Schnyder woods generalized to higher genus

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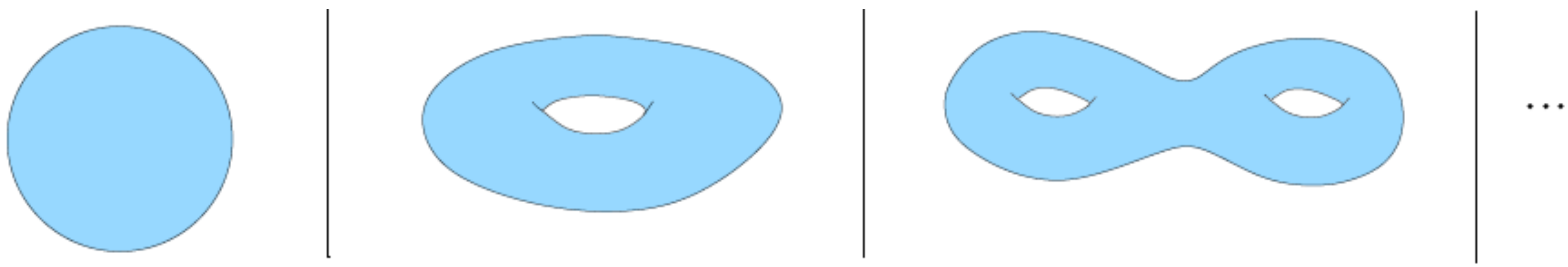
joint work with Luca Castelli Aleardi and Thomas Lewiner

Combinatorics of maps

Surfaces

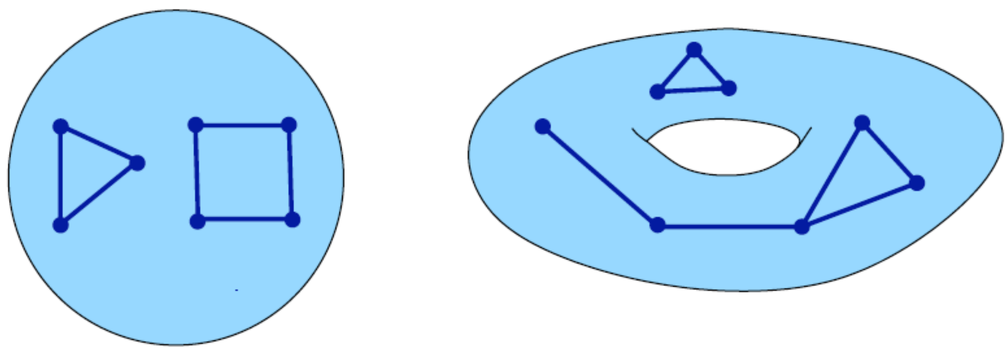
- All surfaces here are closed and orientable
- Classification: one surface in each genus *g*



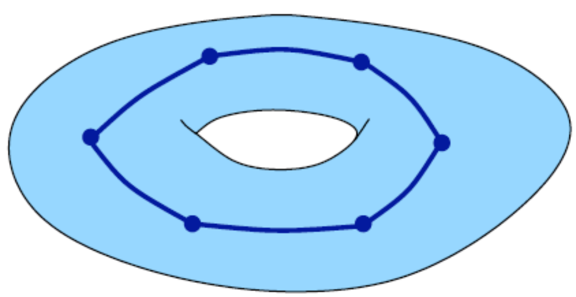
g=0 (sphere) *g*=1 (torus) *g*=2

Graphs on surfaces, maps

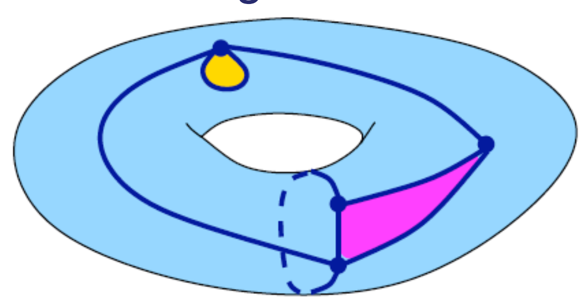
- Graph on surface = graph *G embedded* on a surface S_g
  (no edge-crossings)



- G is a map if the components of G\ S_g are topological disks



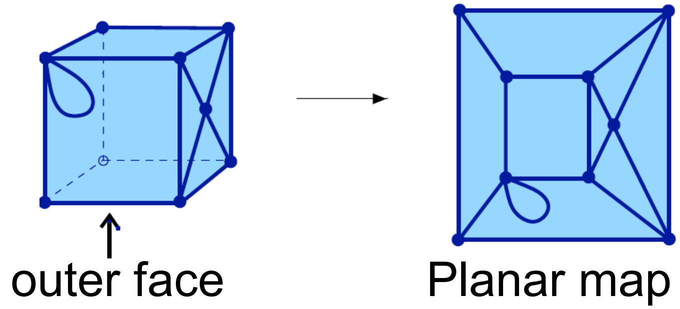
Not a map
(cylindric component)



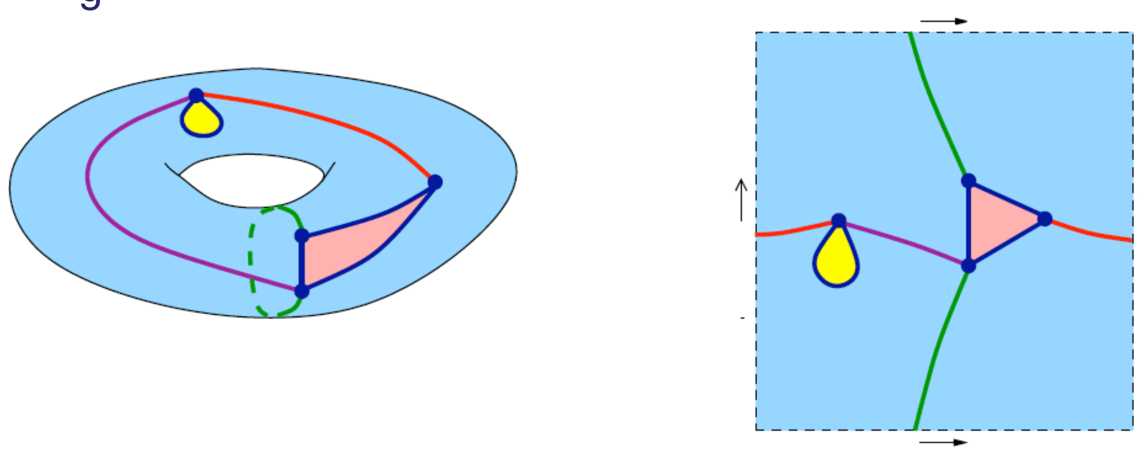
A map
(3 faces)

How to display a map ?

