Bijective Links on Planar Maps via Orientations

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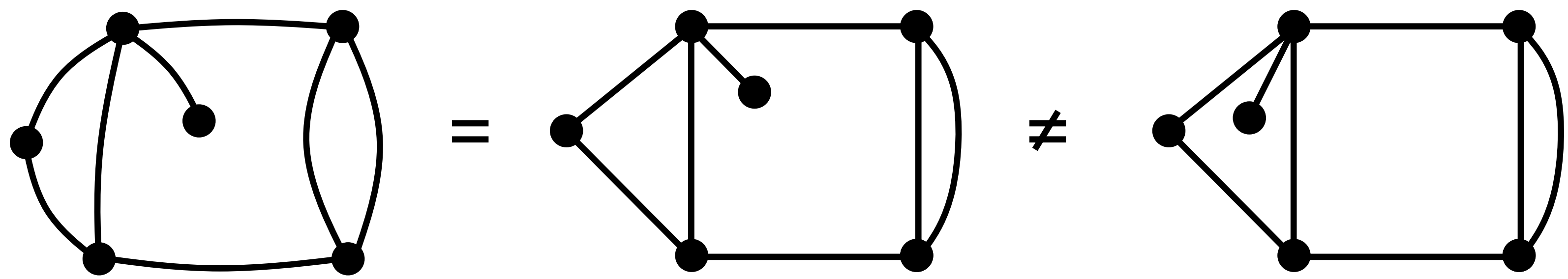
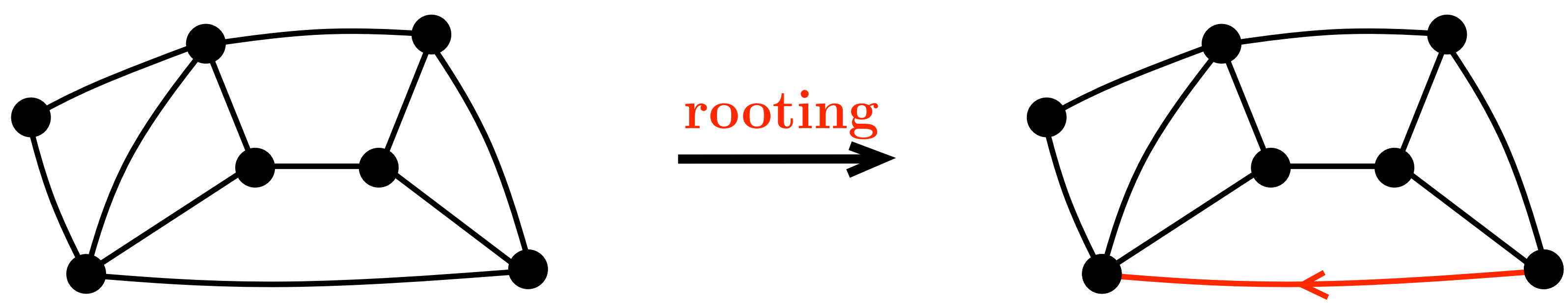
Planar maps

- Planar map = graph drawn in the plane without edge-crossing, taken up to isotopy (continuous structure-preserving transformation)

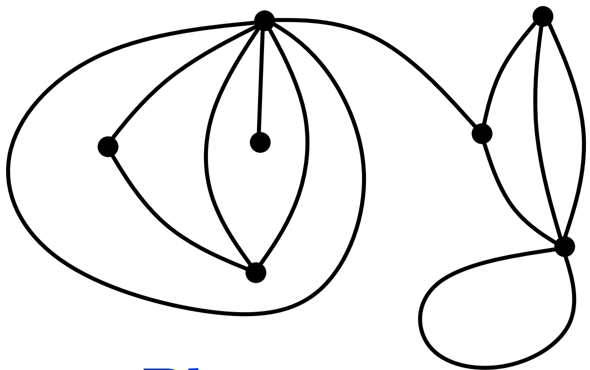


- Rooted map = map + root edge

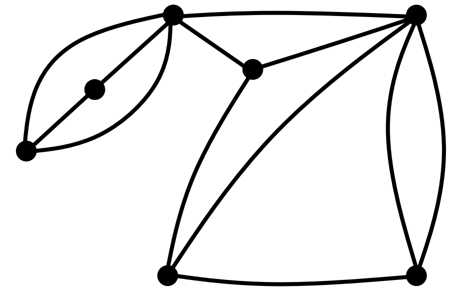


- Motivations: mesh compression, graph drawing + nice combinatorial properties

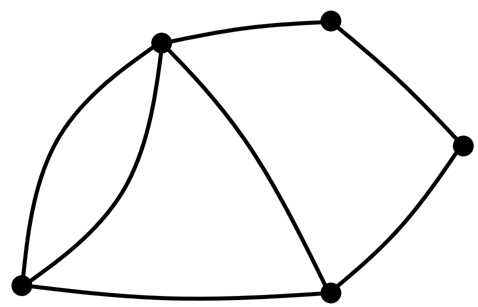
Families of planar maps



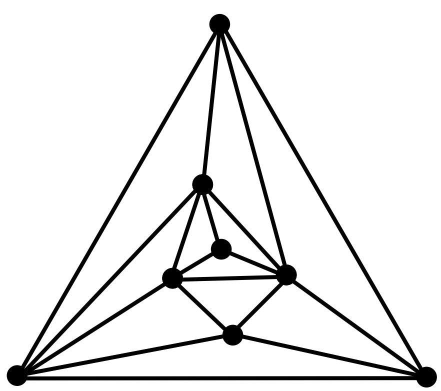
Planar map



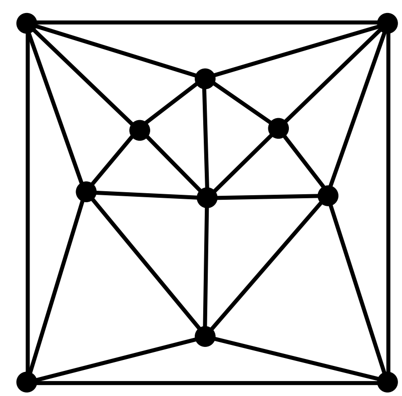
Loopless map



Nonseparable map
(no separating vertex)

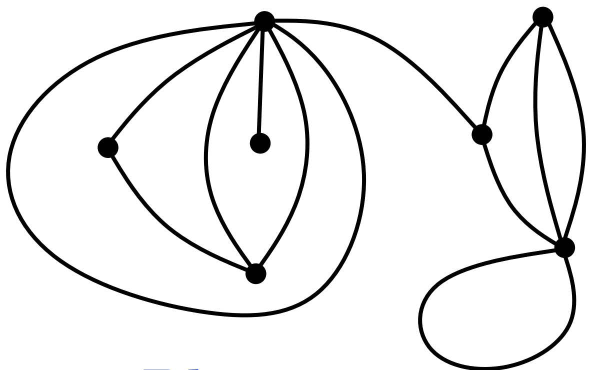


Triangulation



Irreducible triangulation
(no separating triangle)

Families of planar maps



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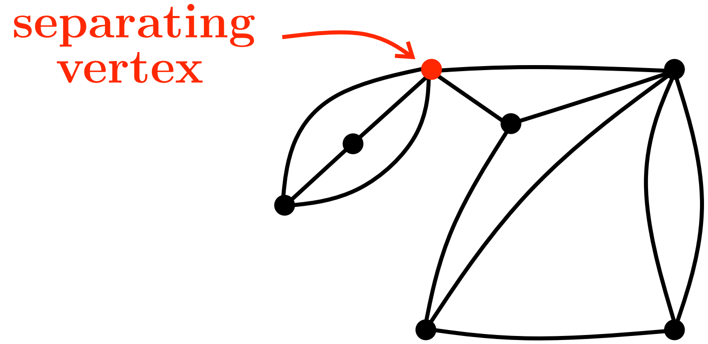
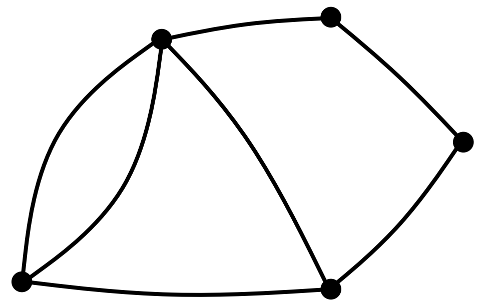


Loopless map



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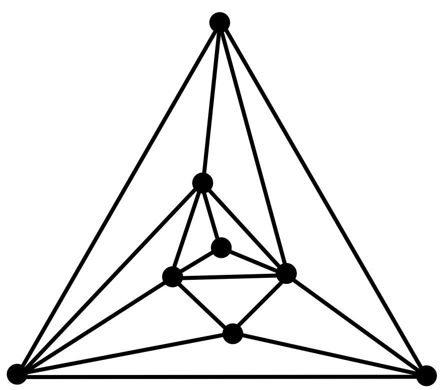
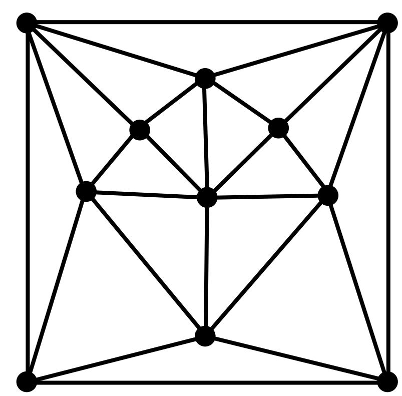


Triangulation



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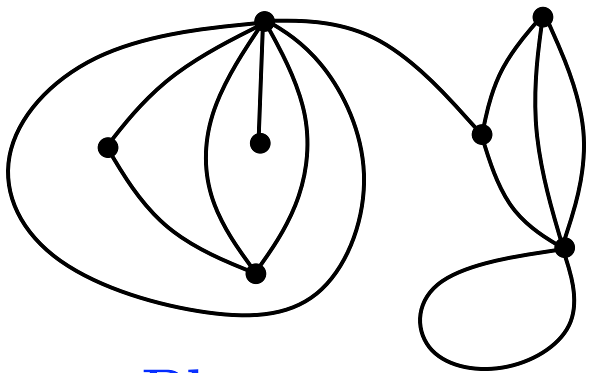
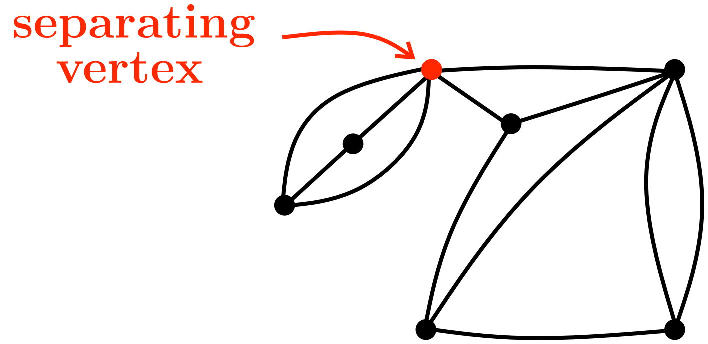
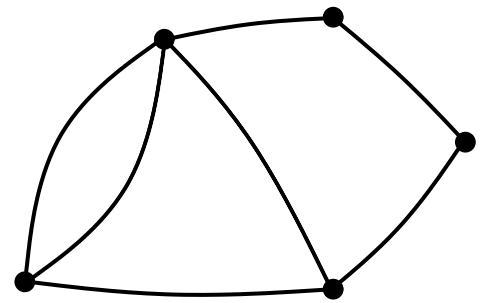


Planar map



Loopless map



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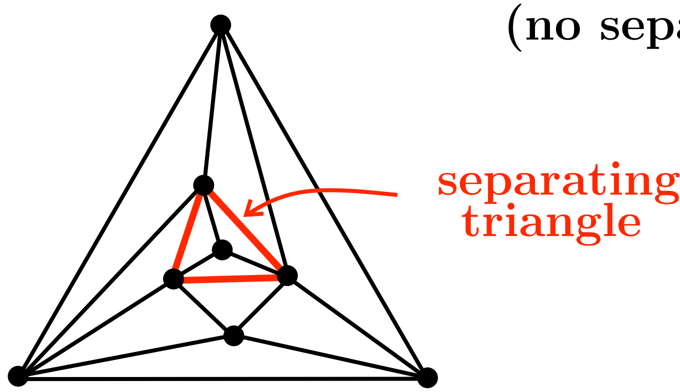

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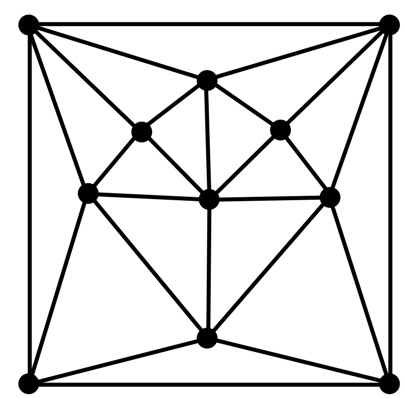

Irreducible triangulation
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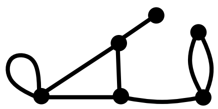
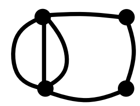
Enumeration of planar maps

- Symbolic approach: Tutte, Brown
- Bijective approach: Cori, Schaeffer, Bouttier-Di Francesco-Guitter



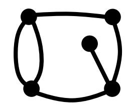
Planar maps

$\boxed{\#(\mathbf{n}\text{ edges}) = \frac{2\cdot 3^n (2n)!}{(n+2)!n!}}$



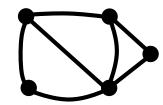
Eulerian

$\boxed{\#(\mathbf{n}\text{ edges}) = \frac{3\cdot 2^{n-1} (2n)!}{(n+2)!n!}}$



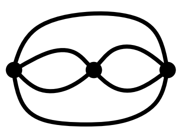
Loopless

$\boxed{\#(\mathbf{n}\text{ edges}) = \frac{2(4n+1)!}{(n+1)!(3n+2)!}}$



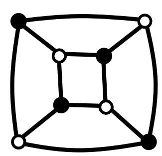
Nonseparable

$\boxed{\#(\mathbf{n}\text{ edges}) = \frac{4(3n-3)!}{(n-1)!(2n)!}}$



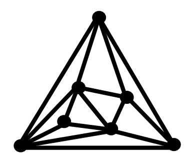
4-regular

$\boxed{\#(\mathbf{n}\text{ vert.}) = \frac{2\cdot 3^n (2n)!}{(n+2)!n!}}$



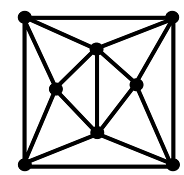
Bicubic

$\boxed{\#(\mathbf{2n}\text{ vert.}) = \frac{3\cdot 2^{n-1} (2n)!}{(n+2)!n!}}$



Triangulations

$\boxed{\#(\mathbf{n+3}\text{ vert.}) = \frac{2(4n+1)!}{(n+1)!(3n+2)!}}$



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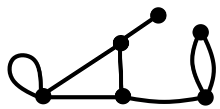
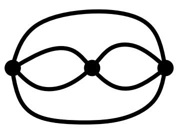


Planar maps

$\#(\mathbf{n}\text{ edges}) = \frac{2\cdot 3^{n}(2n)!}{(n+2)!n!}$

1



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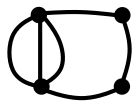

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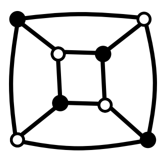
2



Bicubic

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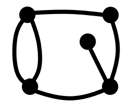

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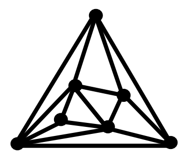
3



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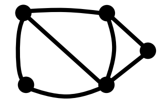

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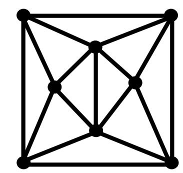
4



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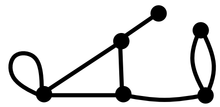

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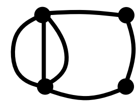

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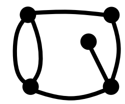

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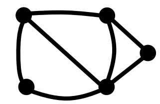

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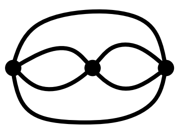
1 2 3 4



4-regular

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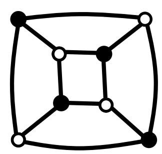

Bicubic

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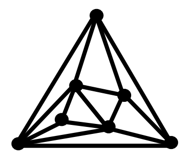

Triangulations

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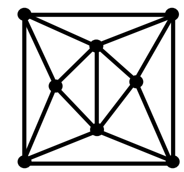


Irreducible

$\#(\mathbf{n+3} \text{ vert.}) = \frac{4(3n-3)!}{(n-1)!(2n)!}$

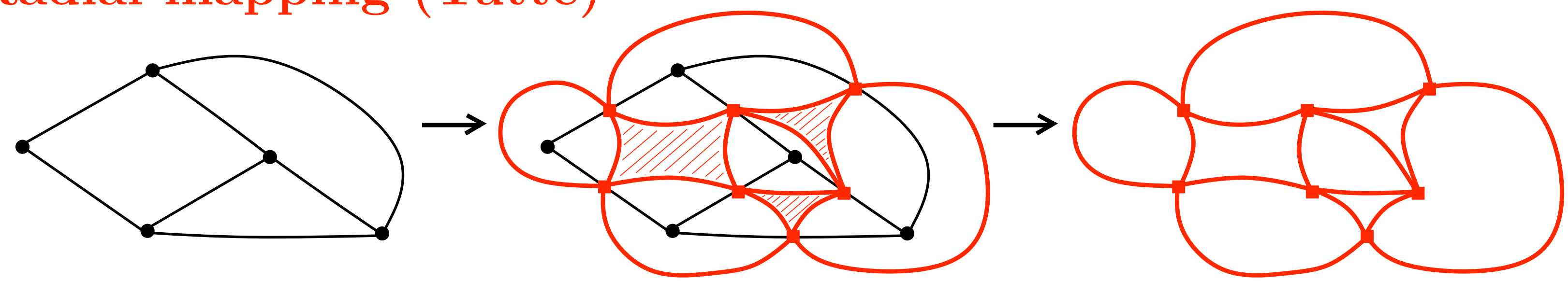
1 , 2 well known bijections (Tutte)

3 recursive bijection (Wormald)

This talk:
- new bijective construction for 3
- first bijective construction for 4

Well known bijections

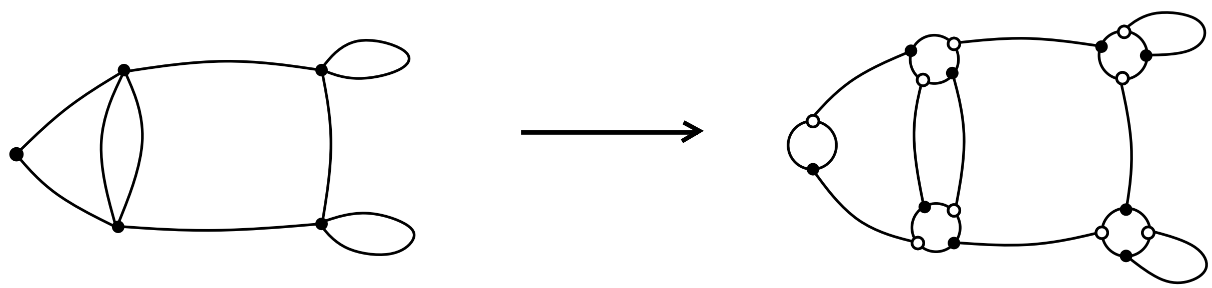
① Radial mapping (Tutte)



Planar maps n edges

4-regular n vertices

② Trinity mapping (Tutte)

