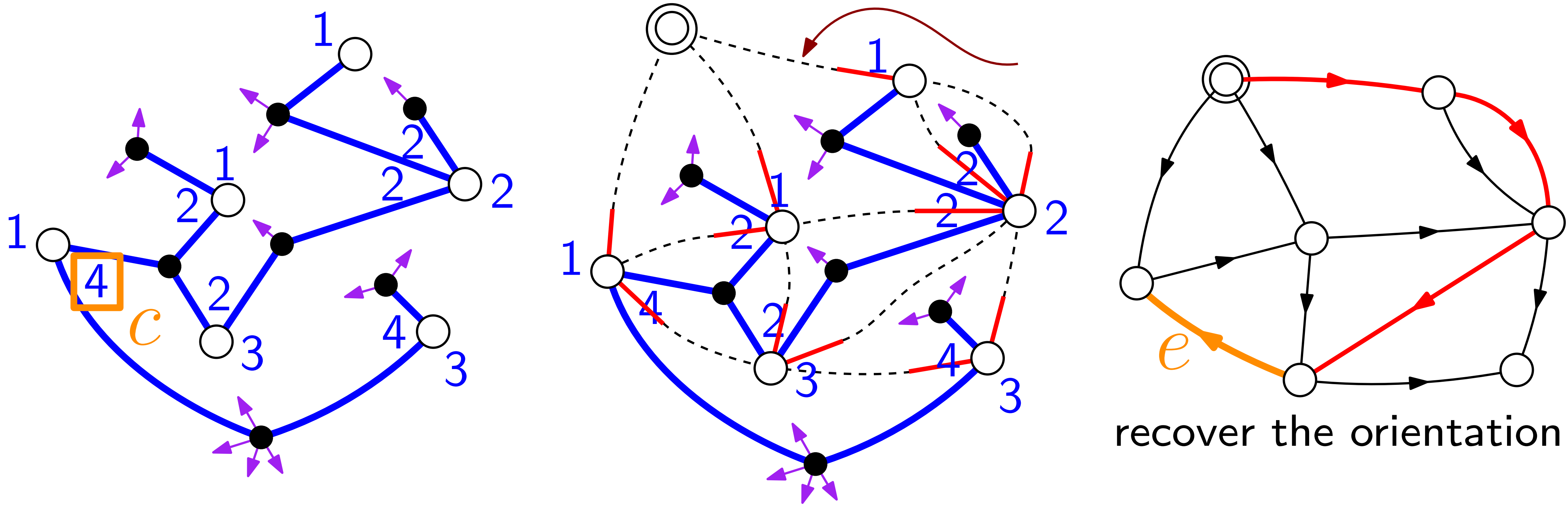


recover the orientation

white corner c ⟷ edge e

label of c ⟷ length of 'rightmost walk $P(e)$ starting from e

Distances from the meta-bijection Φ ?