- g=0: project on the plane



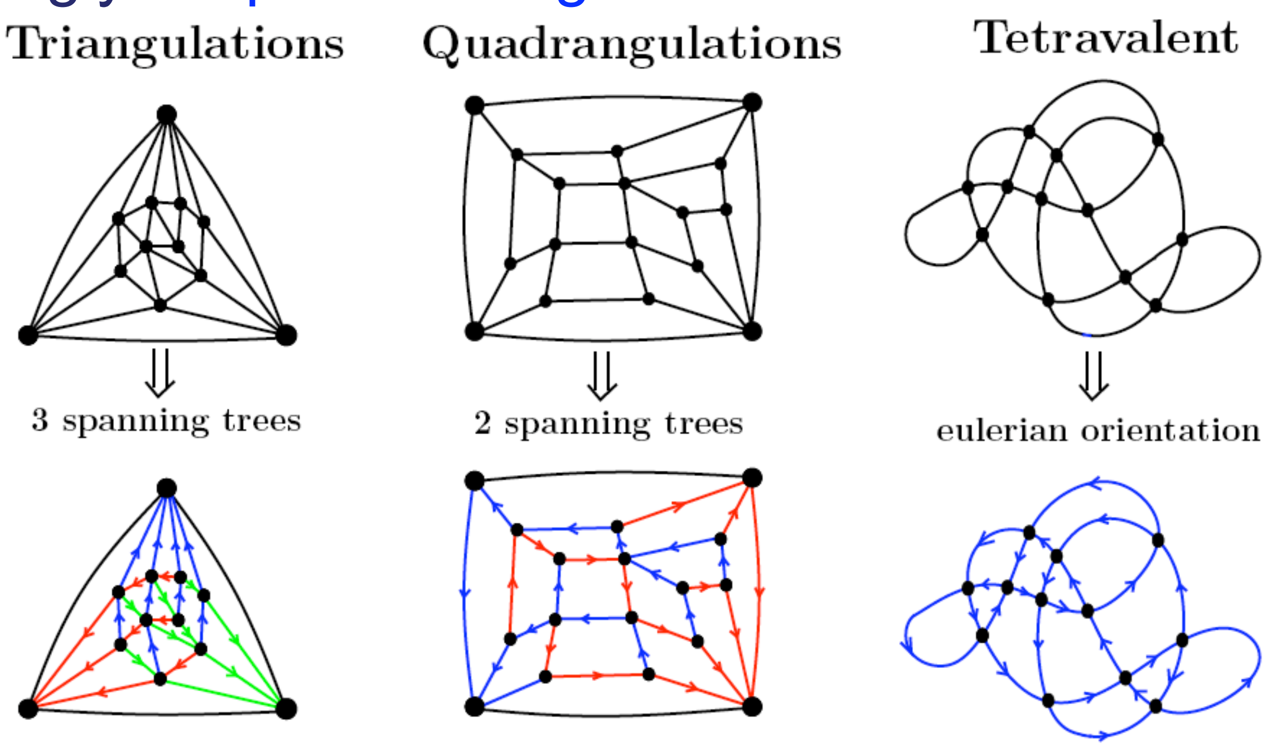
- g=1 : S_g like a square with identified opposite sides



- g>1 : S_g like a $4g$-polygon + identifications of sides

Enumeration of planar maps

- Strikingly simple counting formulas



$|\mathcal{T}_n| = \frac{2(4n-3)!}{n!(3n-2)!}$ $|\mathcal{Q}_n| = \frac{2(3n-3)!}{n!(2n-2)!}$ $|\mathcal{E}_n| = \frac{2\cdot 3^n(2n)!}{n!(n+2)!}$

- Recursive method: [Tutte 60's]
  Bijective method: [Cori-Vauquelin'84], [Schaeffer'97]
  (bijections rely on combinatorial structures: orientations,…)

Counting maps in higher genus

- No exact counting formula known, but
  - Can write recurrences [Bender-Canfield'84]
  - Some bijections work [Chapuy-Marcus-Schaeffer'98]
  - Simple asymptotic pattern [Bender-Canfield'86, Gao'93]]

$\mathcal{M} = \cup_{g,n} \mathcal{M}_g[n]$ **a map family (e.g. triangulations)**

$$\textbf{Then} \quad |\mathcal{M}_g[n]| \underset{n\to\infty}{\sim} c_g \gamma^n n^{5(g-1)/2}.$$

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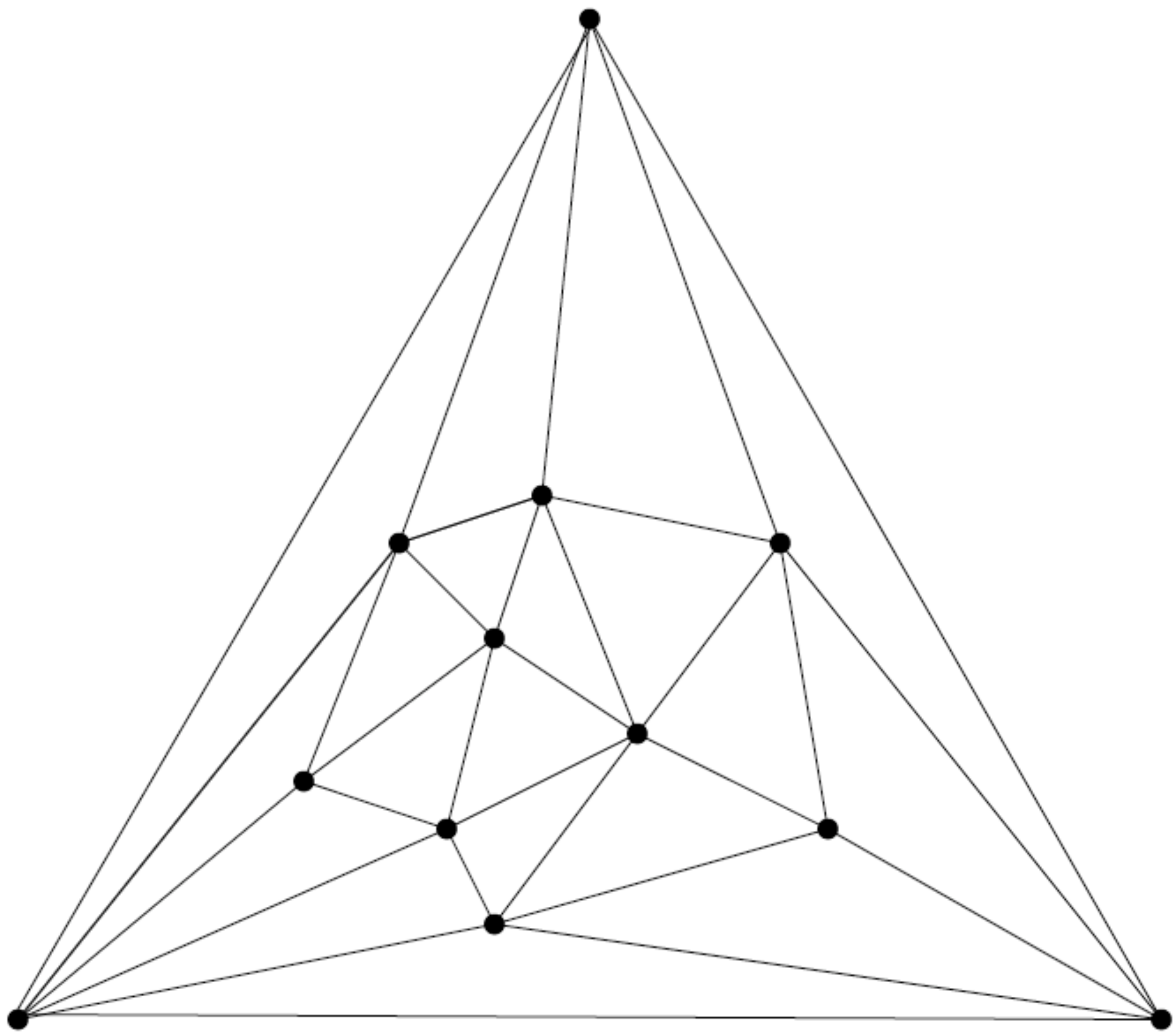
A map in genus *g* 'is like' a planar map with Θ(g) marked edges