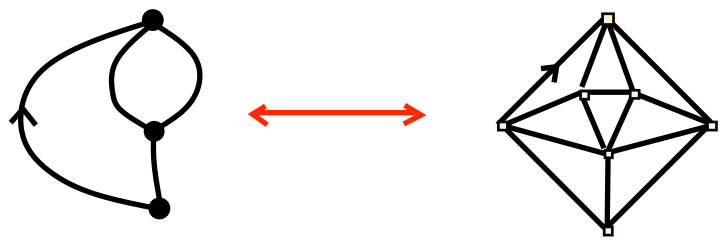


Eulerian maps n edges

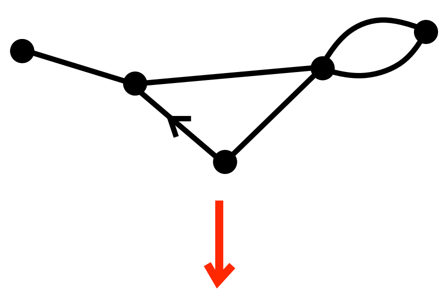
Bicubic maps $2n$ vertices

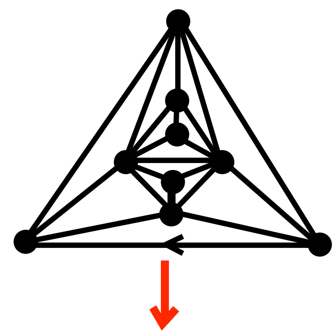
Overview of the talk

1) Bijection nonseparable maps ≃ irreducible triang
+ new duality relation for bipolar orientations

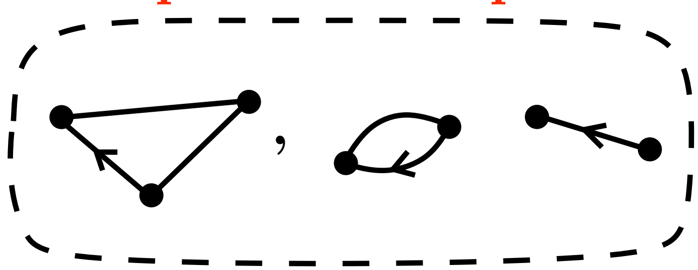


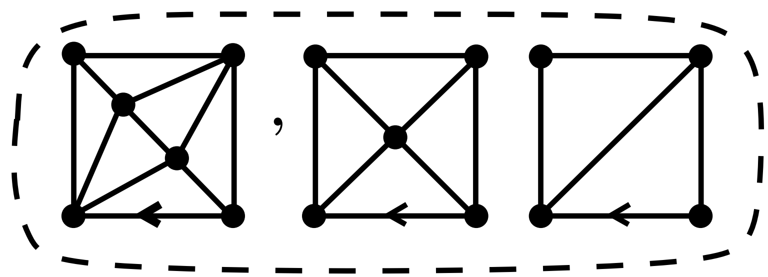
2) Bijection loopless maps ≃ triangulations





nonseparable components **irreducible components**



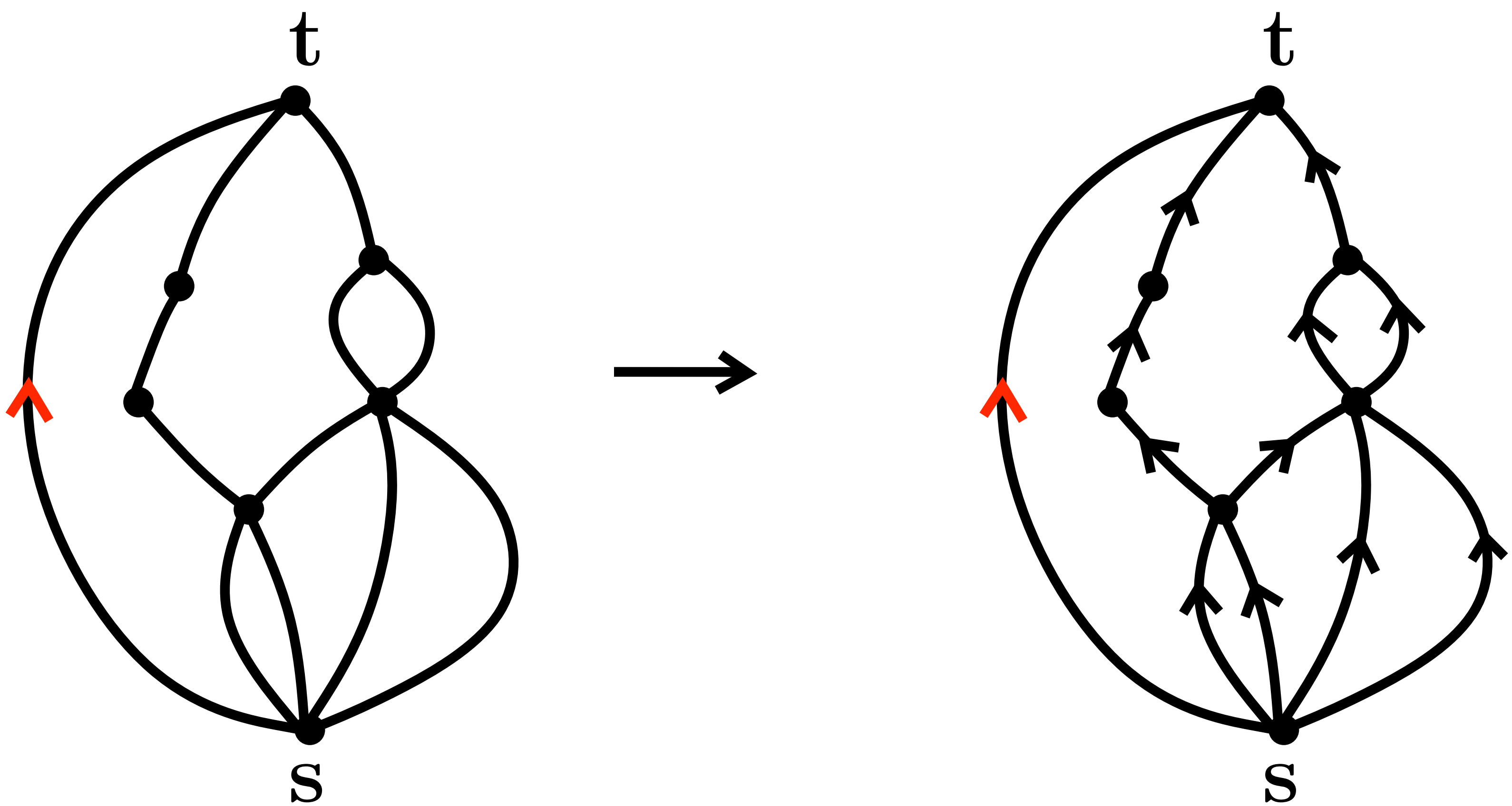


3) Applications to random generation and encoding

Bijection between nonseparable maps and irreducible triangulations

Bipolar orientations

Bipolar orientation = **acyclic orientation** with a unique source and a unique sink



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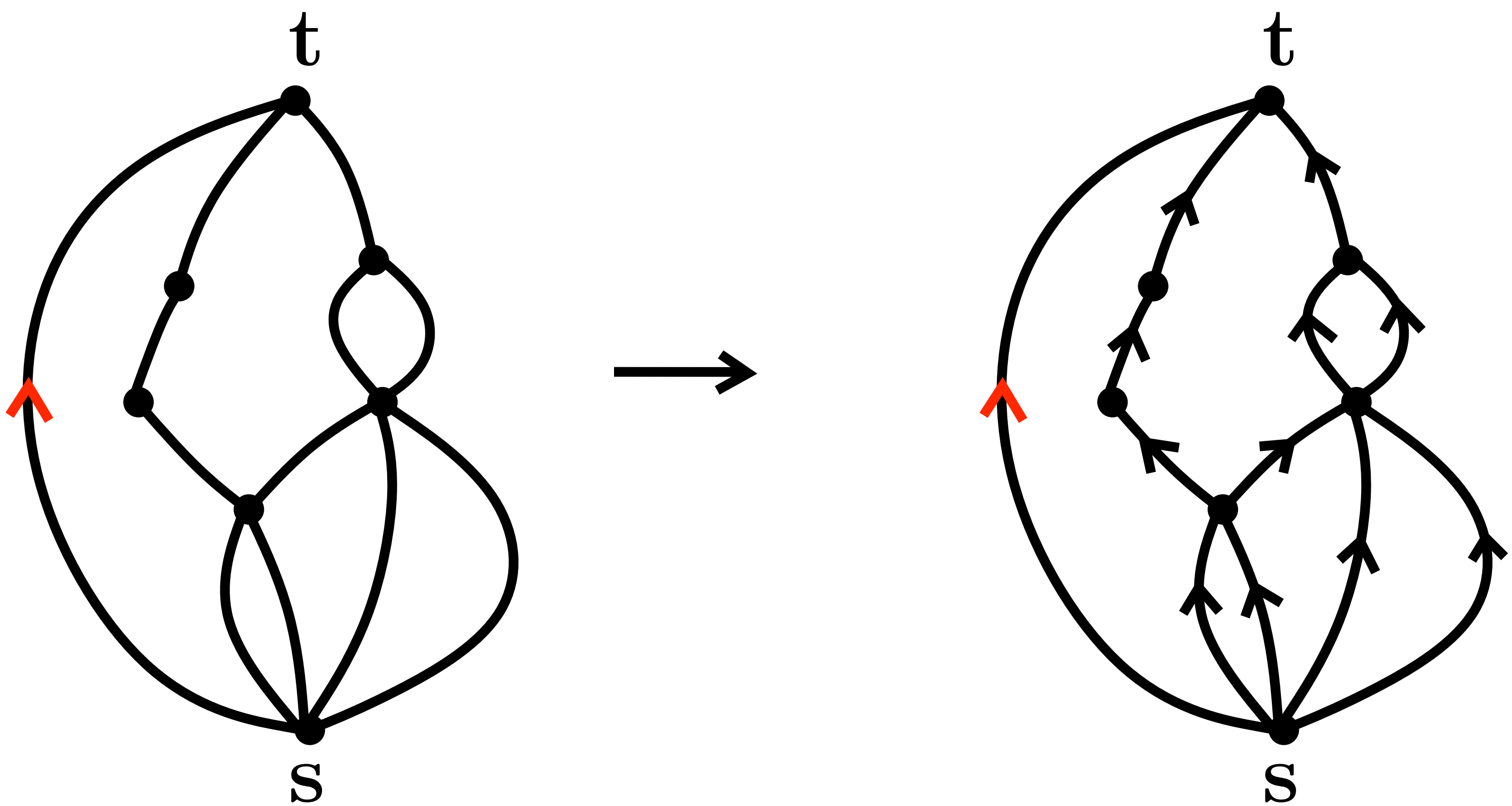
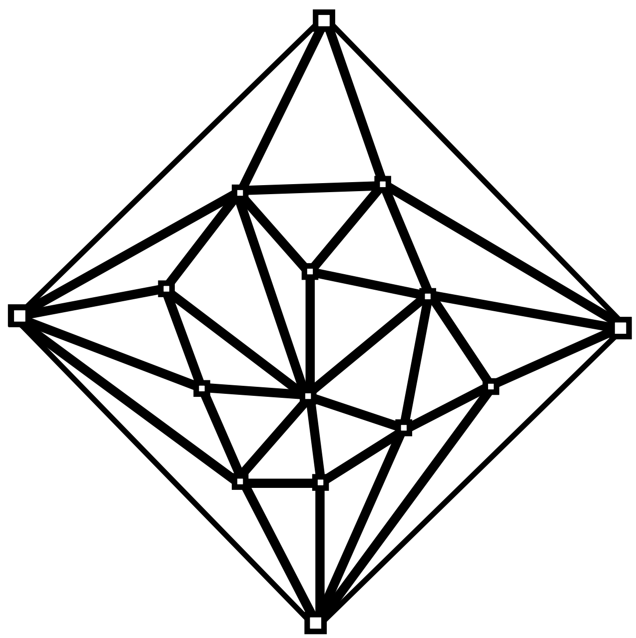
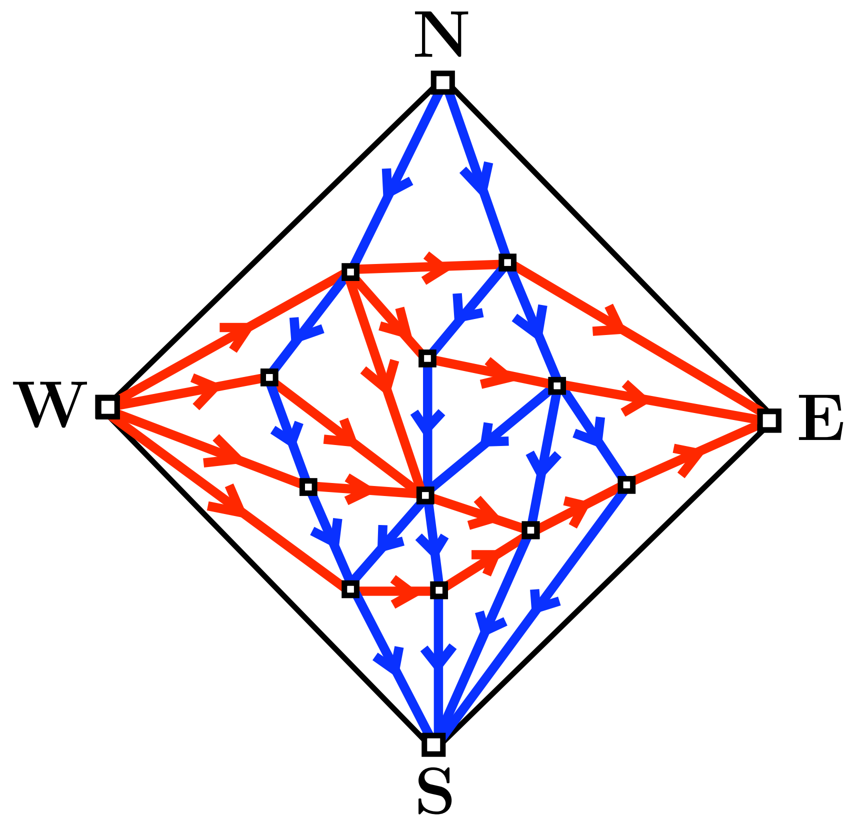


A map admits a bipolar orientation iff there is no separating vertex (nonseparable)

Transversal structures

Transversal structure = partition of inner edges into a red and a blue bipolar orientations that are transversal (introduced by Xin He'93)





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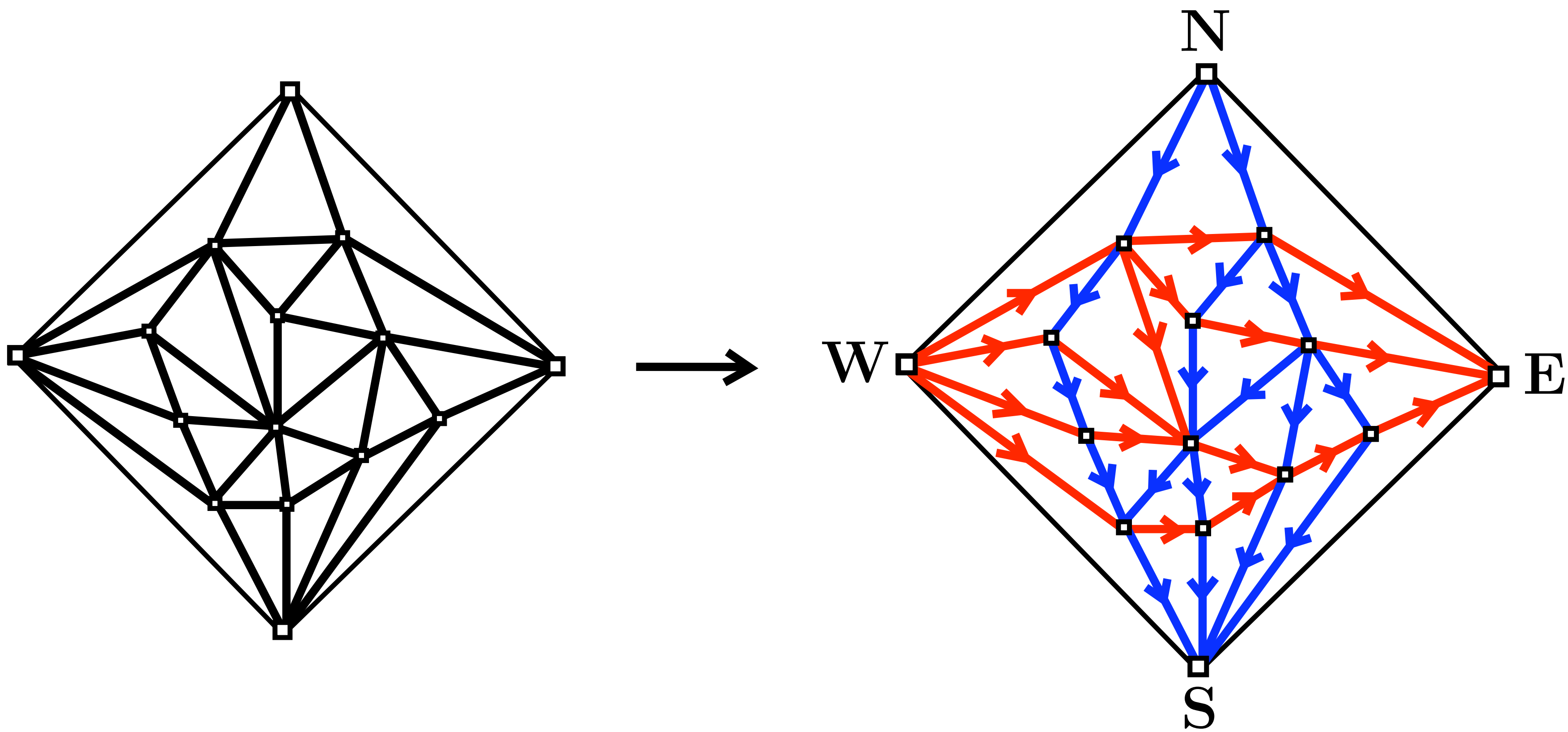
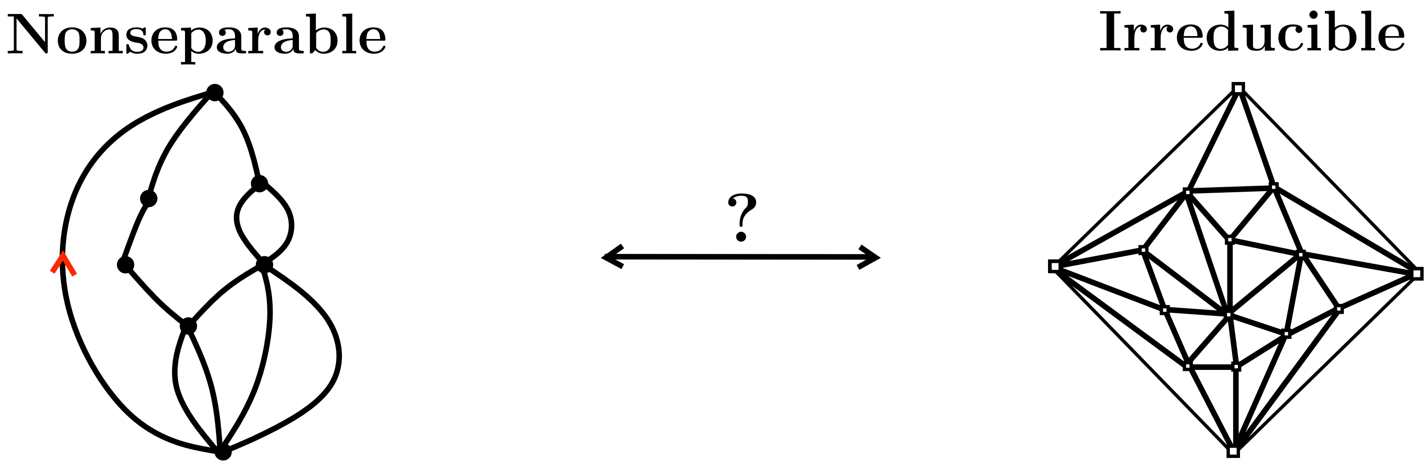
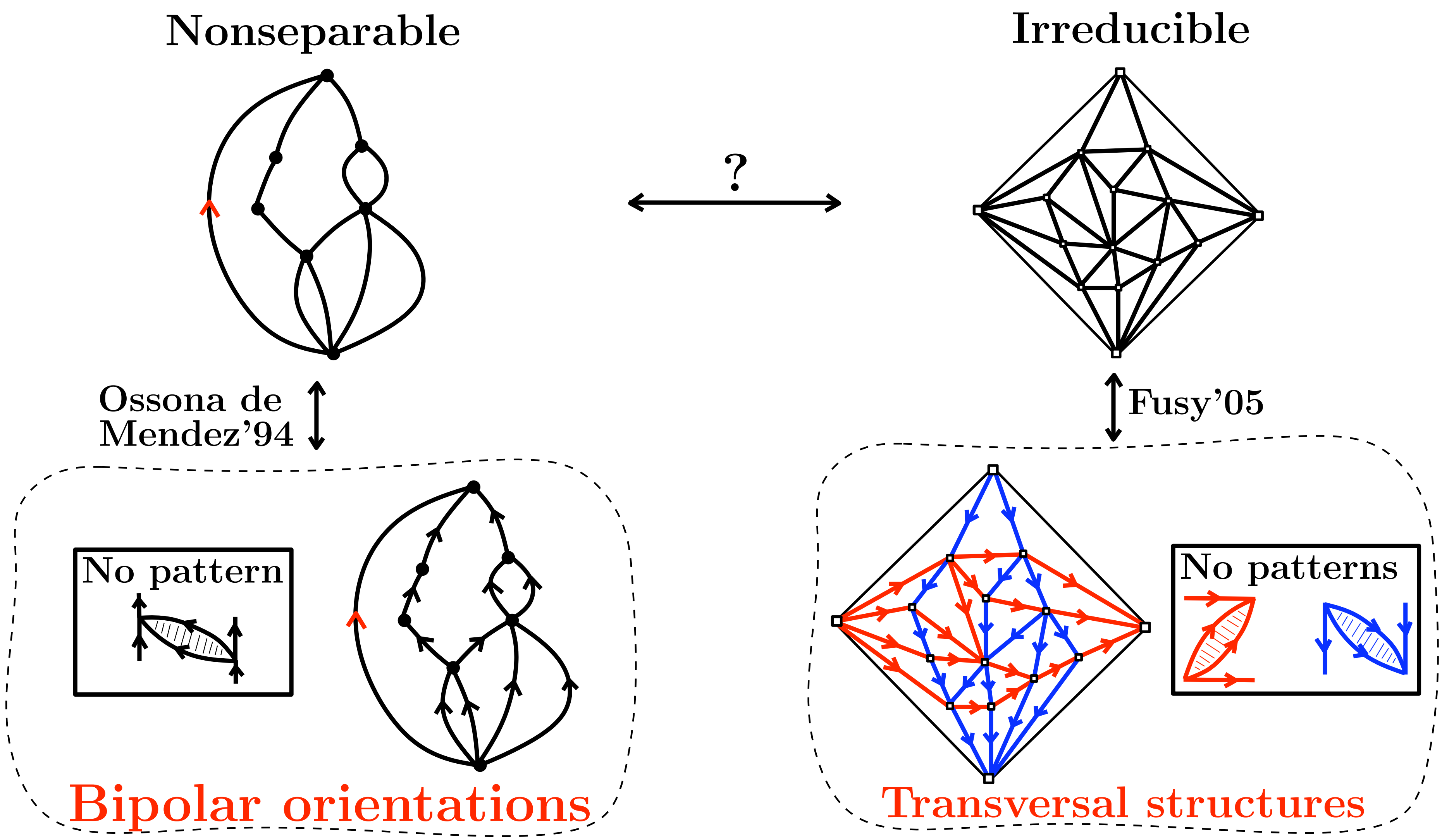