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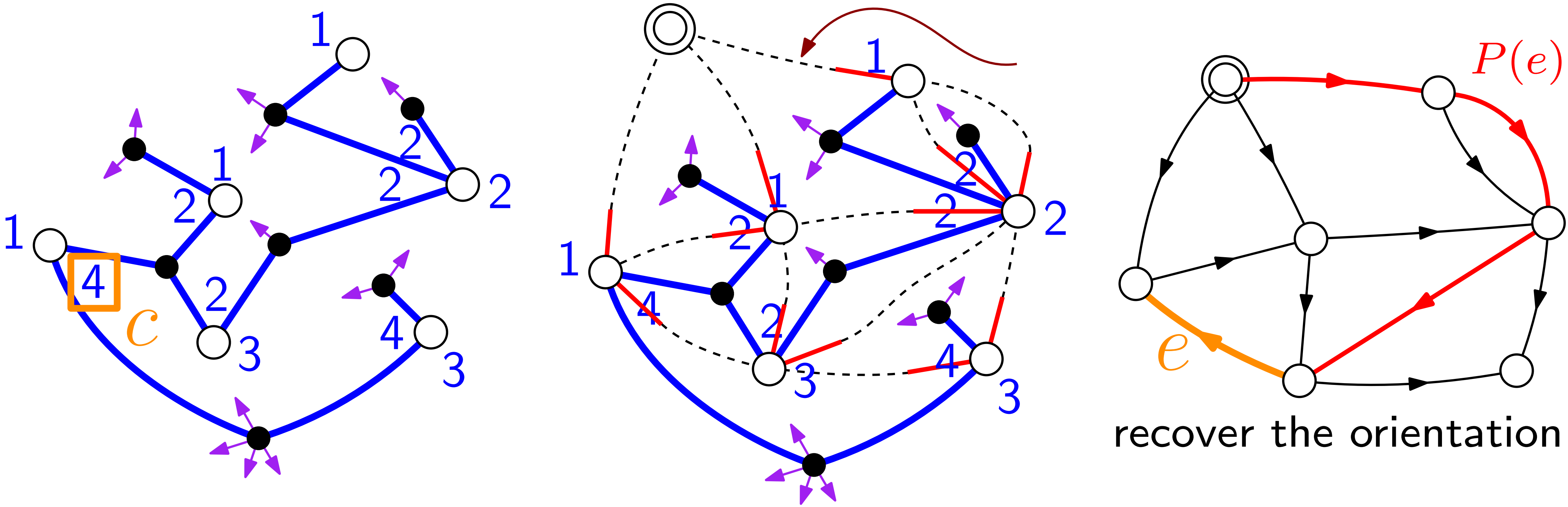

recover the orientation

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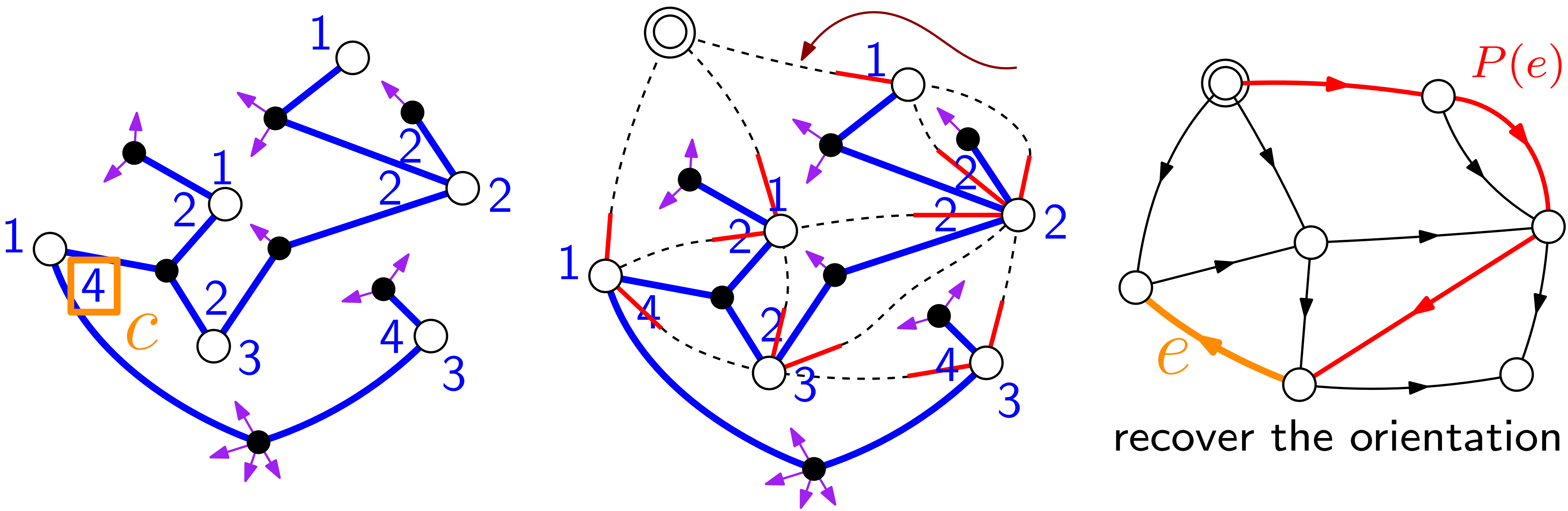
label of c ⟷ length of 'rightmost walk $P(e)$ starting from e