- 'is like' can be made rigorous in some cases:
  - Counting maps with a unique face [Chapuy'08]
  - **This talk**: Schnyder woods can be extended to genus g>0 by allowing Θ(g) 'special edges' [Castelli, F, Lewiner'08]

Schnyder woods for planar triangulations

Planar triangulations

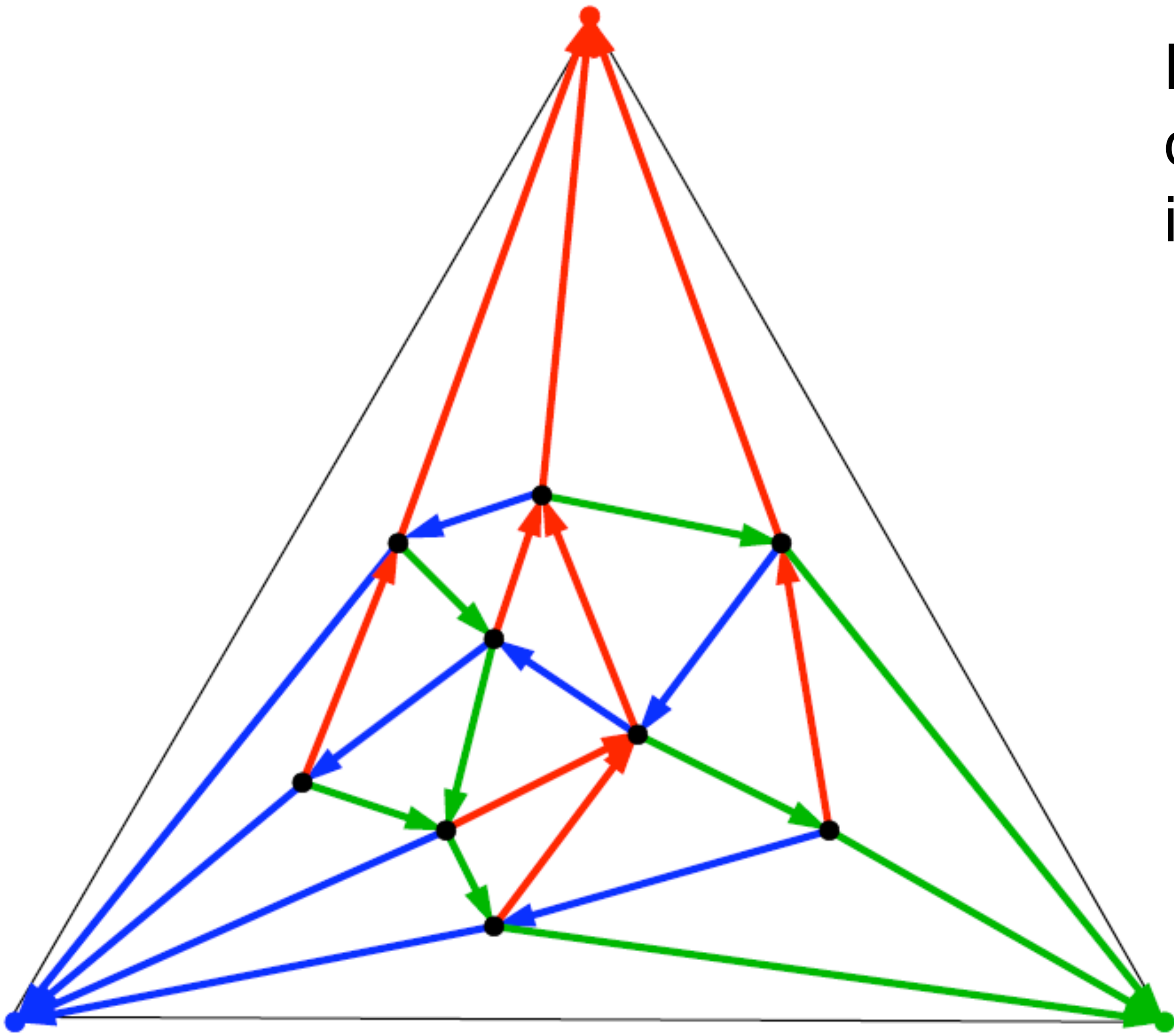


n inner vertices

↓

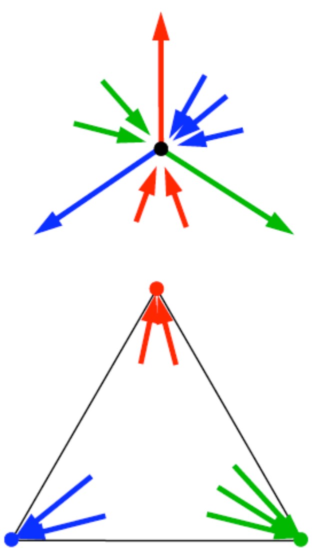
3n inner edges

Definition of Schnyder woods



Each inner edge is directed and colored in red, green or blue