A triangulation of the 4-gon admits a transversal structure iff there is no separating triangle (irreducible)

Reformulating the bijection



Reformulating the bijection



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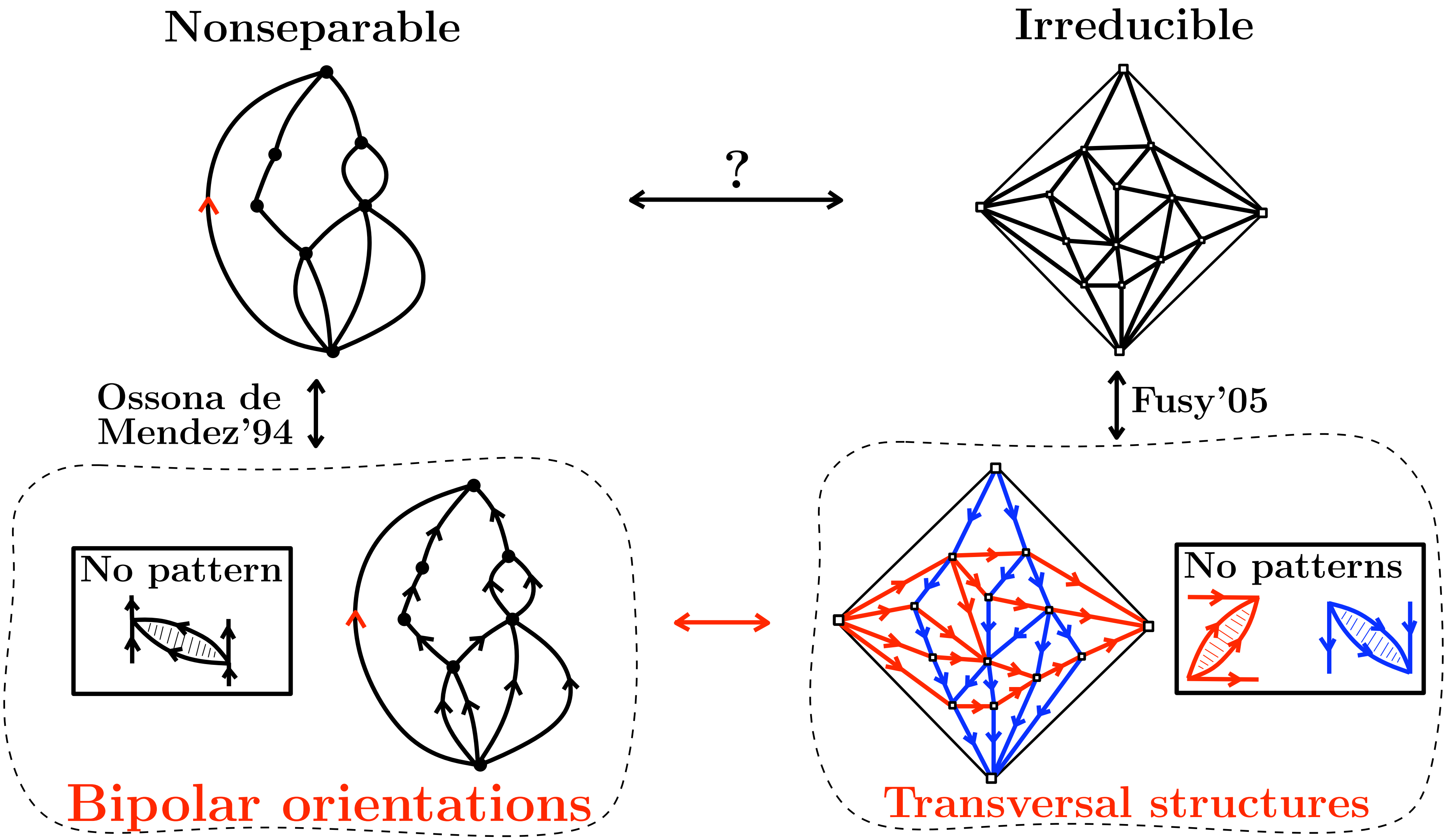
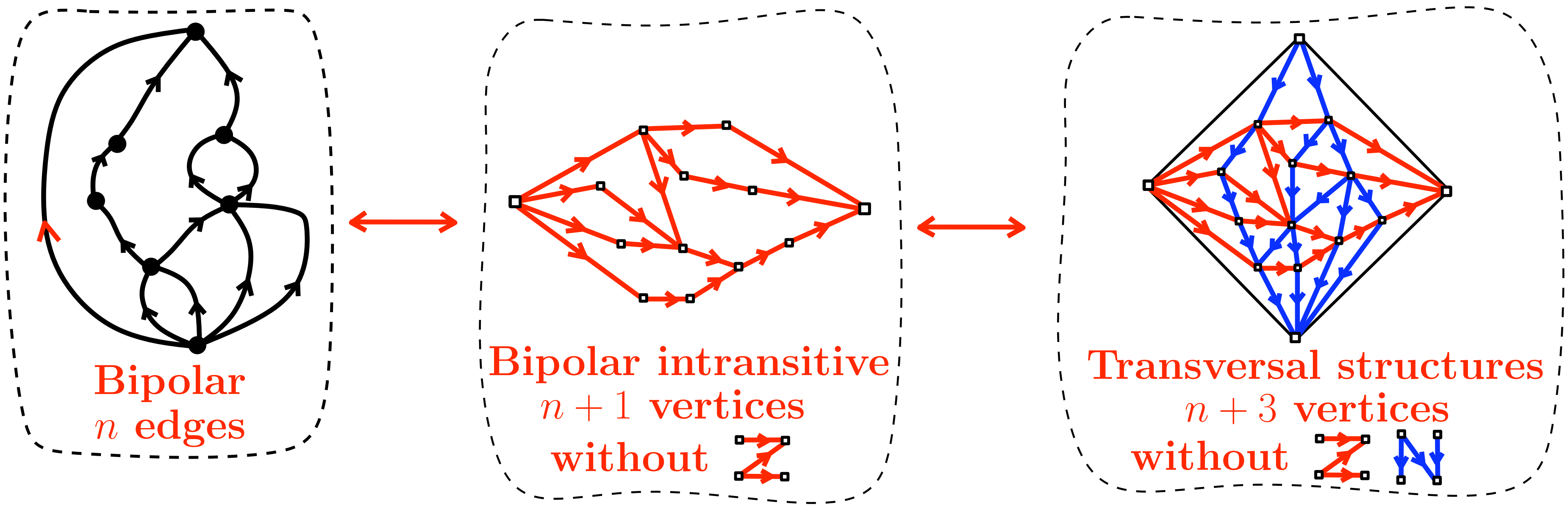


How the bijection works



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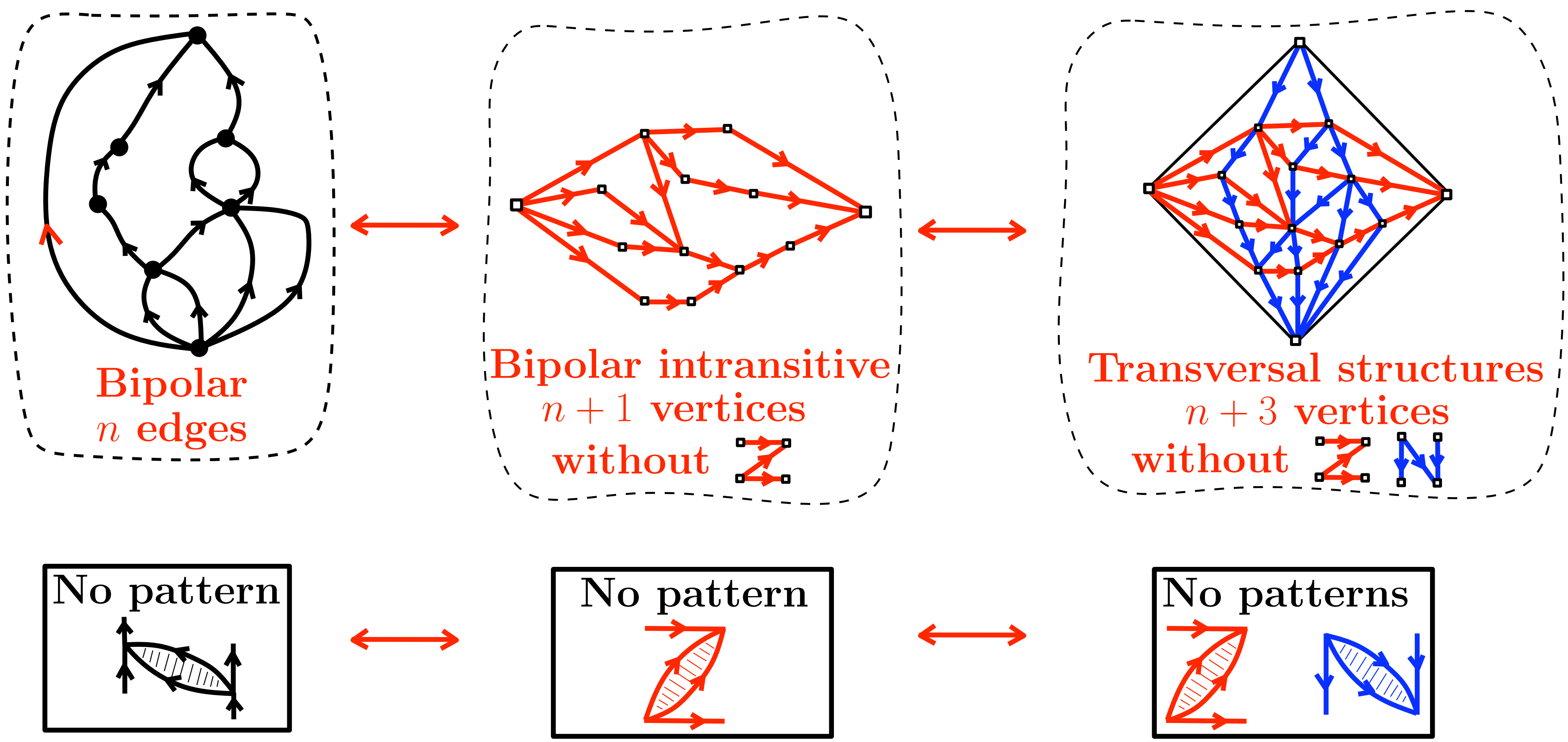
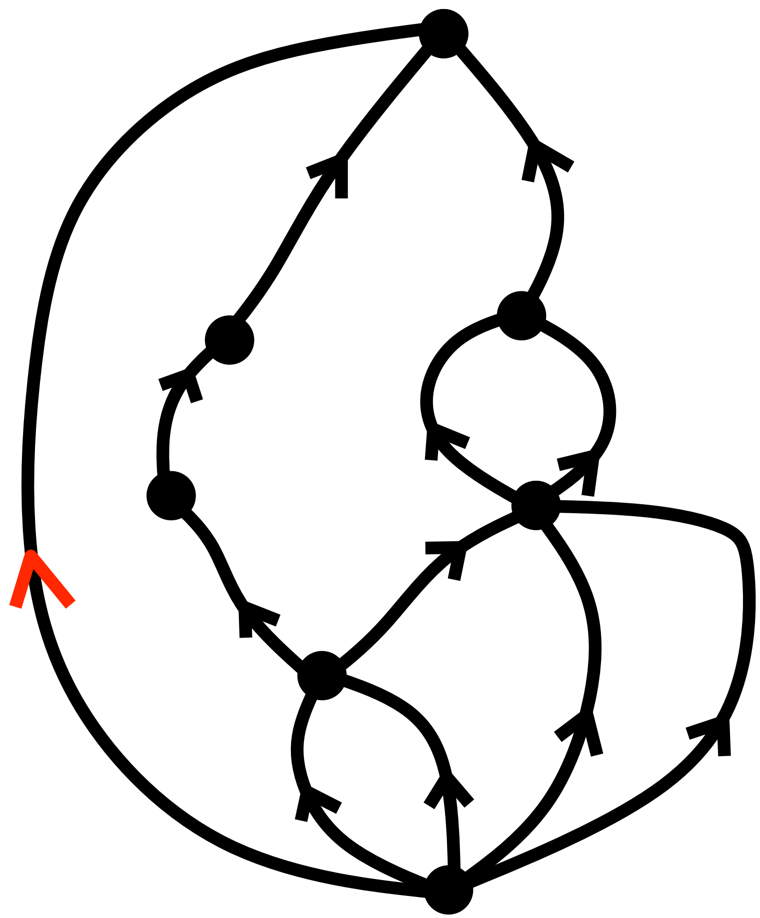