(at least geodesic length $L(e)$)

Distances from the meta-bijection Φ ?



white corner c ⟷ edge e

label of c ⟷ length of 'rightmost walk $P(e)$ starting from e

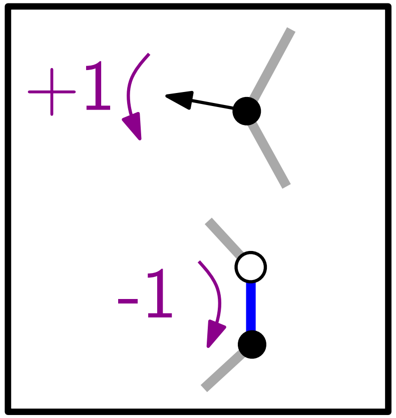
(at least geodesic length $L(e)$)

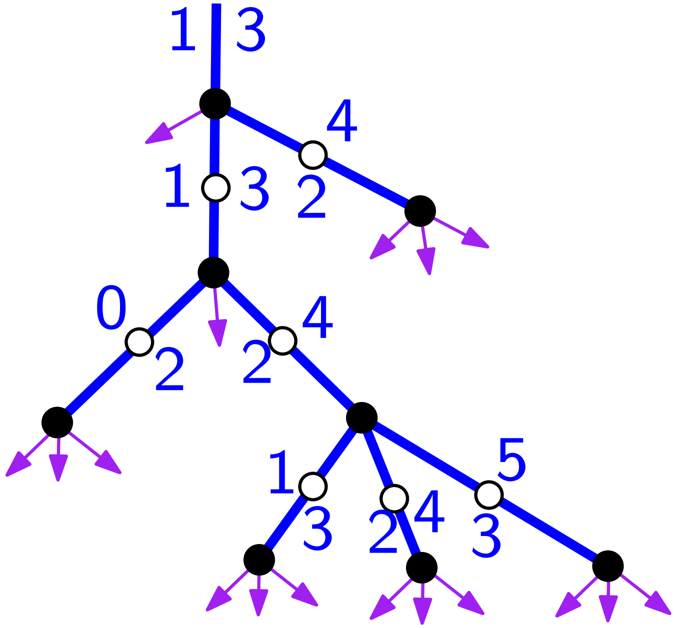
[Addario-Berry&Albenque'13]: for G_n a random simple triangulation (or random simple quadrangulation) on n vertices, and e a random edge of G_n,

$$\text{length}(P(e)) \sim L(e)$$

(in their proof that G_n converges to Brownian map)

'Quasi' 2-point function for simple quadrangulations

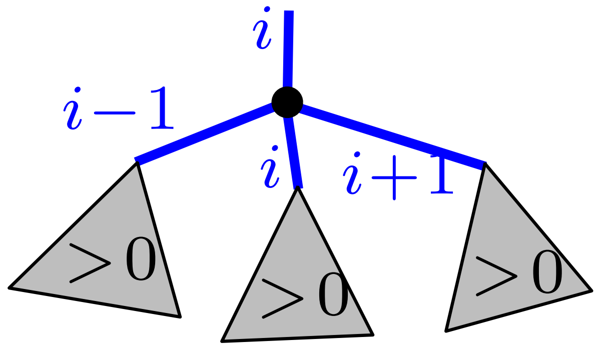




the 2-point function w.r.t. length of rightmost walk is

$$G_i(s) = r_i(s) - r_{i-1}(s)$$

where $r_i(s) = 1 + s \cdot r_{i-1}(s) r_i(s) r_{i+1}(s)$



similar expression for $r_i(s)$ as for $R_i(t)$ (cf [Bouttier,Guitter'10])