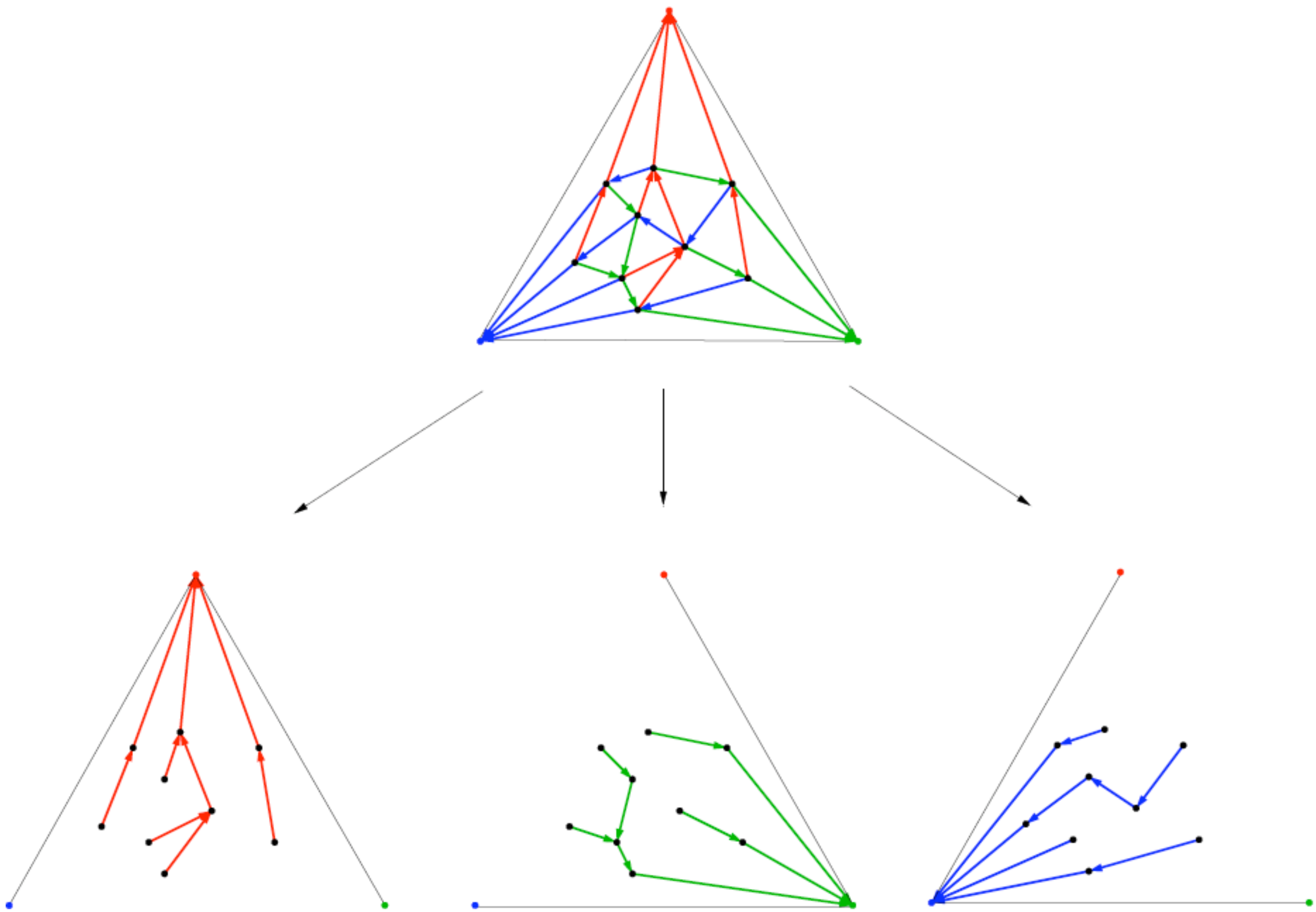
Local rules:



- Every planar triangulation admits a Schnyder wood [Schnyder'89]

Fundamental property

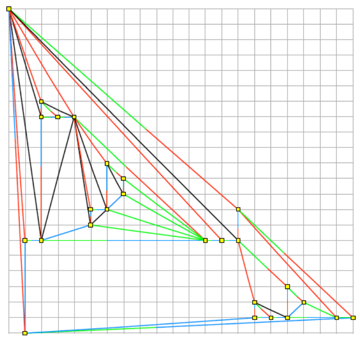
- Schnyder wood → 3 spanning trees (one for each color)



Applications of Schnyder woods

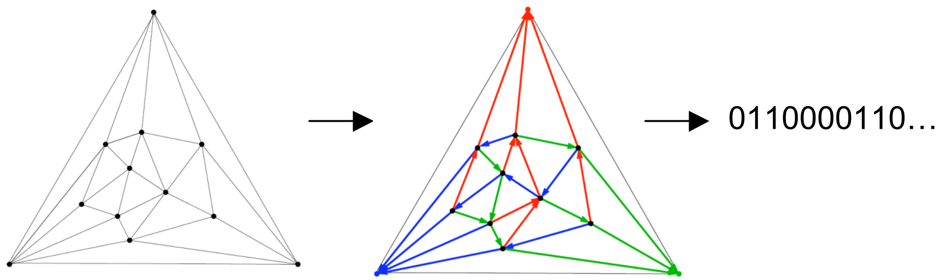
Graph drawing

[Schnyder'90, Bonichon-Felsner-Mosbah'04]



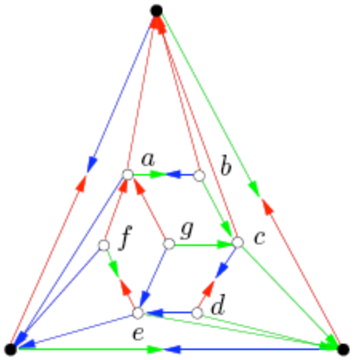
Coding

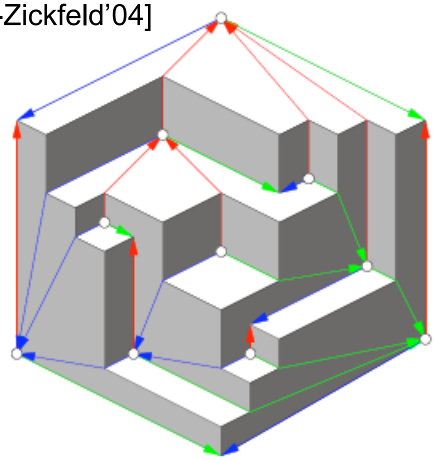
[He-Kao-Lu'99, Bernardi-Bonichon'07, Poulalhon-Schaeffer'03]