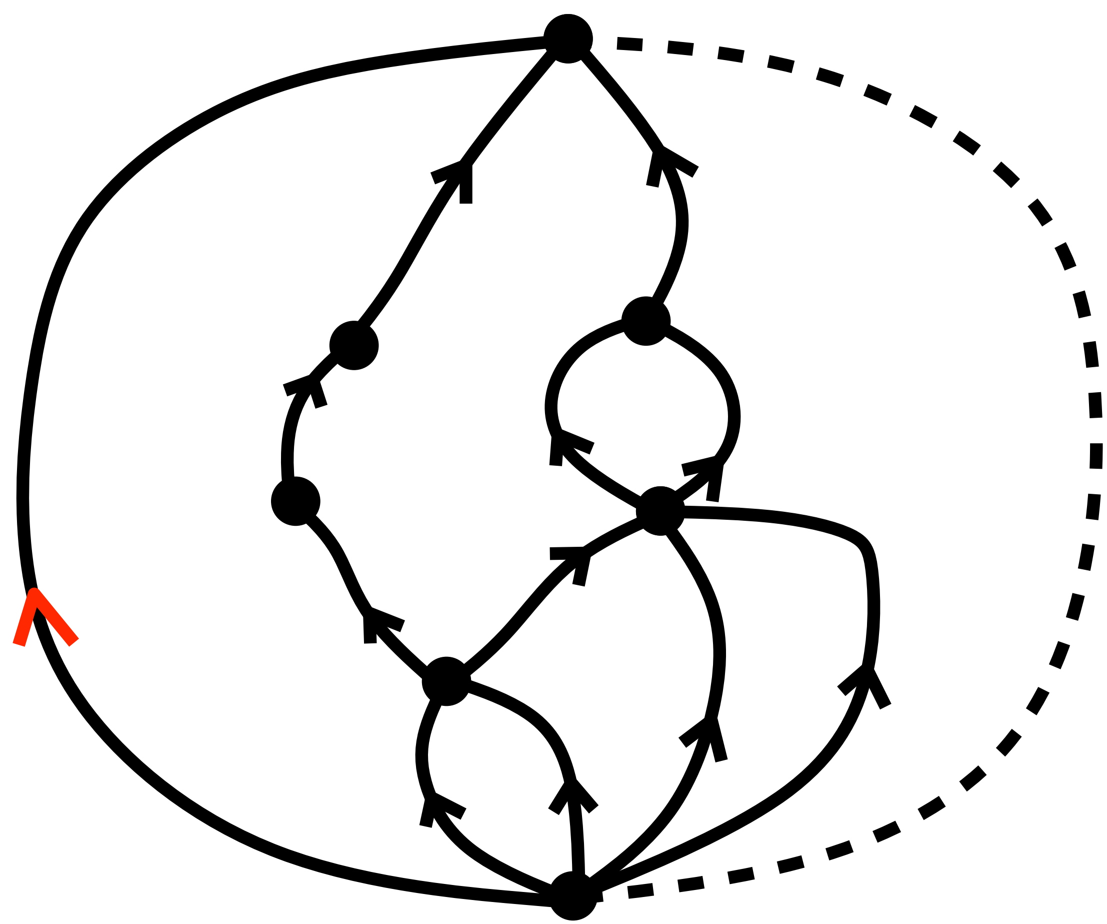
Bipolar→Bipolar intransitive

Start with a plane bipolar orientation



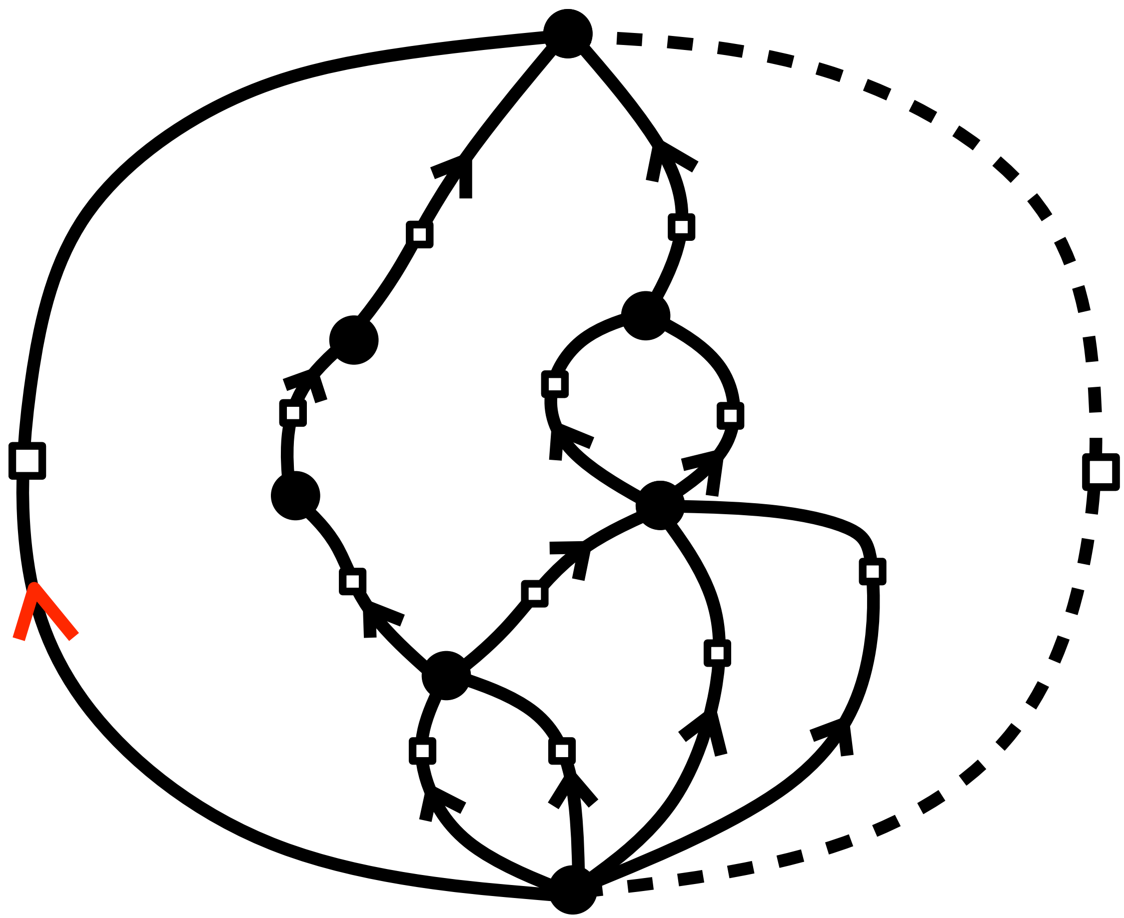
Bipolar→Bipolar intransitive

Double the root edge



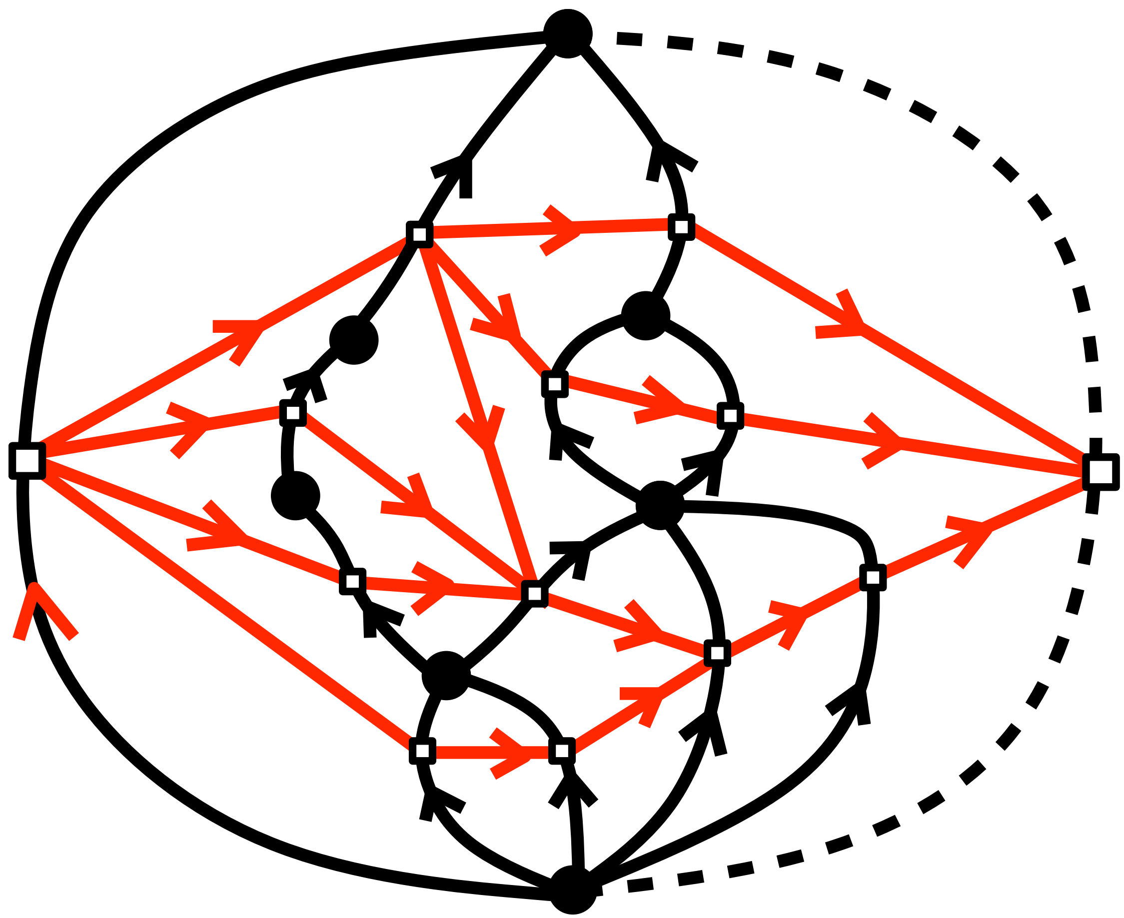
Bipolar→Bipolar intransitive

Insert a white vertex in each edge

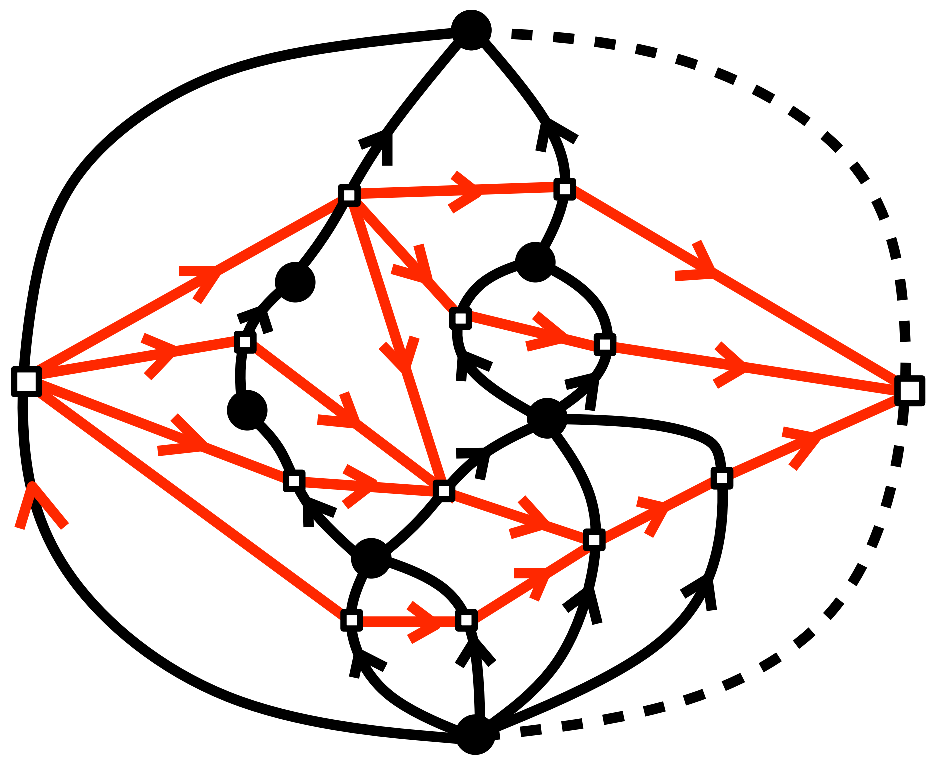


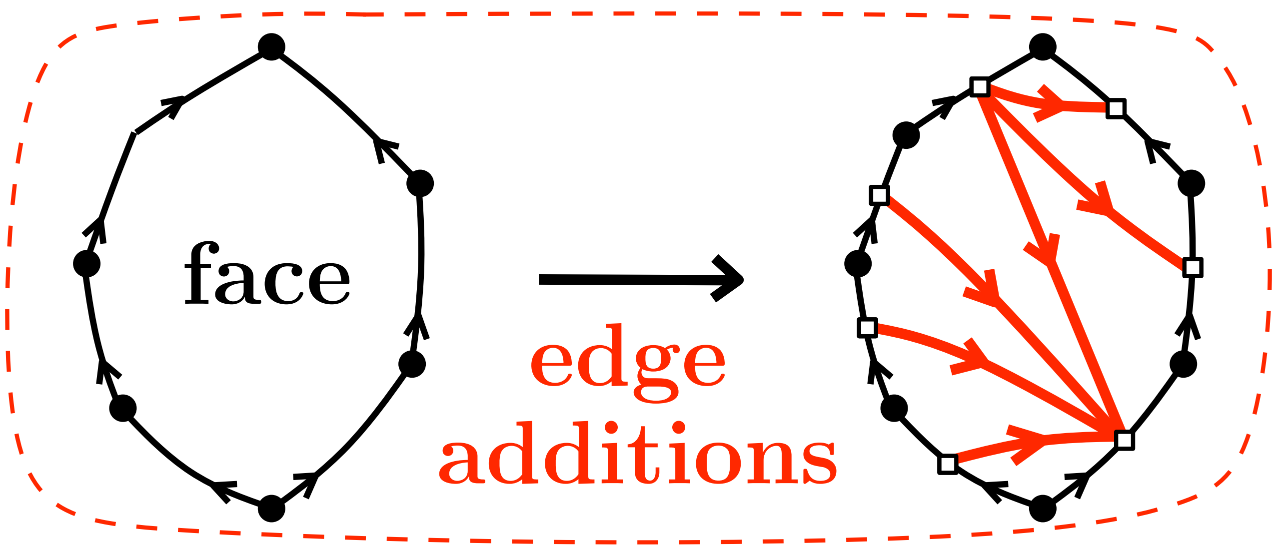
Bipolar→Bipolar intransitive

Triangulate the faces by red edges

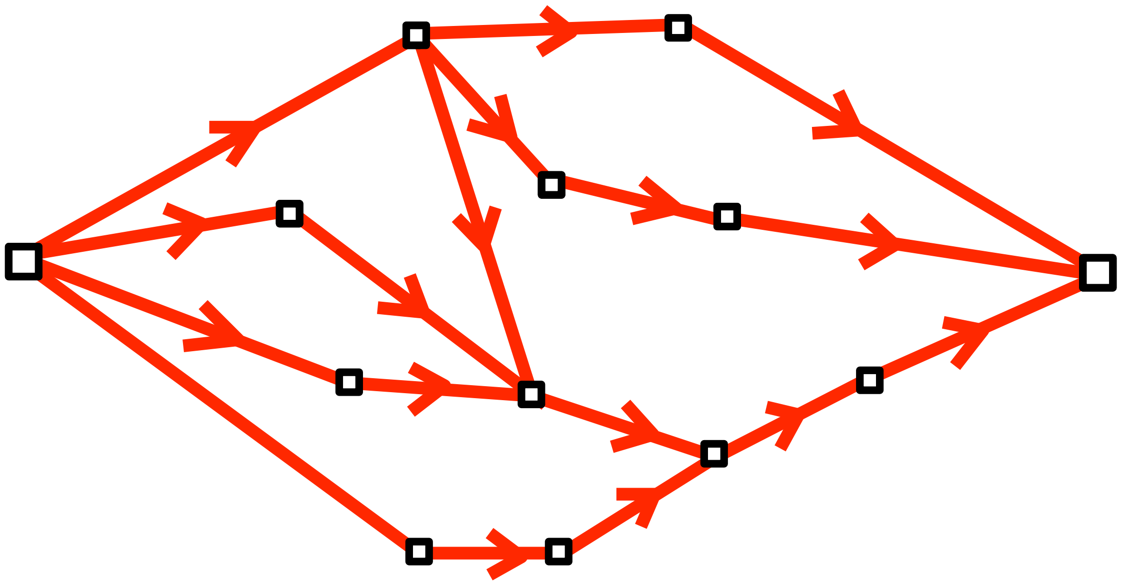


Bipolar→Bipolar intransitive



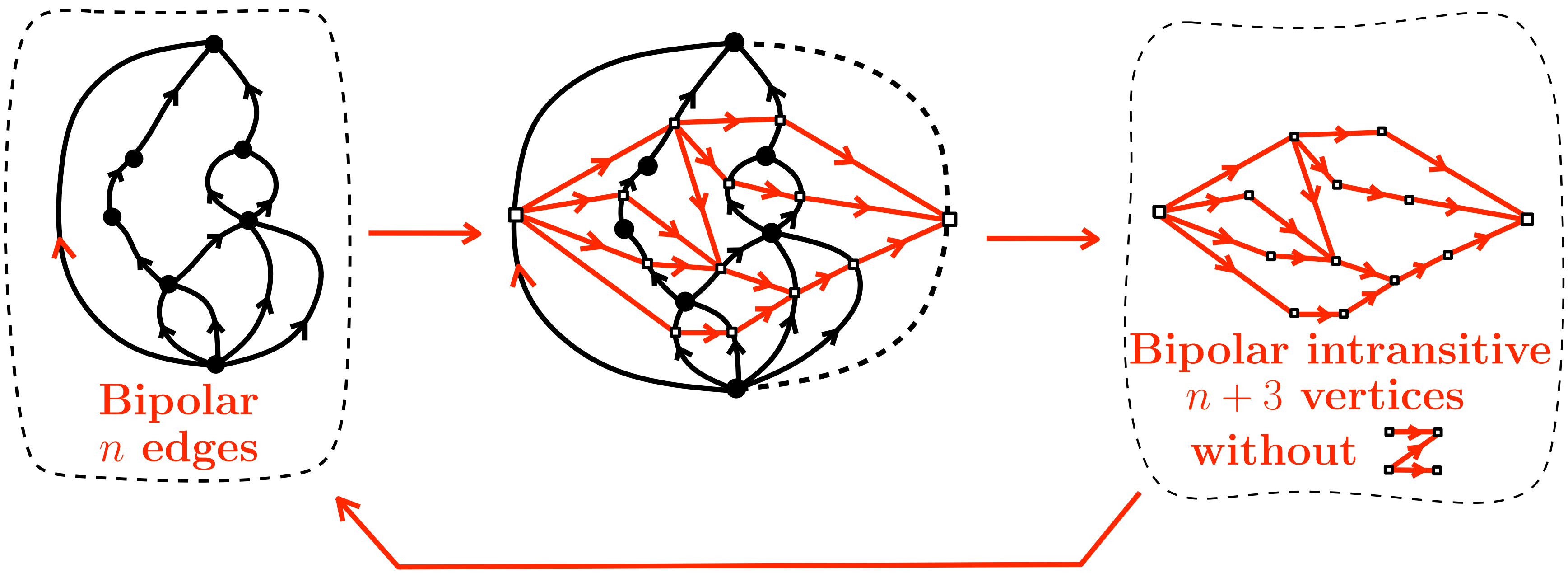


Bipolar→Bipolar intransitive

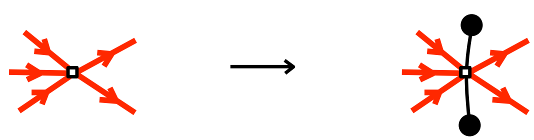


Bipolar→Bipolar intransitive

A first bijection:

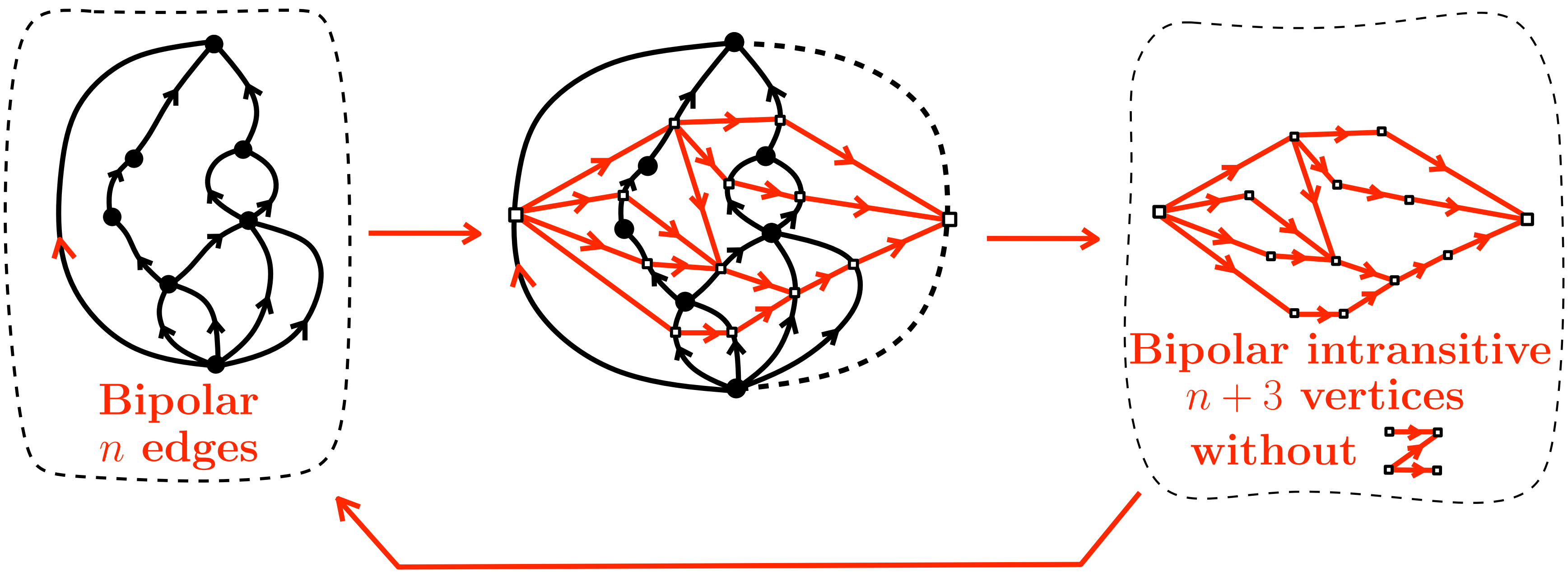


1) place a black vertex in each face
2) insert one black edge for each white vertex

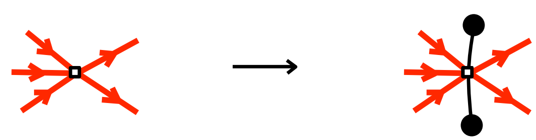


Bipolar→Bipolar intransitive

A first bijection:



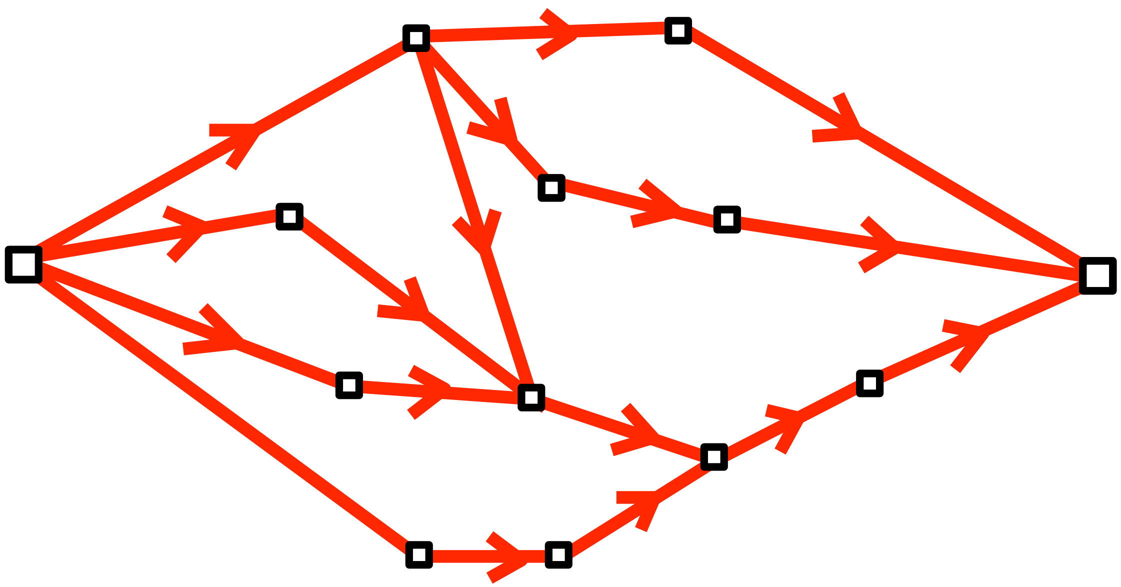
1) place a black vertex in each face
2) insert one black edge for each white vertex