Planarity criterion

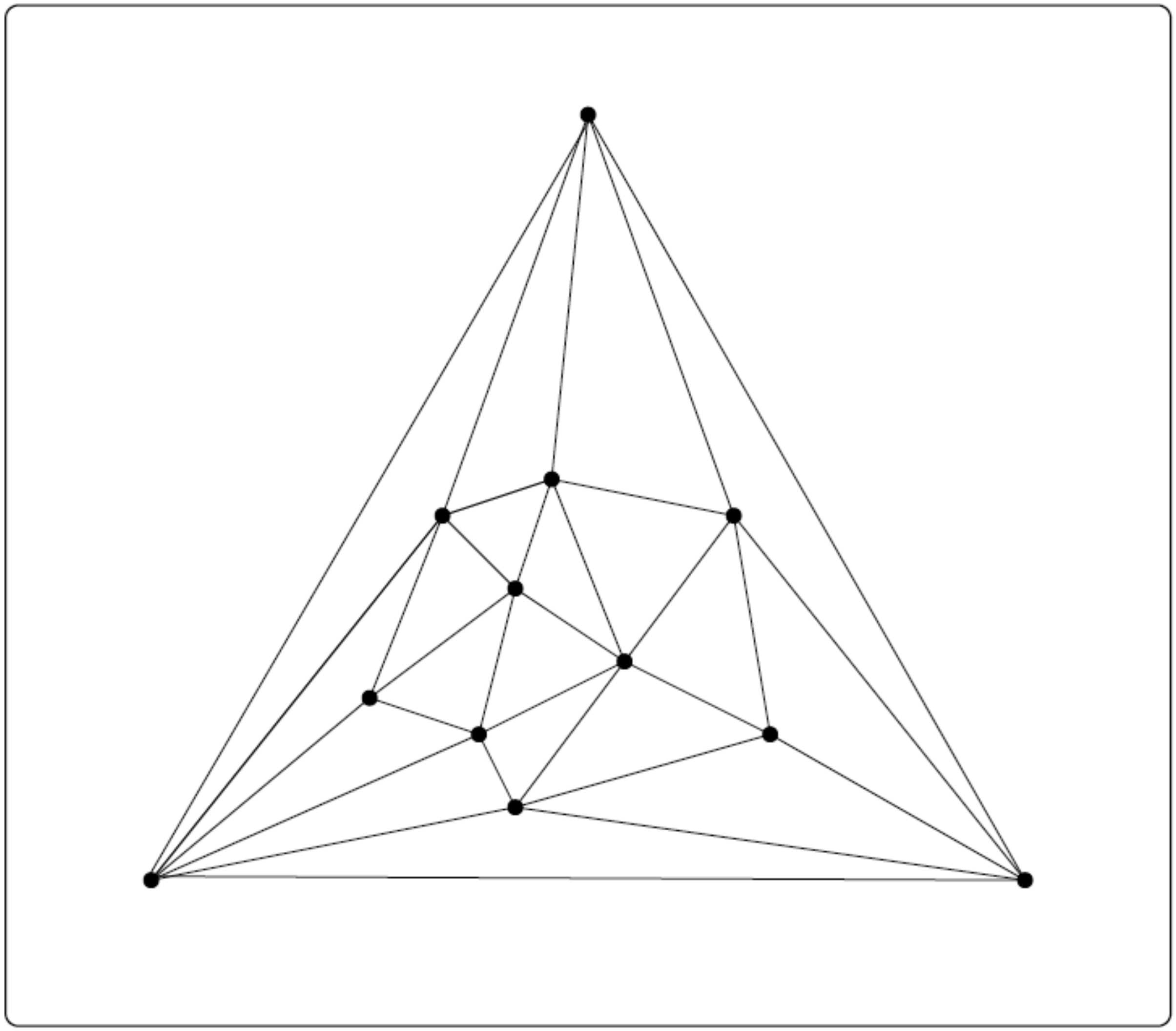
[Schnyder'89, Felsner-Zickfeld'04]





$$\mathbf{G} = (\mathbf{V}, \mathbf{E}) \text{ planar} \Leftrightarrow (V \cup E, \subseteq) \text{ has dimension} \leq 3$$

Computing a Schnyder wood



Traversal algorithm: the faces are conquerred progressively

[Schnyder'89] reformulated by [Brehm'03]

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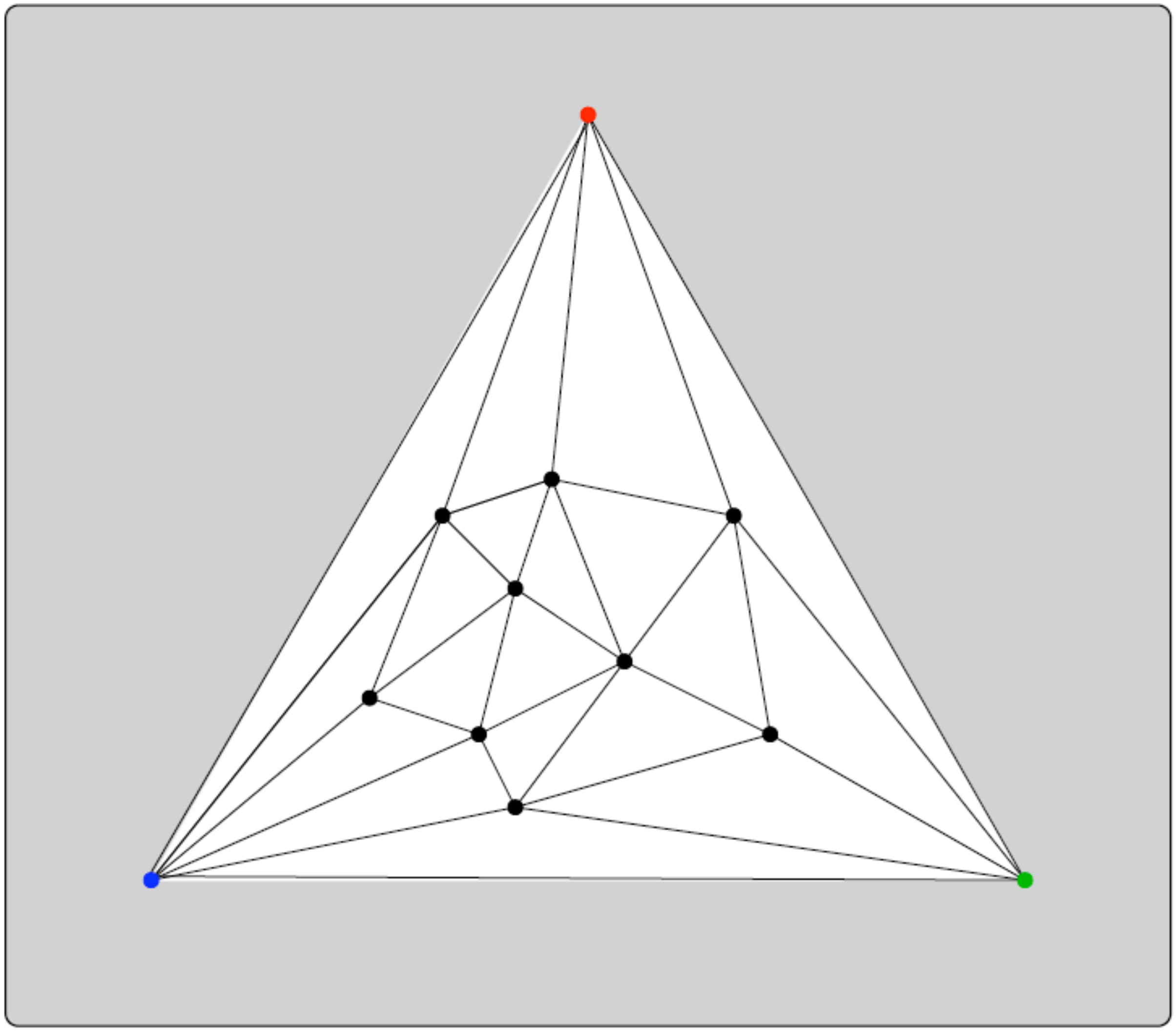
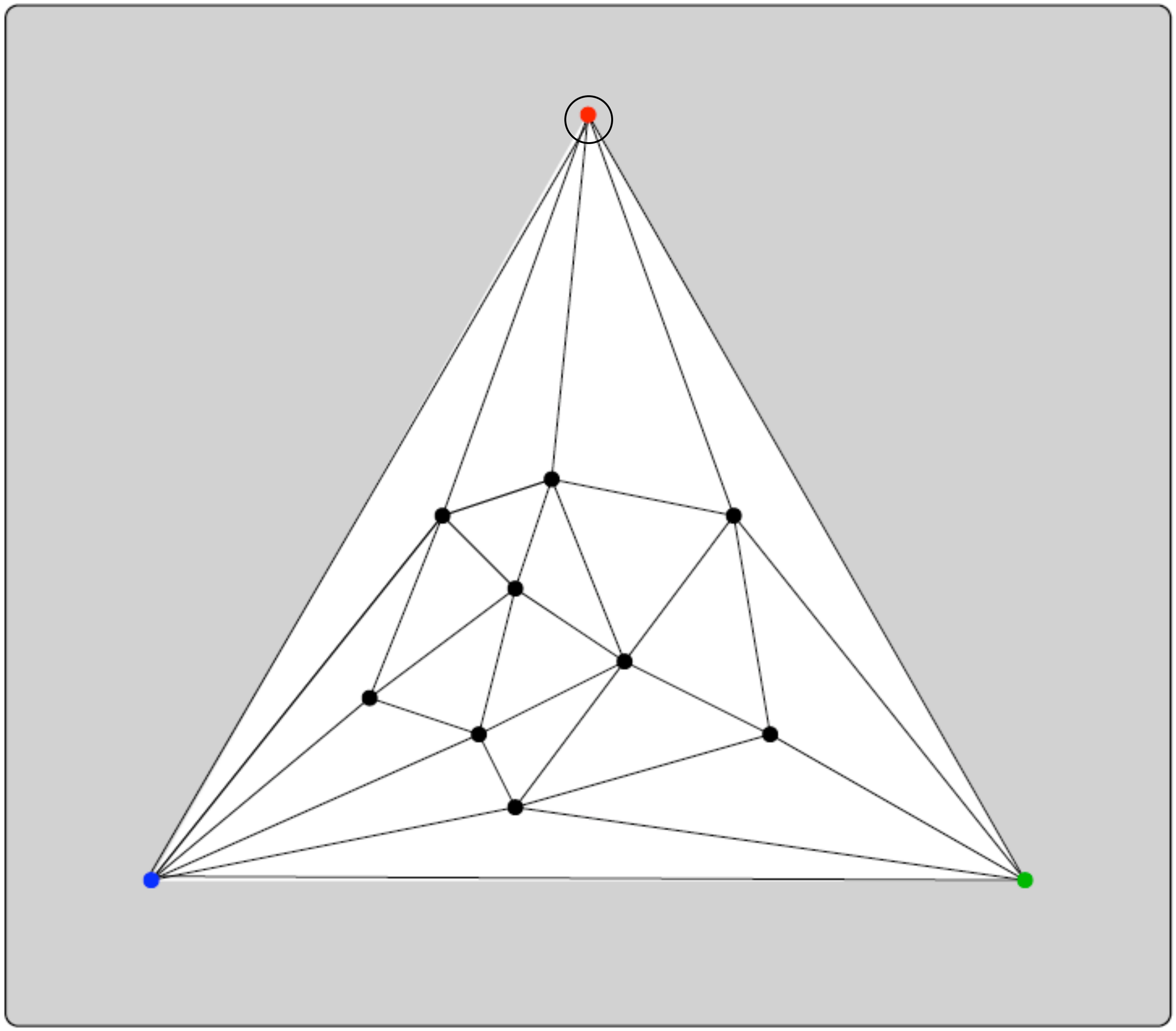


First step: Conquer
The outer face

Computing a Schnyder wood



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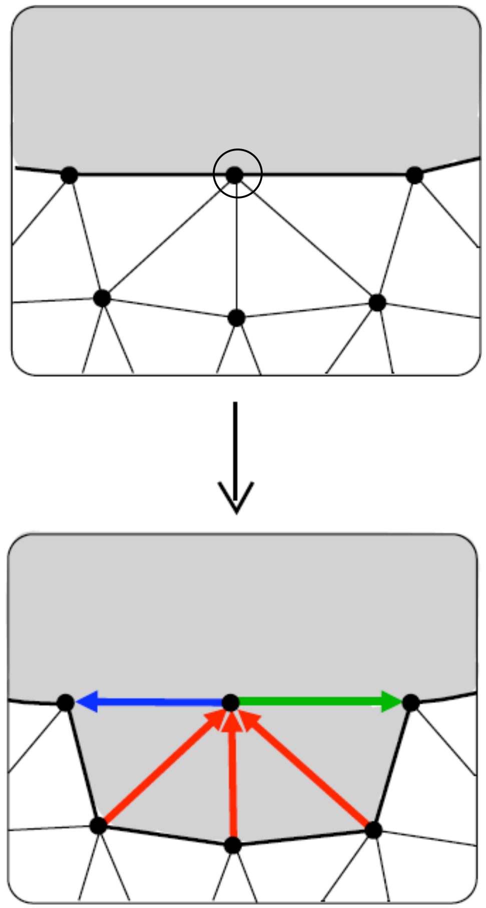