Remark: These are counted by the Baxter number: $\boxed{B_n = \frac{2}{n(n+1)^2}\sum_{k=0}^{n-1}\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}$

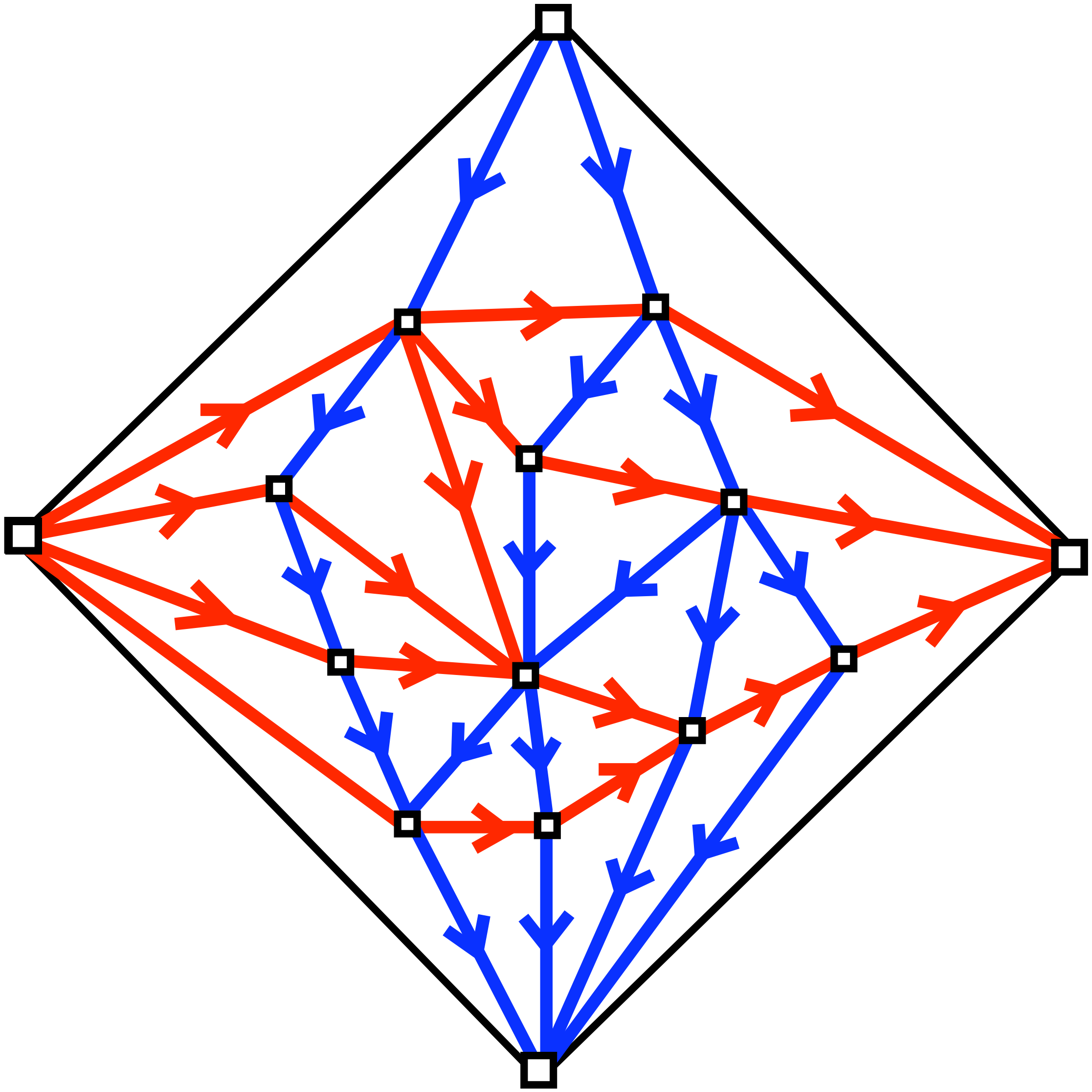
Bipolar intransitive$\longrightarrow$Trans. struct.

Start with an intransitive bipolar orientation

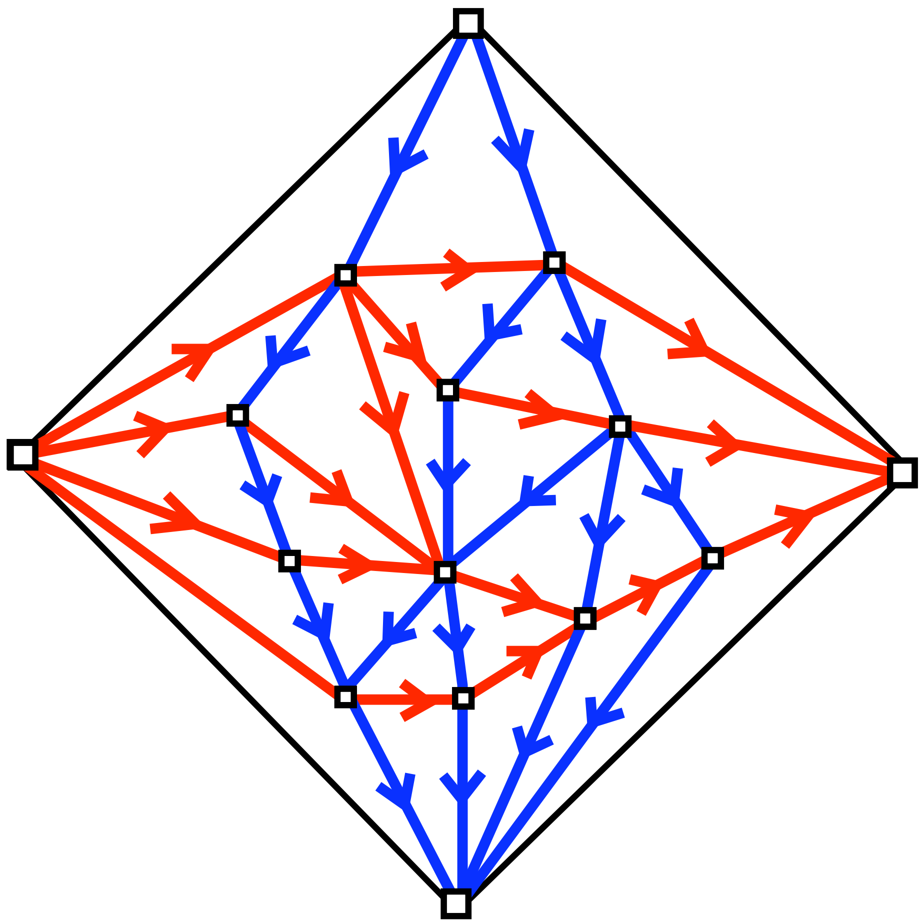


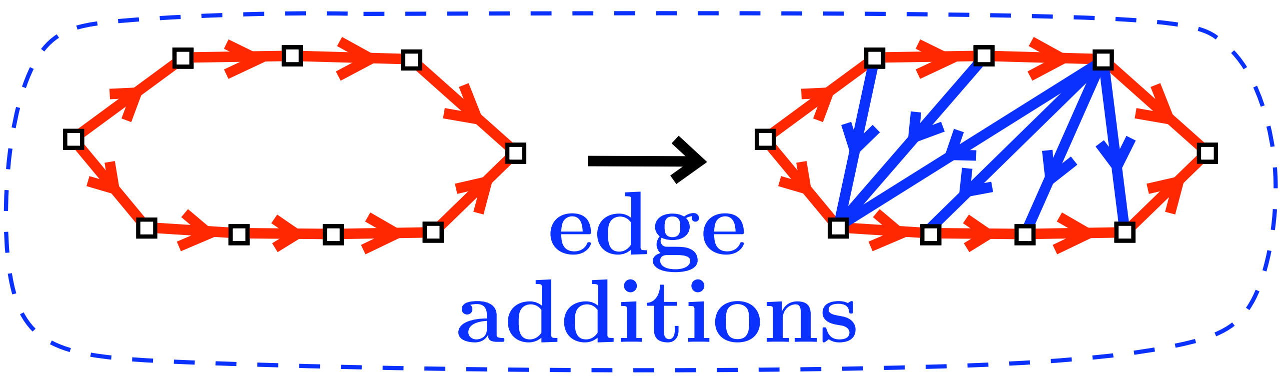
Bipolar intransitive⟶Trans. struct.

Triangulate the faces by blue edges



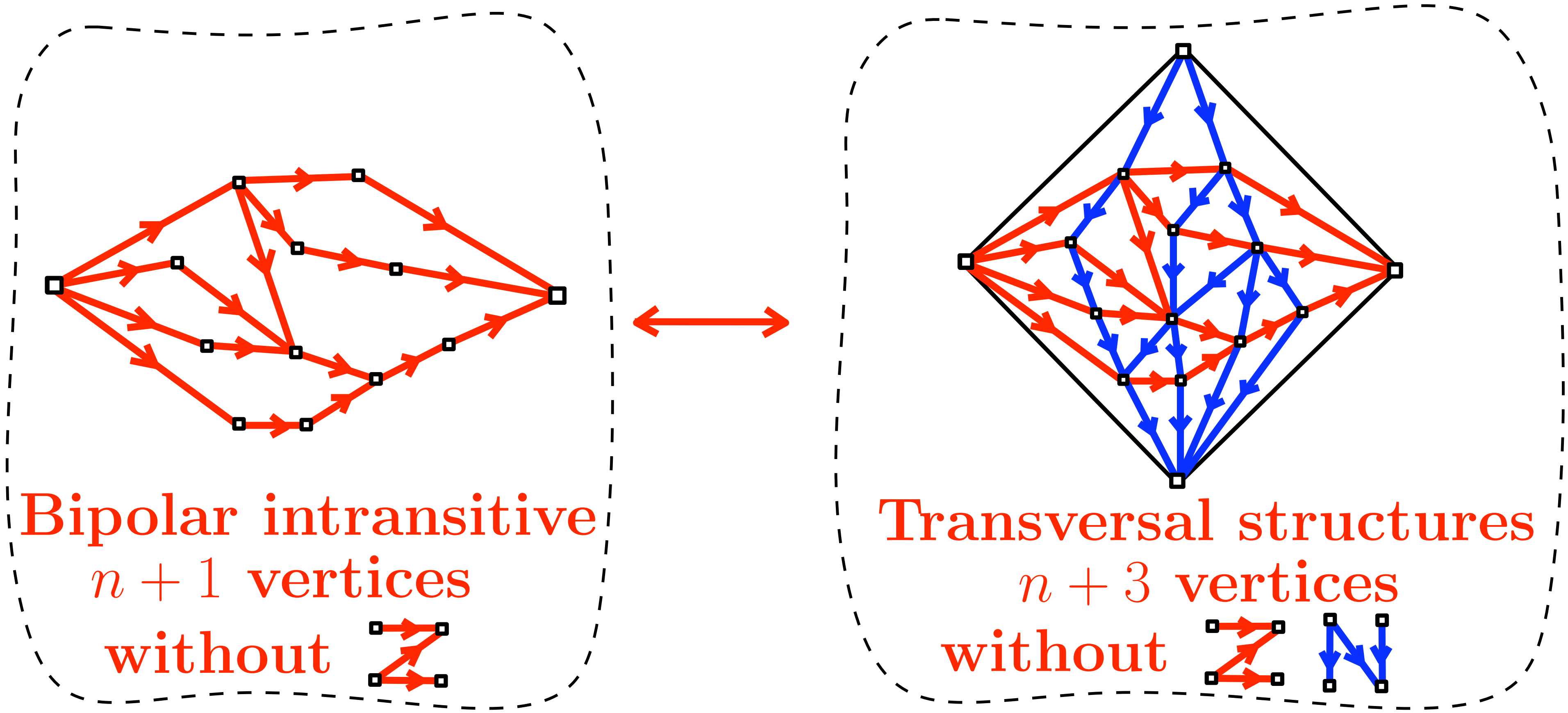
Bipolar intransitive→Trans. struct.



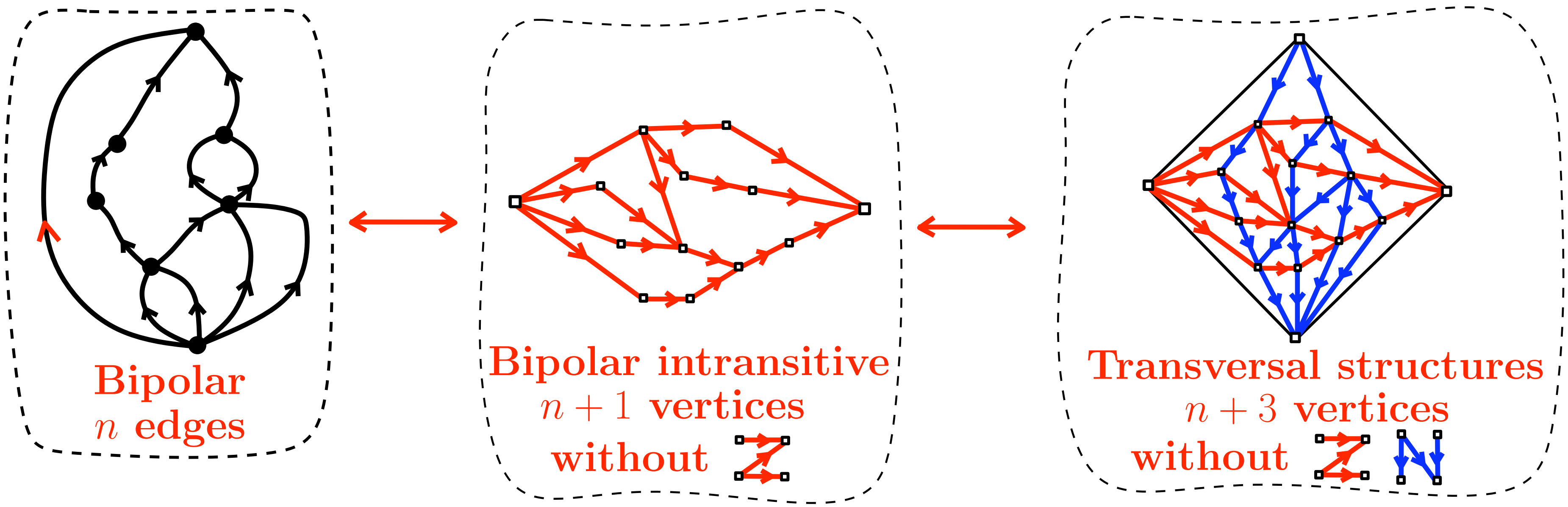


Bipolar intransitive→Trans. struct.

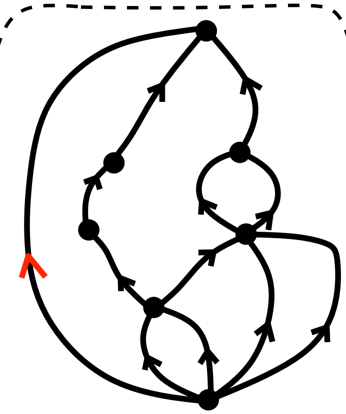
A second bijection:



The bijection

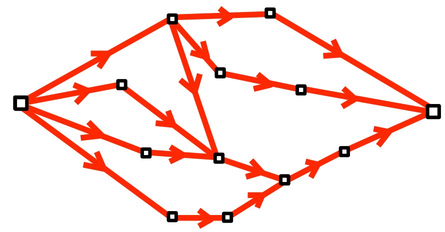


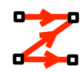
The bijection



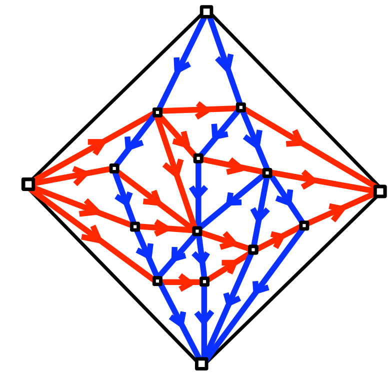
Bipolar
n **edges**

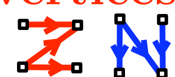
⟷



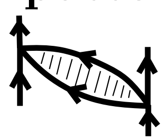
Bipolar intransitive
$n+1$ **vertices**
without 

⟷



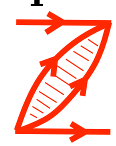
Transversal structures
$n+3$ **vertices**
without 