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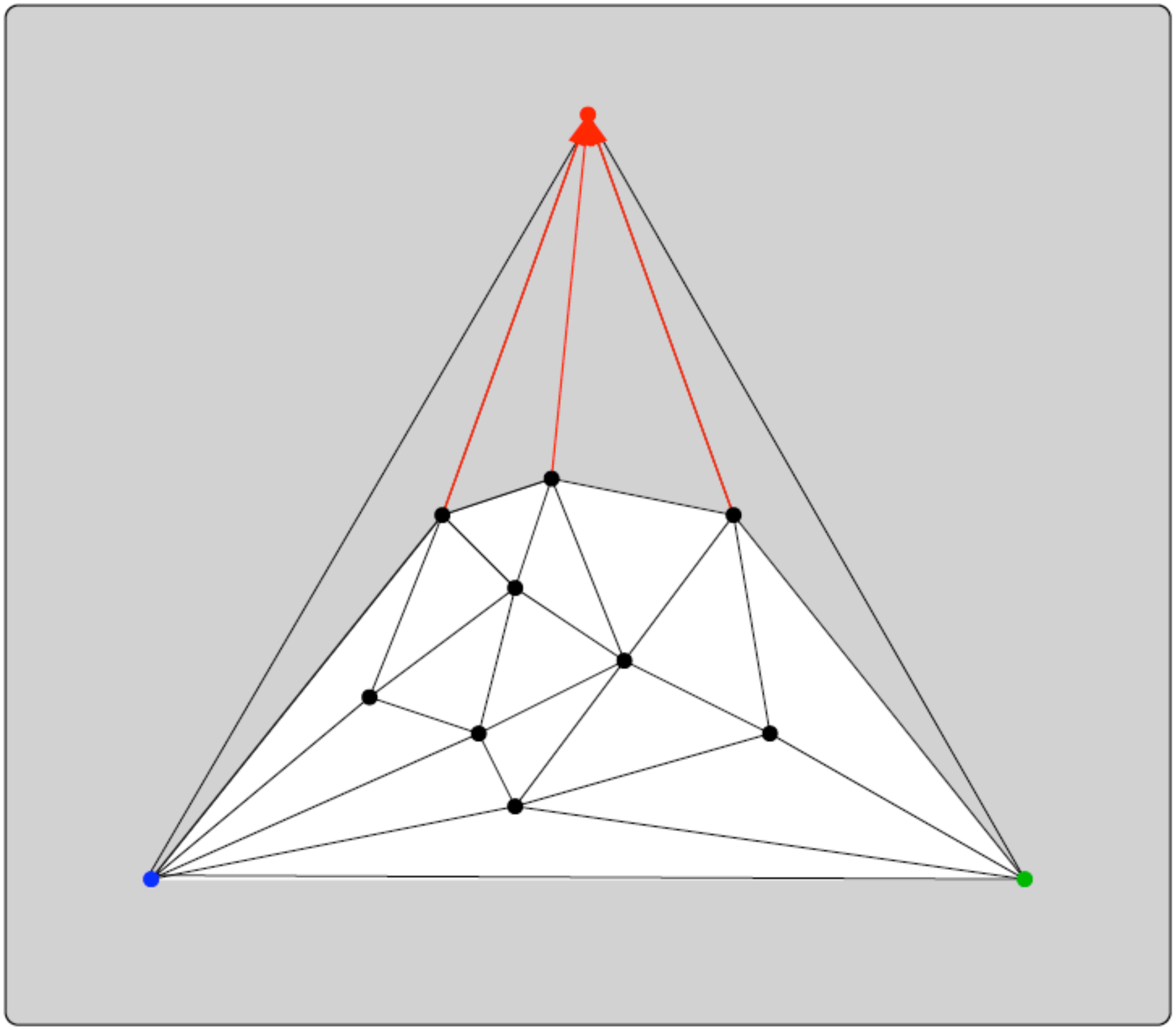

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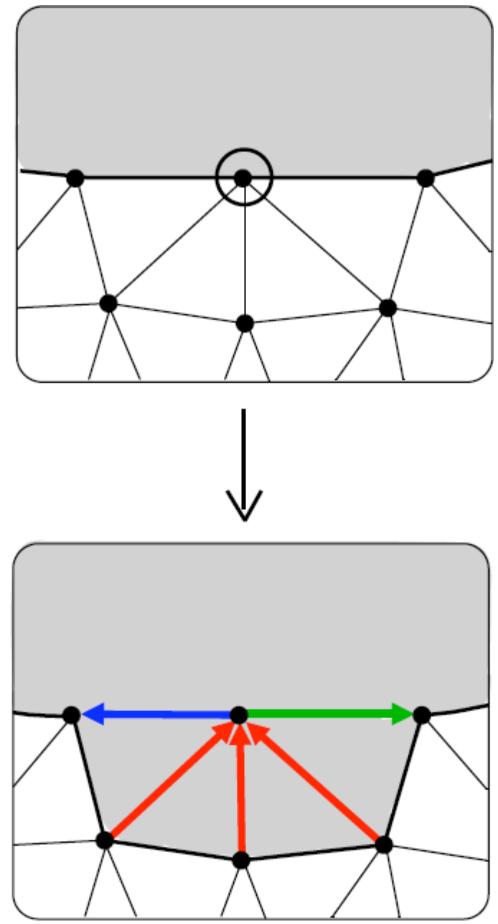

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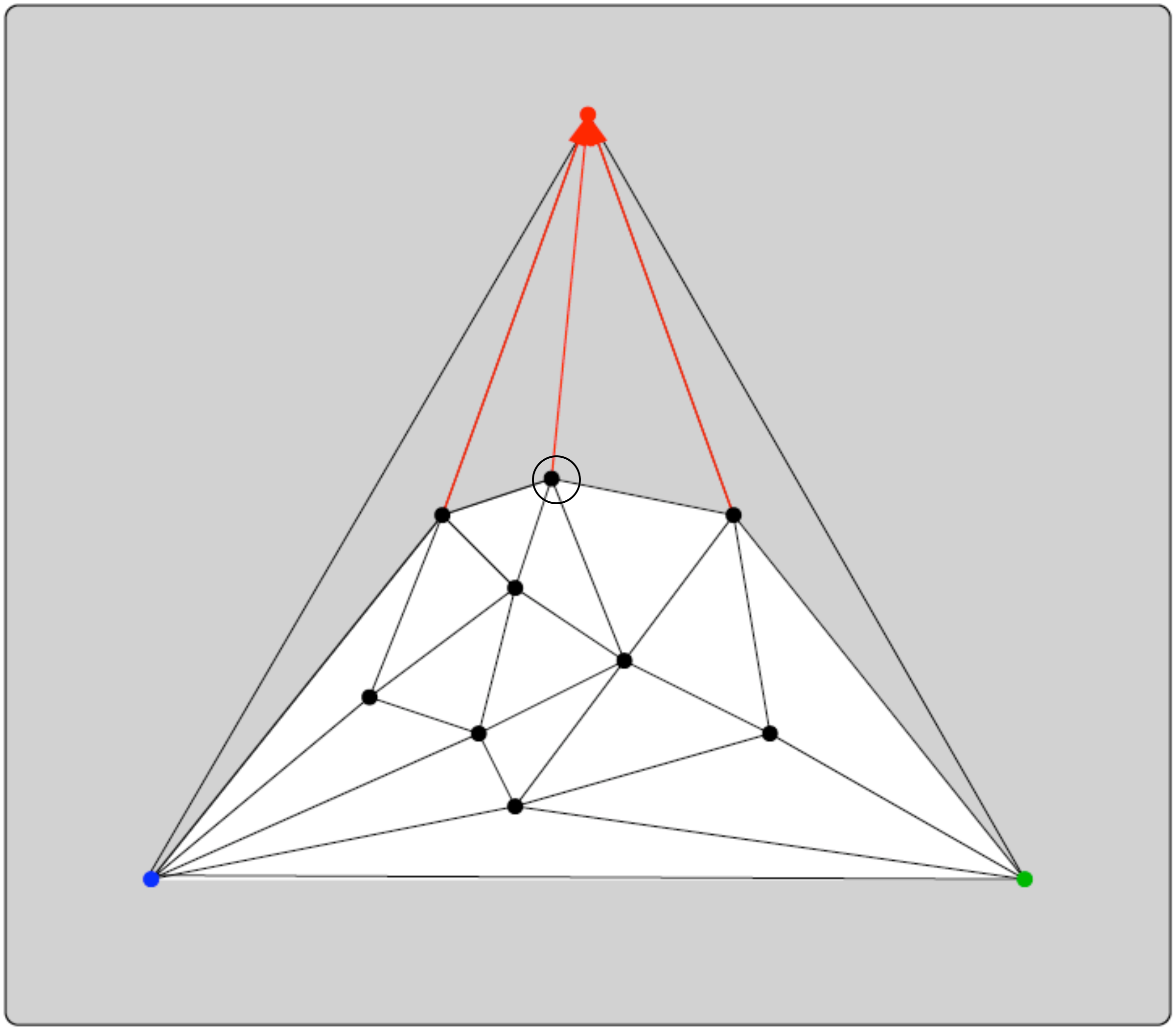