No pattern



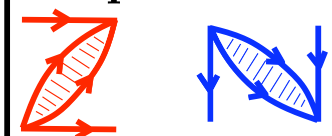
⟷

No pattern

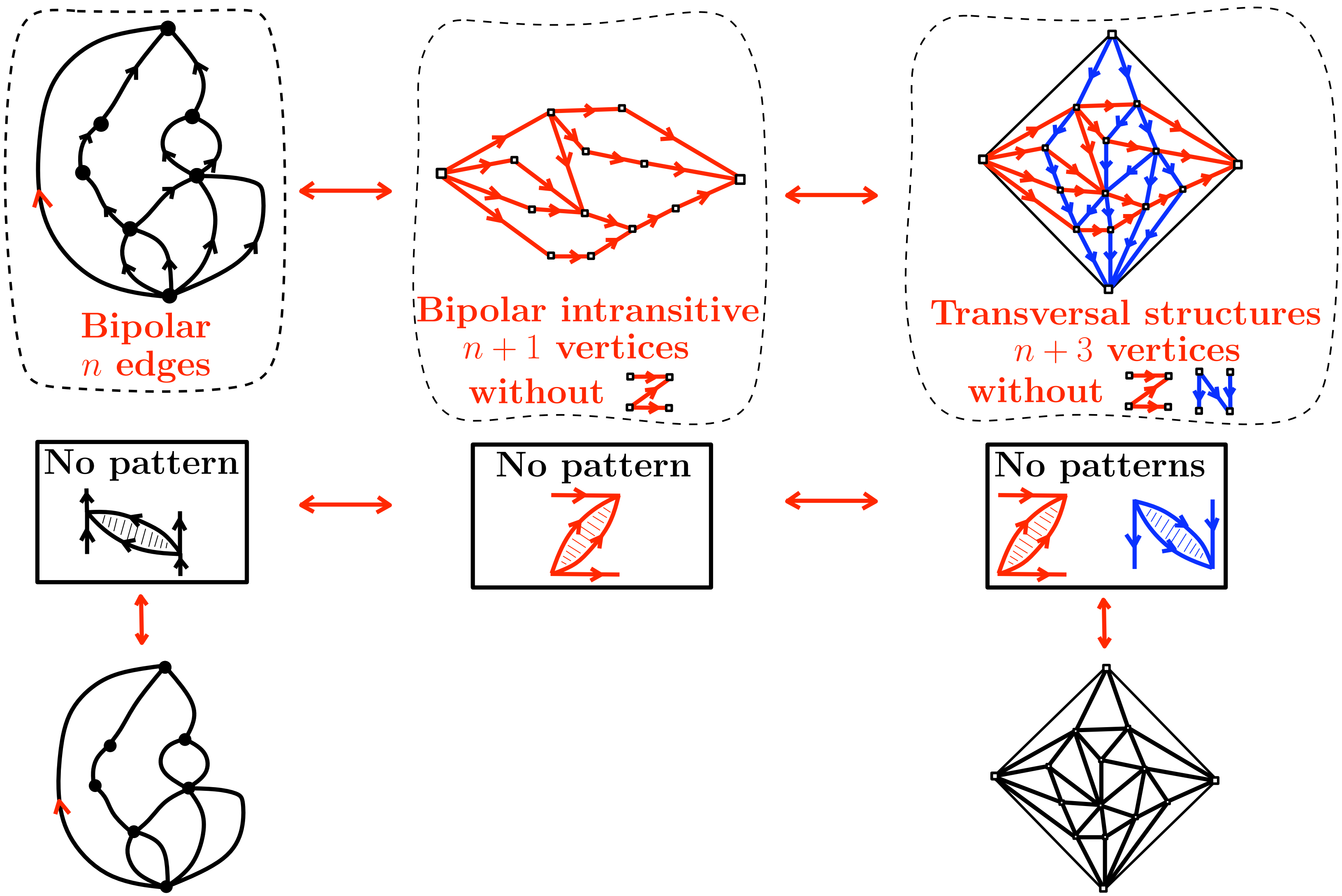


⟷

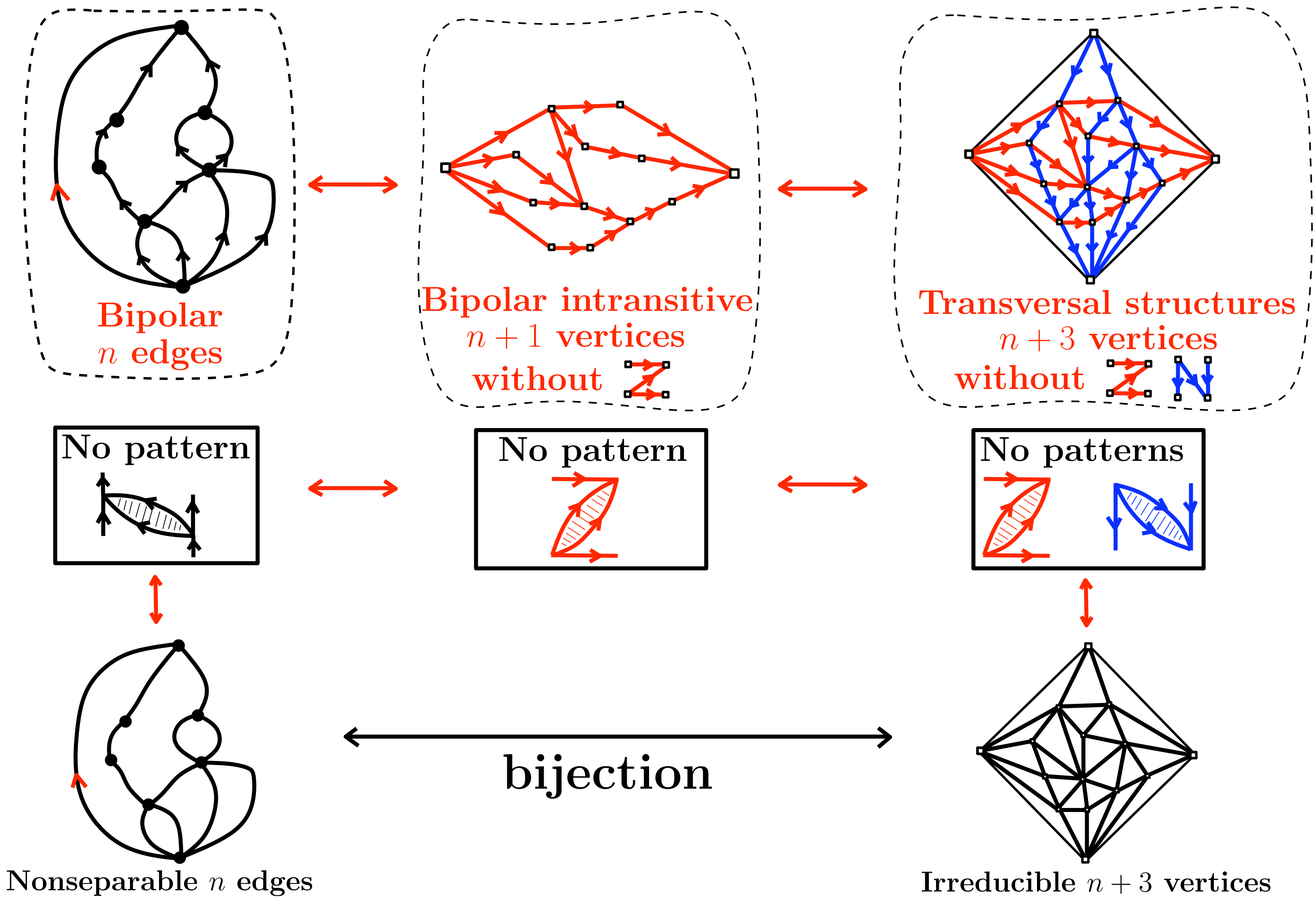
No patterns



The bijection



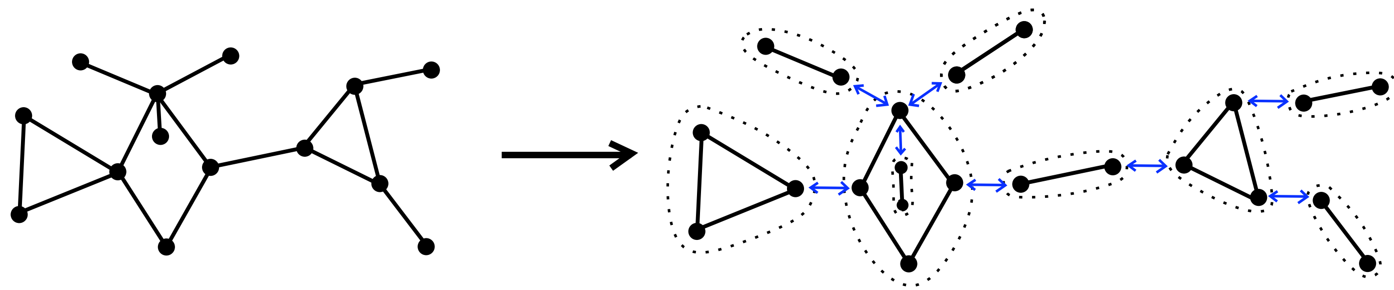
The bijection



Bijection between loopless maps and triangulations

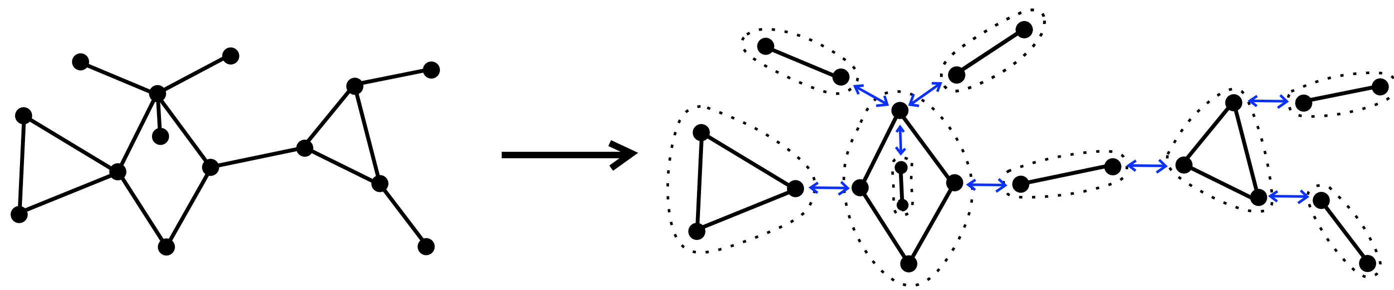
Decomposing a loopless map

- **Block decomposition**

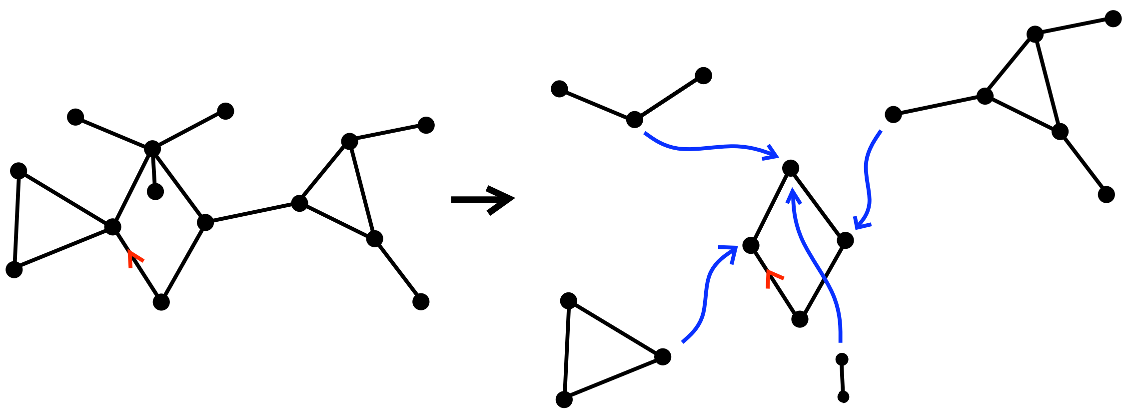


Decomposing a loopless map

- Block decomposition



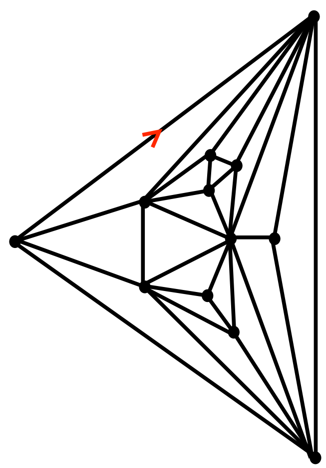
- For rooted loopless maps:



Nonseparable core where each corner is possibly occupied by a loopless map

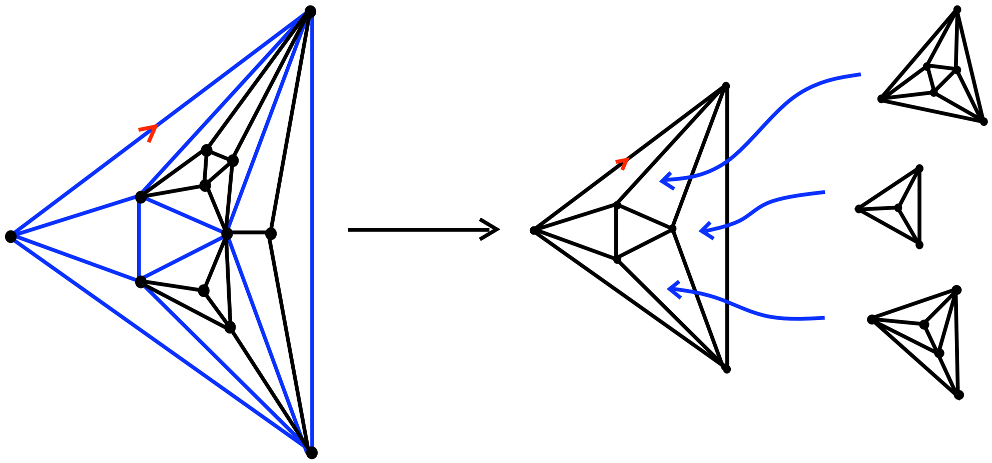
Decomposing a triangulation

- **Classical decomposition at separating triangles**



Decomposing a triangulation

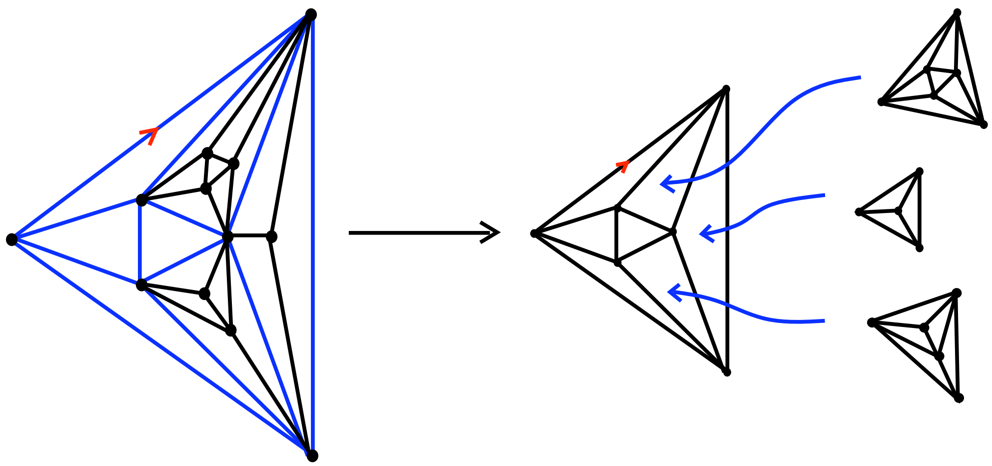
- **Classical decomposition at separating triangles**



4-connected triangulation where each face is possibly occupied by a triangulation

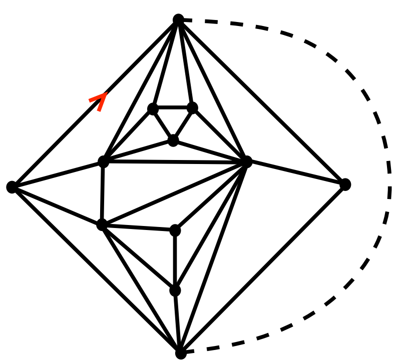
Decomposing a triangulation

- **Classical decomposition at separating triangles**



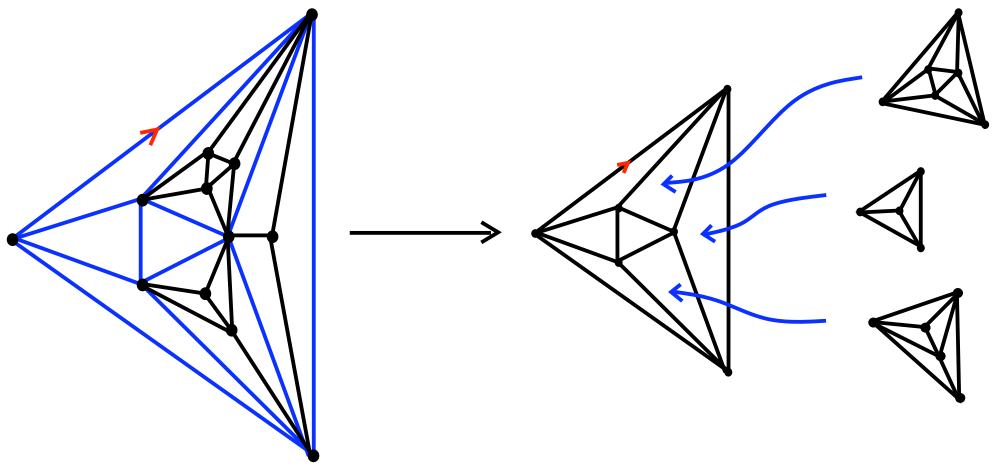
4-connected triangulation where each face is possibly occupied by a triangulation

- **Here: the same after deleting an outer edge**



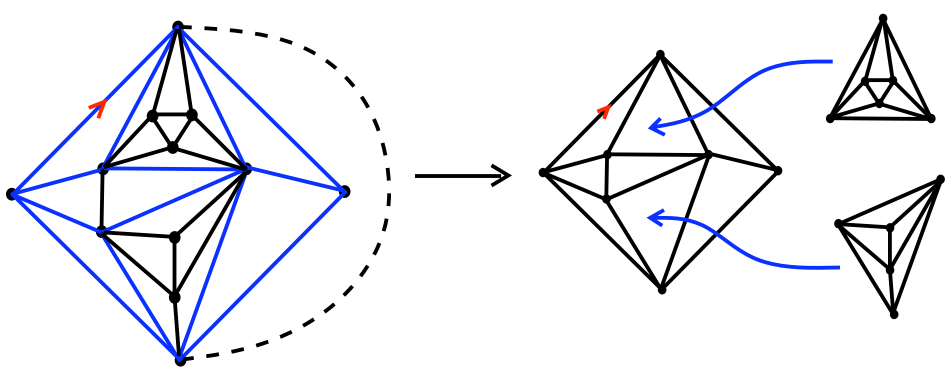
Decomposing a triangulation

- **Classical decomposition at separating triangles**



4-connected triangulation where each face is possibly occupied by a triangulation

- **Here: the same after deleting an outer edge**



Irreducible triangulation where each face is possibly occupied by a triangulation

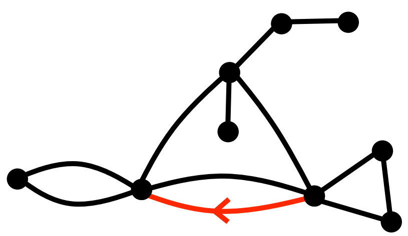
The decompositions are parallel

A loopless map M ($|M| = \#$ **edges**) **is**

i) the vertex-map $\bullet$

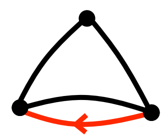
or

ii) $|\mathbf{M}| \geq \mathbf{1}$



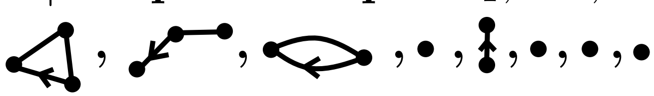
$=$

Nonseparable core C



+

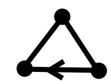
$2|C|$ **loopless maps** $M_1, \ldots, M_{2|C|}$



$$\boxed{|\mathbf{M}| = |\mathbf{C}| + \sum_{\mathbf{i=1}}^{\mathbf{2|C|}} |\mathbf{M_i}|}$$

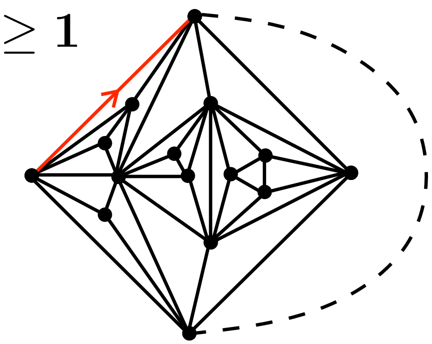
A triangulation T ($|T| = \#$ **vert. -3**) **is**

i) the triangle-map



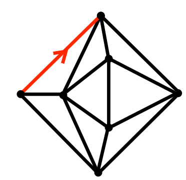
or

ii) $|\mathbf{T}| \geq \mathbf{1}$



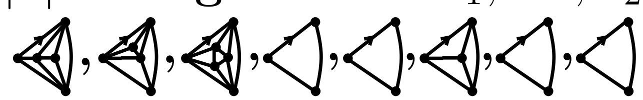
$=$

Irreducible core I



+

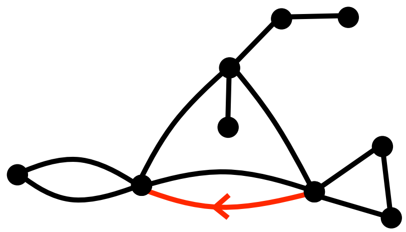
$2|I|$ **triangulations** $T_1, \ldots, T_{2|I|}$



$$\boxed{|\mathbf{T}| = |\mathbf{I}| + \sum_{\mathbf{i=1}}^{\mathbf{2|I|}} |\mathbf{T_i}|}$$

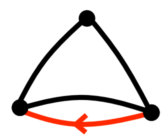
The decompositions are parallel

A loopless map M ($|M| = \#$ edges) is
i) the vertex-map $\bullet$
or
ii) $|\mathbf{M}| \geq \mathbf{1}$



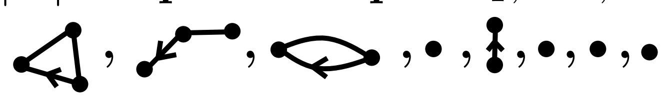
=

Nonseparable core C



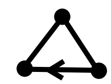
+

$2|C|$ loopless maps $M_1, \dots, M_{2|C|}$

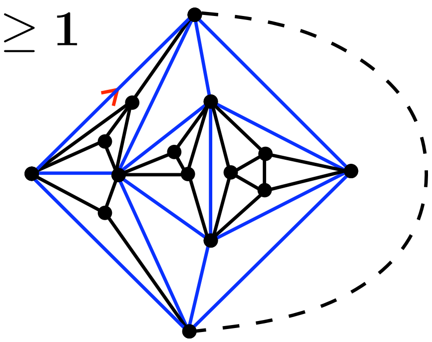


$$\boxed{|\mathbf{M}| = |\mathbf{C}| + \sum_{\mathbf{i=1}}^{\mathbf{2|C|}} |\mathbf{M_i}|}$$

A triangulation T ($|T| = \#$ vert. -3) is
i) the triangle-map

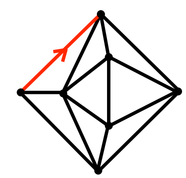


or
ii) $|\mathbf{T}| \geq \mathbf{1}$



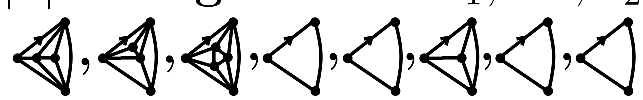
=

Irreducible core I



+

$2|I|$ triangulations $T_1, \dots, T_{2|I|}$



$$\boxed{|\mathbf{T}| = |\mathbf{I}| + \sum_{\mathbf{i=1}}^{\mathbf{2|I|}} |\mathbf{T_i}|}$$

Nonseparable maps n edges $\xleftrightarrow[\textit{bijection}]{}$ Irreducible triang. $n+3$ vertices

$\downarrow$

Loopless maps n edges $\xleftrightarrow[\textit{bijection}]{}$ Triangulations $n+3$ vertices

Results obtained so far

rooted nonseparable maps with n edges

$\Updownarrow$ *bipolar orientations*

rooted irreducible triangulations with $n+3$ vertices

$\Longrightarrow$

rooted loopless maps with n edges

$\Updownarrow$ *parallel decompositions*

rooted triangulations with $n+3$ vertices

Results obtained so far

rooted nonseparable maps with n edges

bipolar orientations

rooted irreducible triangulations with $n+3$ vertices

$$|\mathcal{N}_{\mathbf{n}}| = |\mathcal{I}_{\mathbf{n}}|$$

$\Longrightarrow$

rooted loopless maps with n edges

parallel decompositions

rooted triangulations with $n+3$ vertices

$$|\mathcal{L}_{\mathbf{n}}| = |\mathcal{T}_{\mathbf{n}}|$$

Results obtained so far

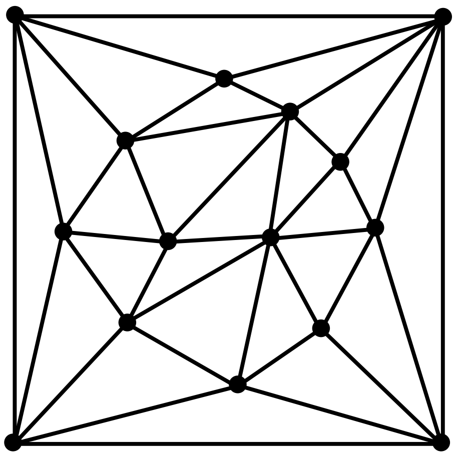
$$\begin{array}{ccc}
\boxed{\begin{array}{c}\textbf{rooted nonseparable}\\ \textbf{maps with } n \textbf{ edges}\end{array}} & & \boxed{\begin{array}{c}\textbf{rooted loopless maps}\\ \textbf{with } n \textbf{ edges}\end{array}} \\
\textit{bipolar orientations} \Big\updownarrow & \Longrightarrow & \Big\updownarrow \textit{parallel decompositions} \\
\boxed{\begin{array}{c}\textbf{rooted irreducible}\\ \textbf{triangulations}\\ \textbf{with } n{+}3 \textbf{ vertices}\end{array}} & & \boxed{\begin{array}{c}\textbf{rooted triangulations}\\ \textbf{with } n{+}3 \textbf{ vertices}\end{array}}
\end{array}$$

$|\mathcal{N}_\mathbf{n}| = |\mathcal{I}_\mathbf{n}| = \mathbf{?}$

$|\mathcal{L}_\mathbf{n}| = |\mathcal{T}_\mathbf{n}| = \mathbf{?}$

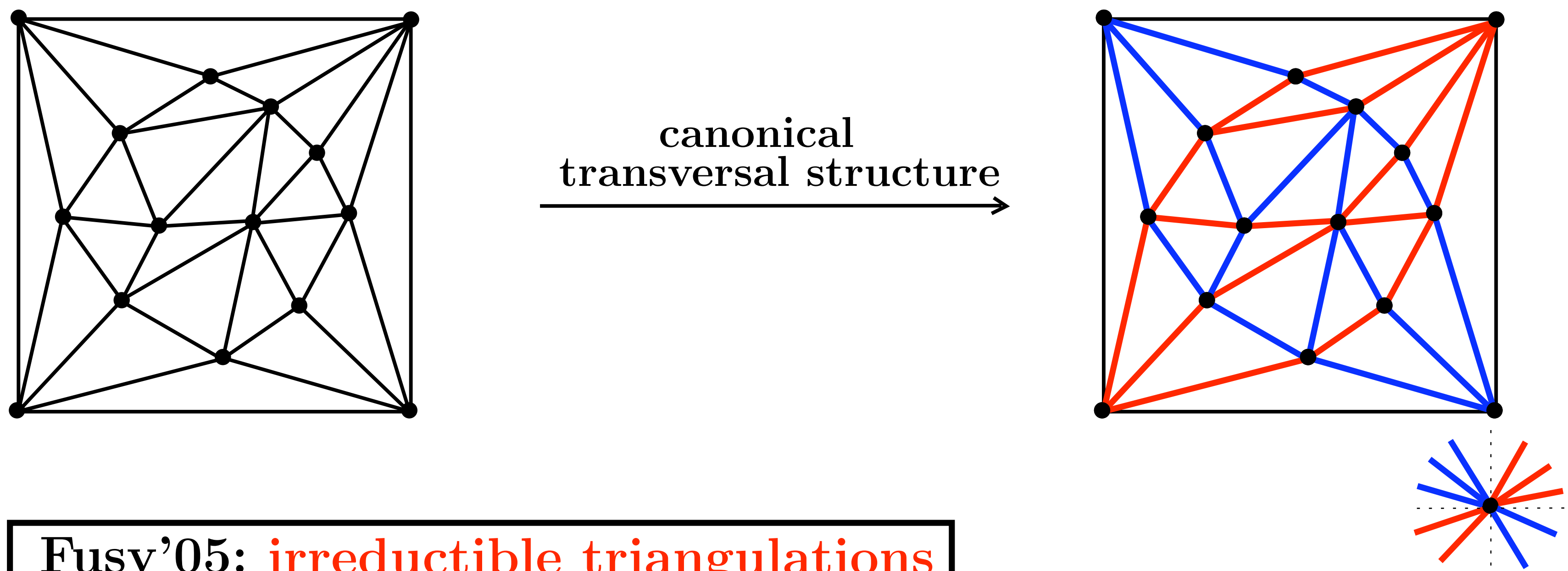
Counting the families

Irreducible triang. $\longleftrightarrow$ ternary trees



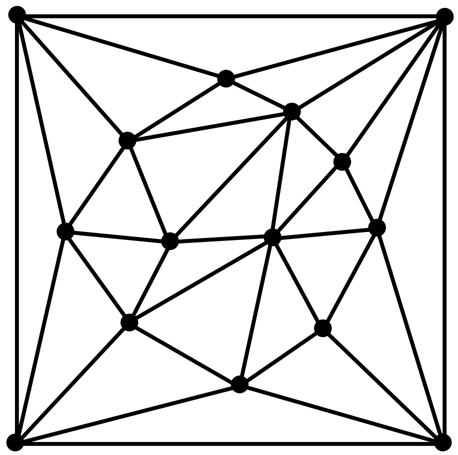
Fusy'05: irreductible triangulations are in bijection with ternary trees

Irreducible triang. $\longleftrightarrow$ ternary trees

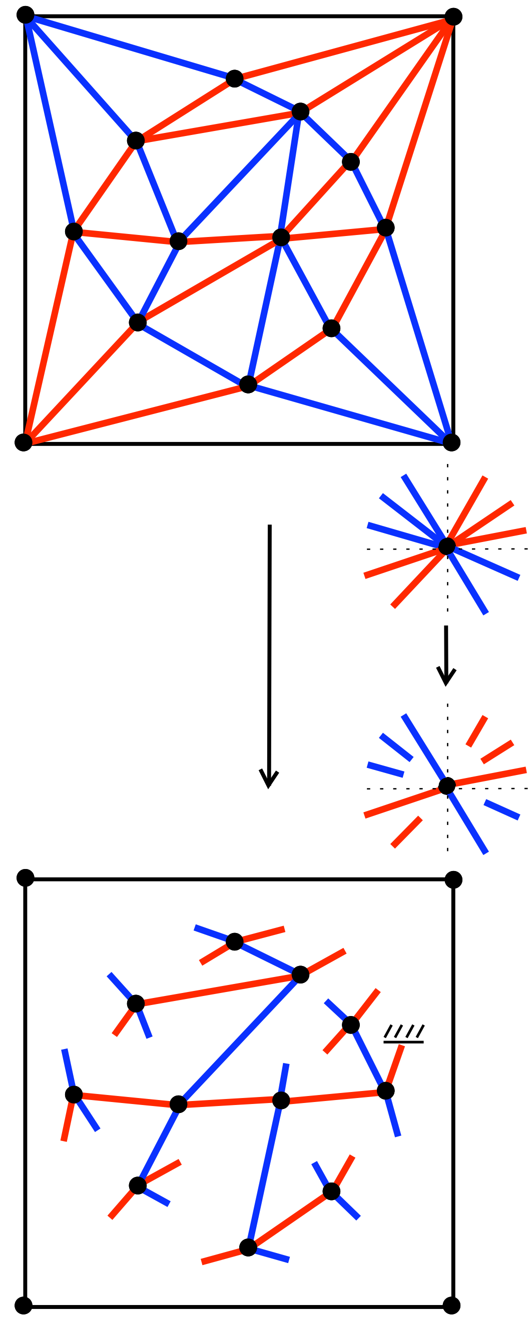


Fusy'05: irreductible triangulations are in bijection with ternary trees

Irreducible triang. $\longleftrightarrow$ ternary trees

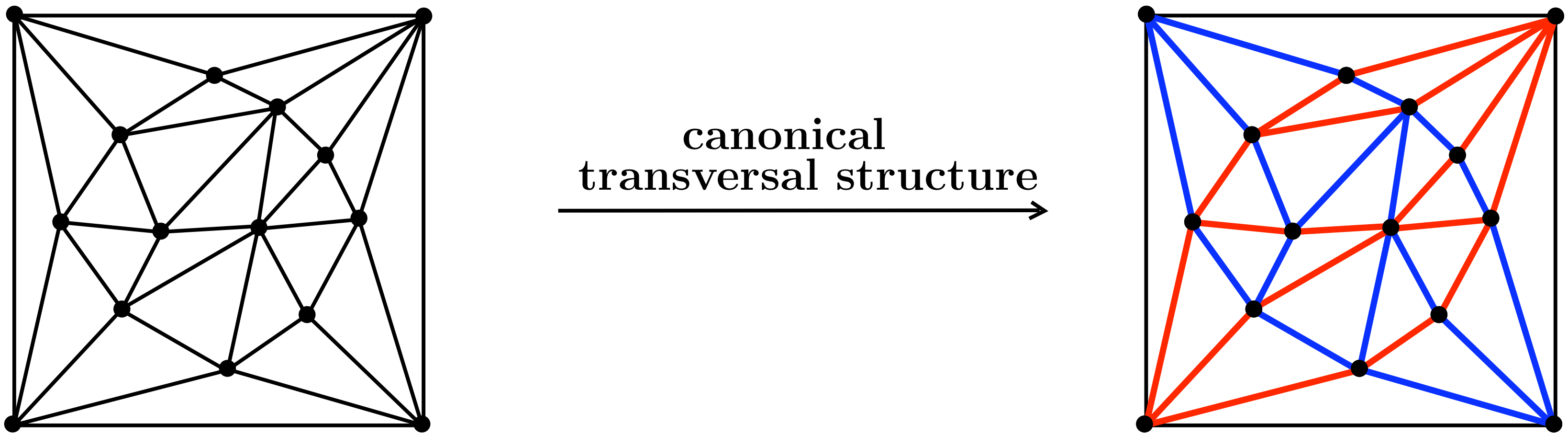


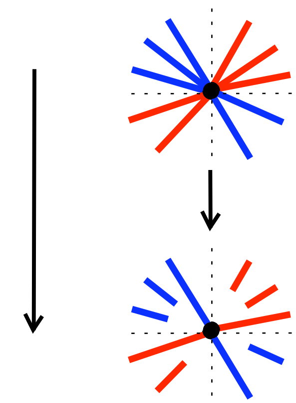
$\xrightarrow{\text{canonical transversal structure}}$



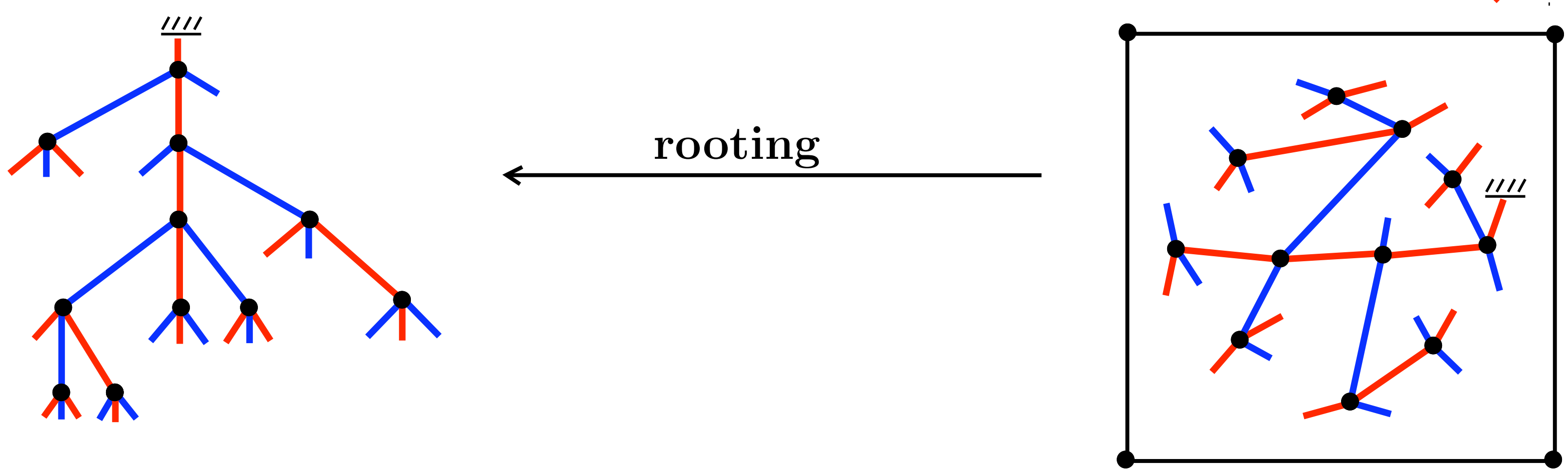
Fusy'05: irreductible triangulations are in bijection with ternary trees

Irreducible triang. $\longleftrightarrow$ ternary trees

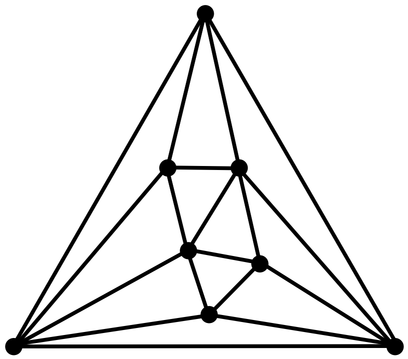




Fusy'05: irreductible triangulations are in bijection with ternary trees

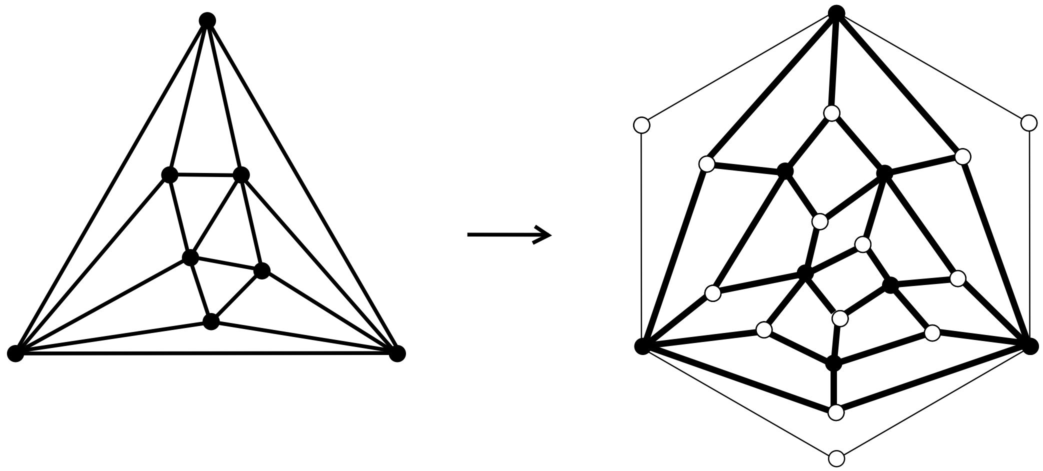


Triangulations$\leftrightarrow$quaternary trees



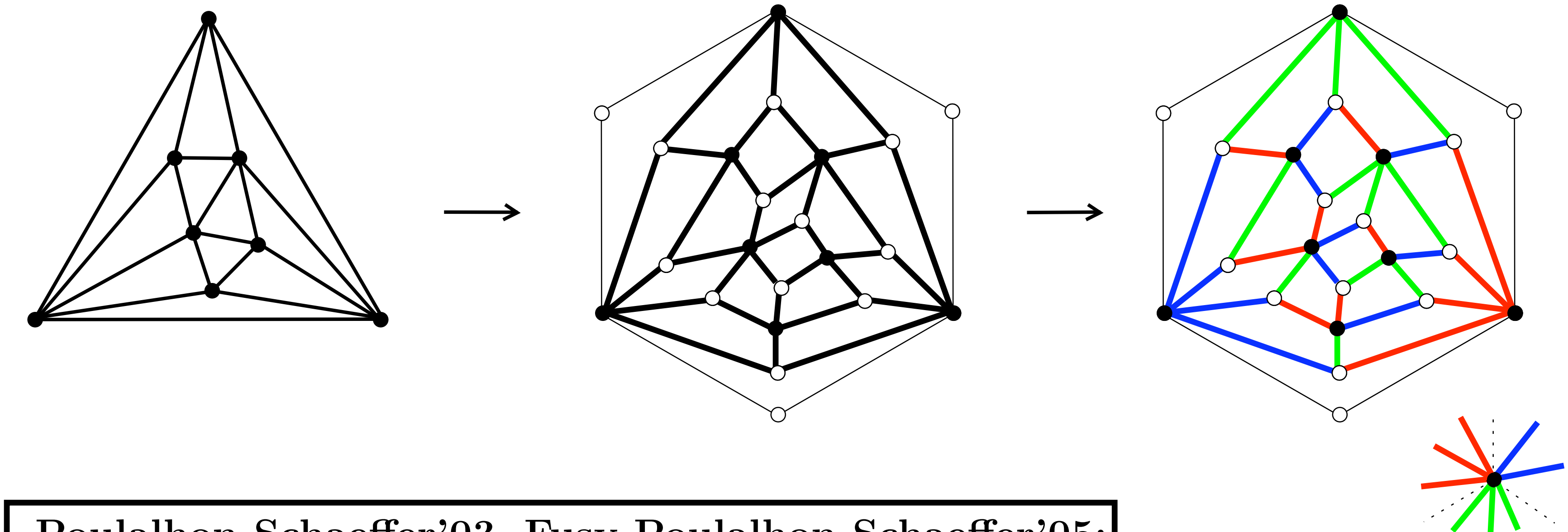
Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05:
triangulations are in bijection with quaternary trees

Triangulations↔quaternary trees



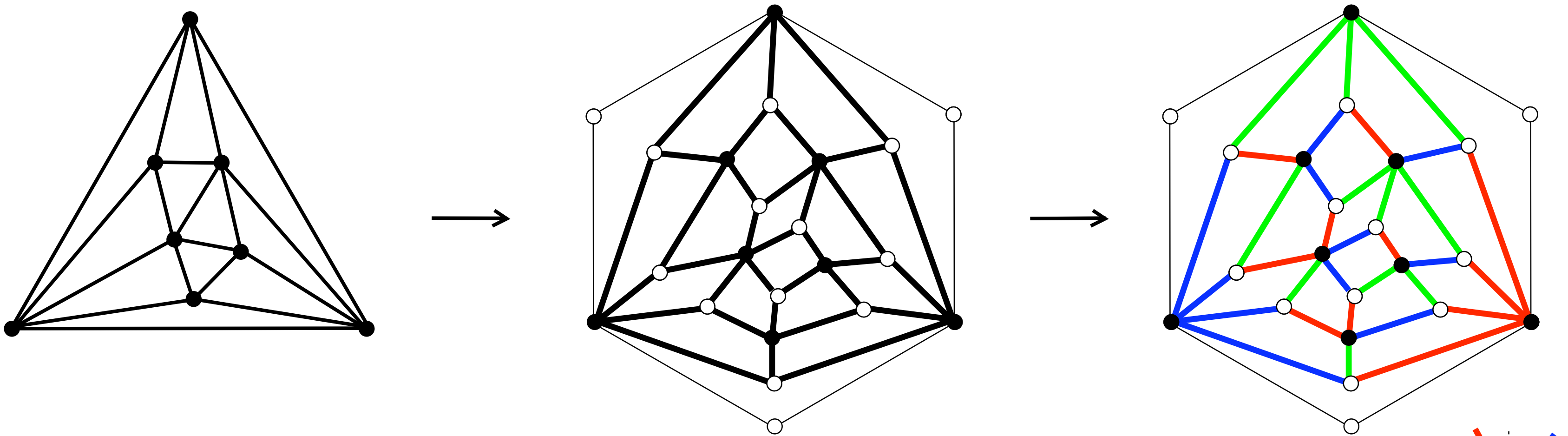
Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05:
triangulations are in bijection with quaternary trees

Triangulations$\leftrightarrow$quaternary trees

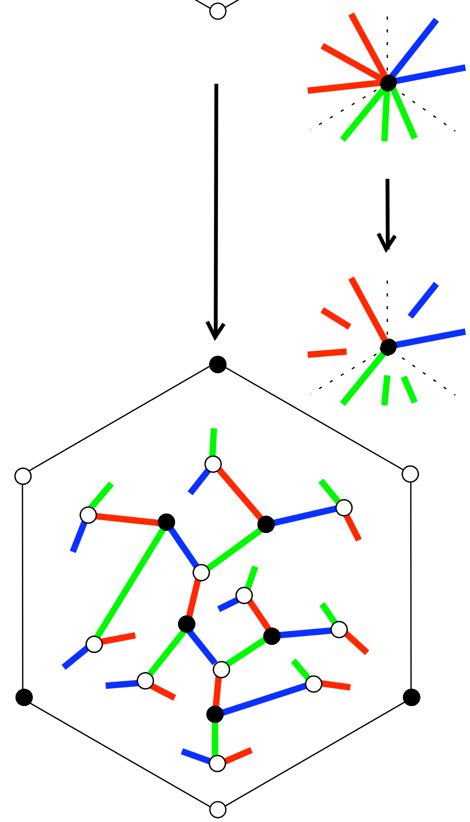


Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05:
triangulations are in bijection with quaternary trees

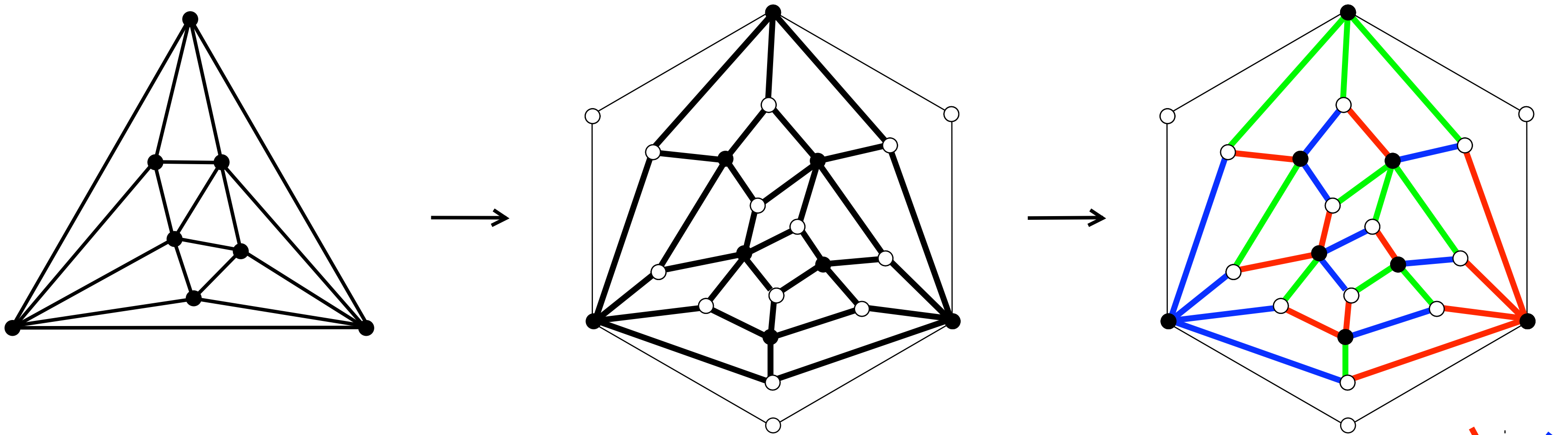
Triangulations↔quaternary trees



Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05:
triangulations are in bijection with quaternary trees



Triangulations↔quaternary trees



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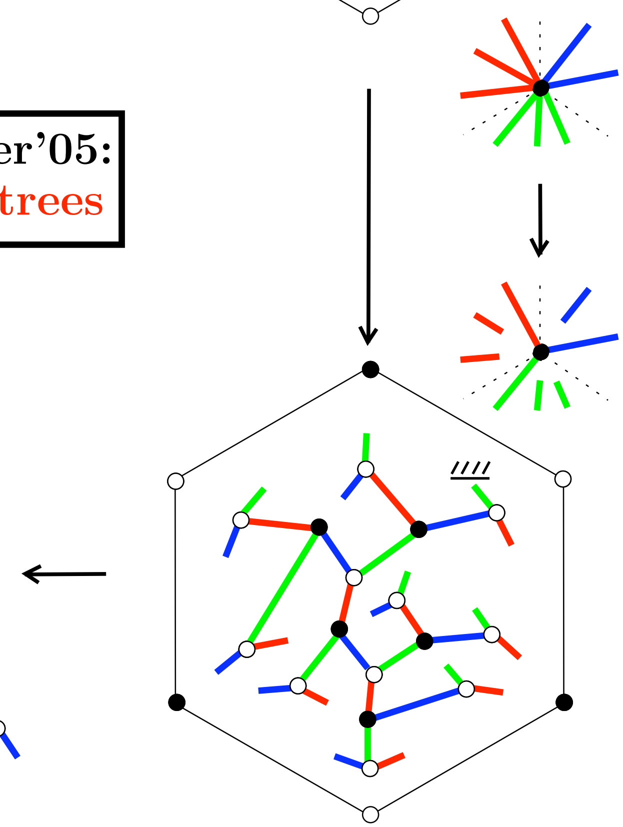

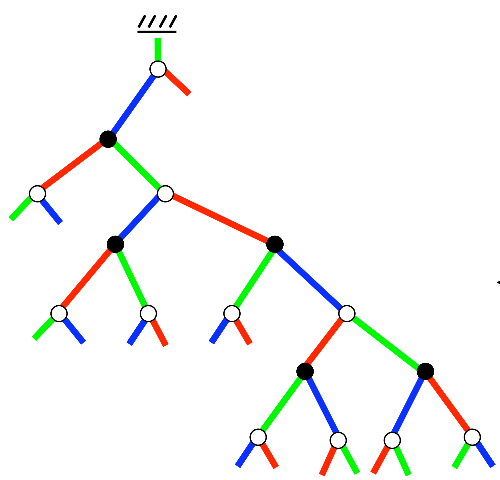

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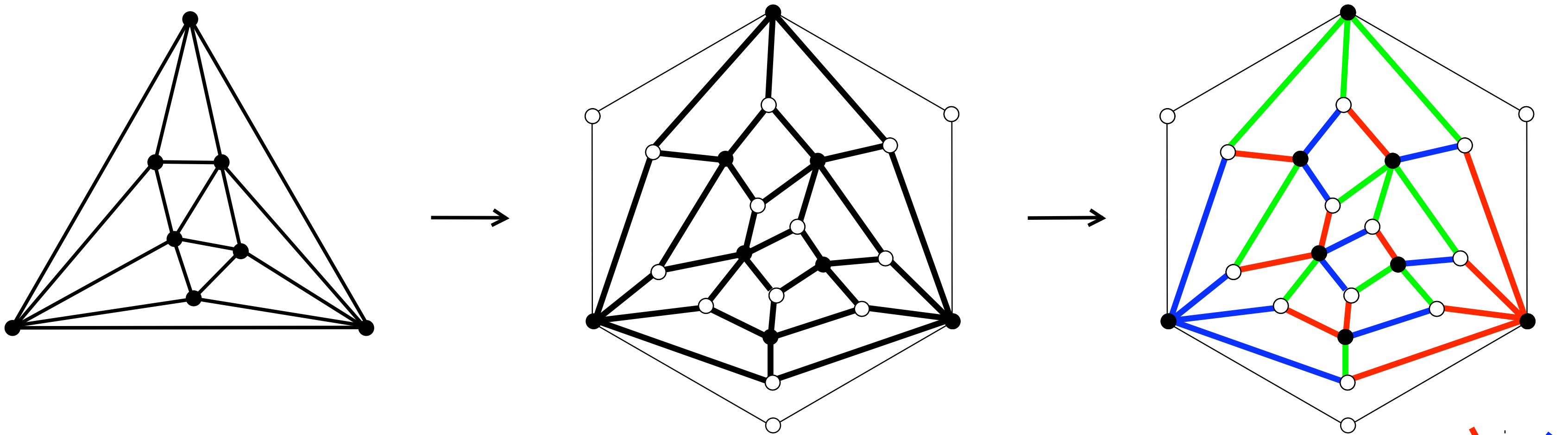

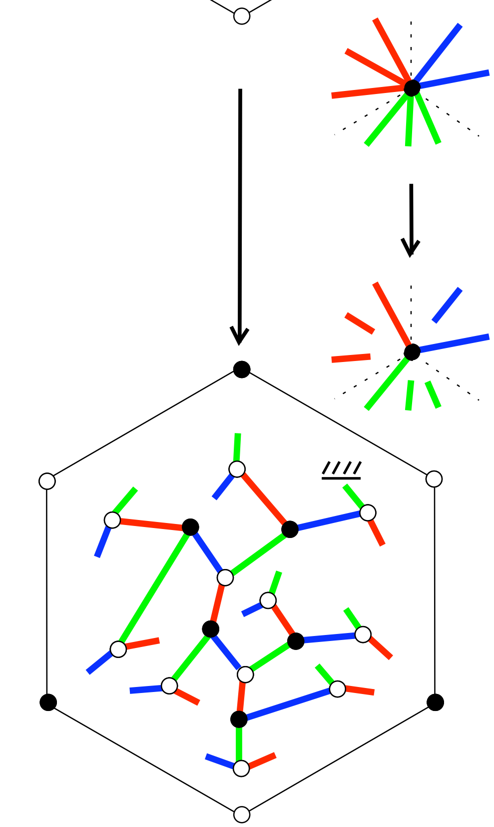

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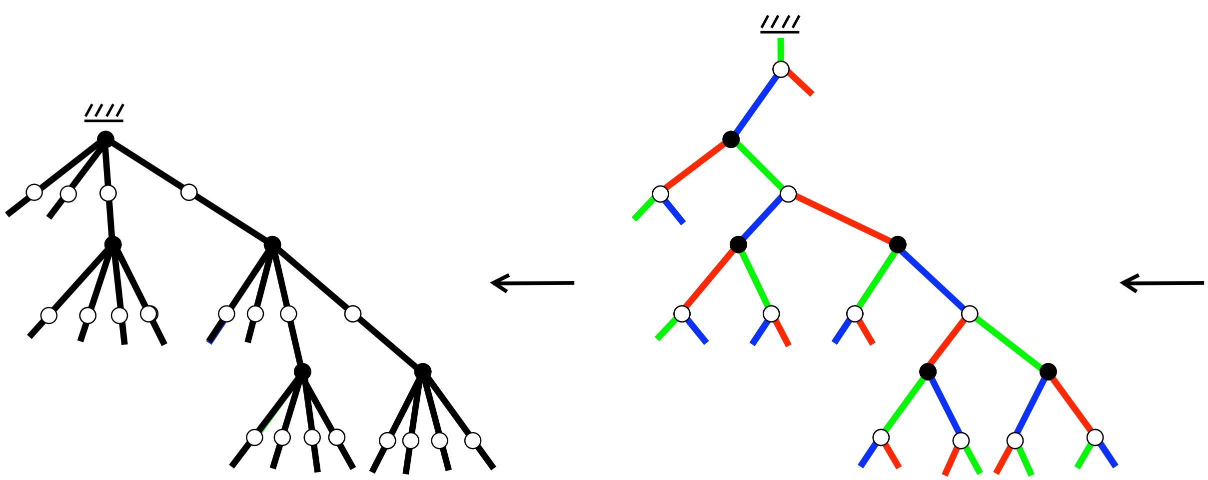

Enumerative Results

rooted nonseparable maps with n edges

$\updownarrow$

rooted irreducible triangulations with $n{+}3$ vertices

$\updownarrow$

rooted ternary trees with $n{-}1$ nodes

$\Longrightarrow$

rooted loopless maps with n edges

$\updownarrow$

rooted triangulations with $n{+}3$ vertices

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Enumerative Results

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$\Downarrow$

$$|\mathcal{N}_{\mathbf{n}}| = |\mathcal{T}_{\mathbf{n}}| = \frac{\mathbf{4(3n-3)!}}{\mathbf{(n-1)!(2n)!}}$$

$\Longrightarrow$

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rooted triangulations with $n{+}3$ vertices

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$\Downarrow$

$$|\mathcal{L}_{\mathbf{n}}| = |\mathcal{I}_{\mathbf{n}}| = \frac{\mathbf{2(4n+1)!}}{\mathbf{(n+1)!(3n+2)!}}$$

Enumerative Results

Schaeffer'99

rooted nonseparable maps with n edges

$\updownarrow$

rooted irreducible triangulations with $n{+}3$ vertices

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?

$\Downarrow$

$$|\mathcal{L}_\mathbf{n}| = |\mathcal{I}_\mathbf{n}| = \frac{\mathbf{2(4n+1)!}}{\mathbf{(n+1)!(3n+2)!}}$$

Applications

Algorithmic applications

- **General scheme: (Schaeffer'99, Poulalhon-Schaeffer'03)**

$$\boxed{\textbf{maps}} \longleftrightarrow \boxed{\textbf{trees}} \longleftrightarrow \boxed{\textbf{``Dyck words''}}$$

Algorithmic applications

- **General scheme: (Schaeffer'99, Poulalhon-Schaeffer'03)**

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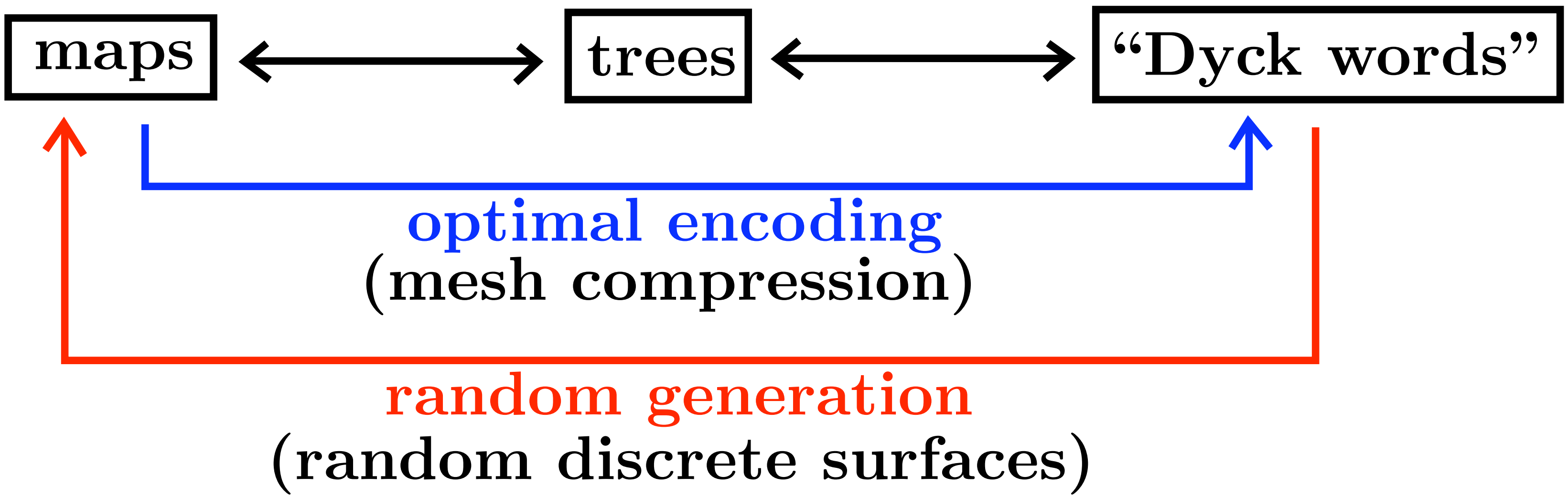
optimal encoding
(mesh compression)

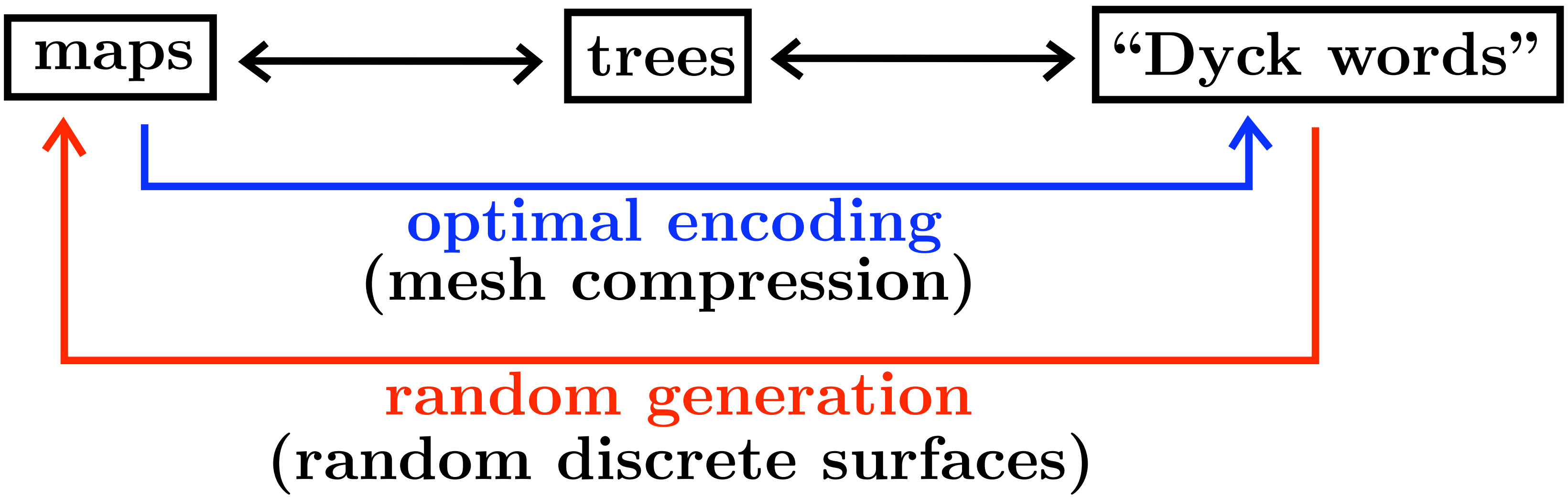
Algorithmic applications

- **General scheme**: (Schaeffer'99, Poulalhon-Schaeffer'03)



Algorithmic applications

- **General scheme: (Schaeffer'99, Poulalhon-Schaeffer'03)**



- **Applies here to :**
  - **irreducible triangulations**
  - **triangulations**

Algorithmic applications

- **General scheme: (Schaeffer'99, Poulalhon-Schaeffer'03)**

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optimal encoding
(mesh compression)

random generation
(random discrete surfaces)

- **Applies here to :**
  - **irreducible triangulations**
  - **triangulations**

  as well as :
  - **loopless maps**
  - **nonseparable maps**