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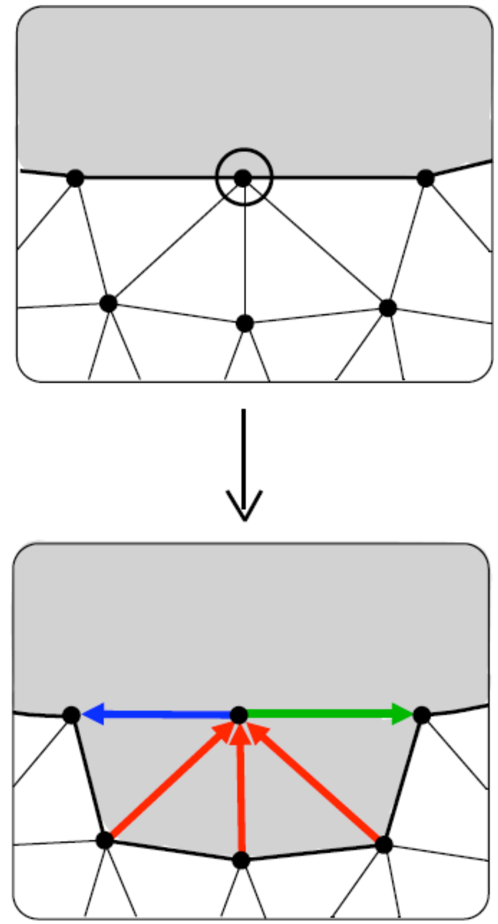

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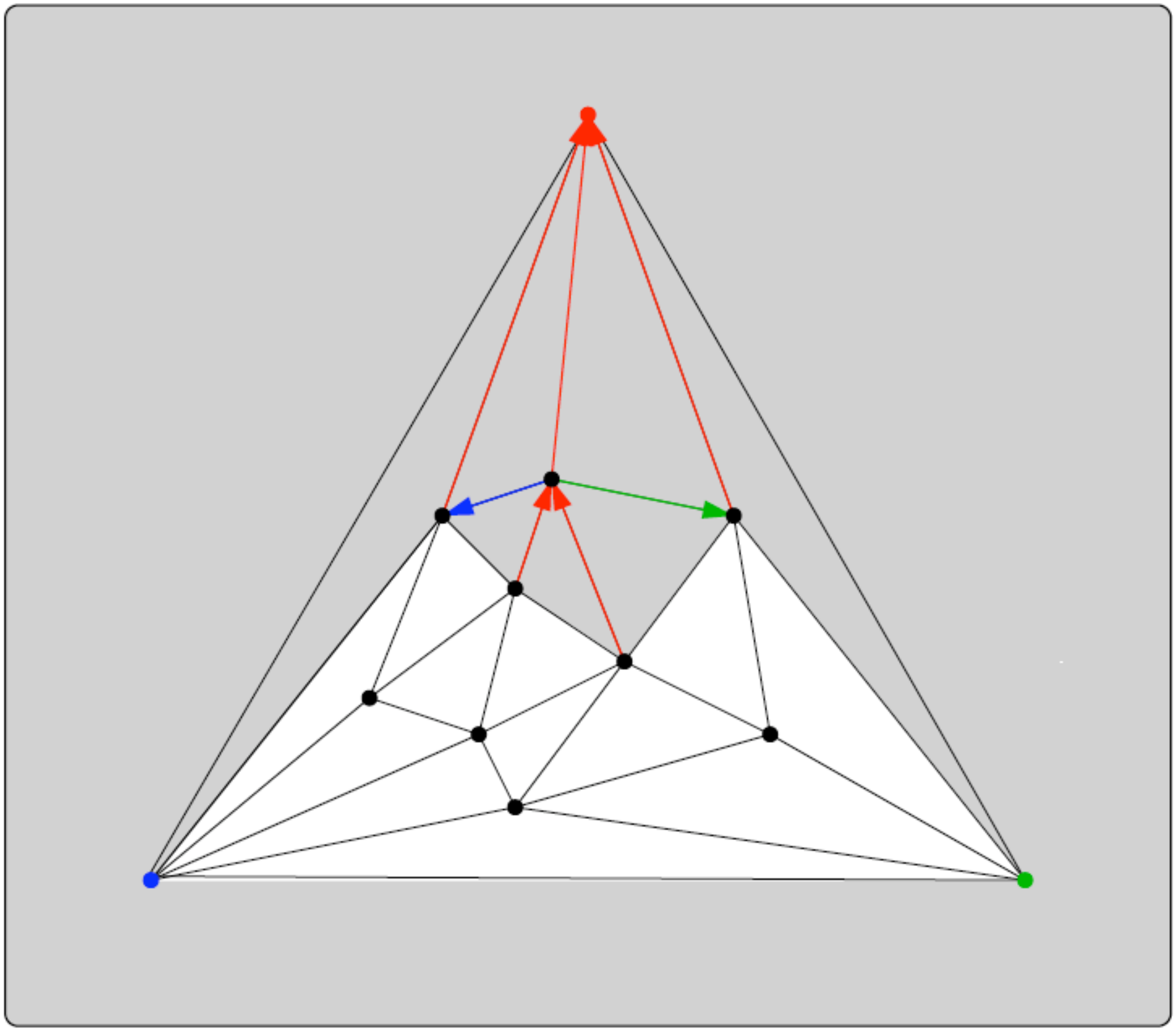

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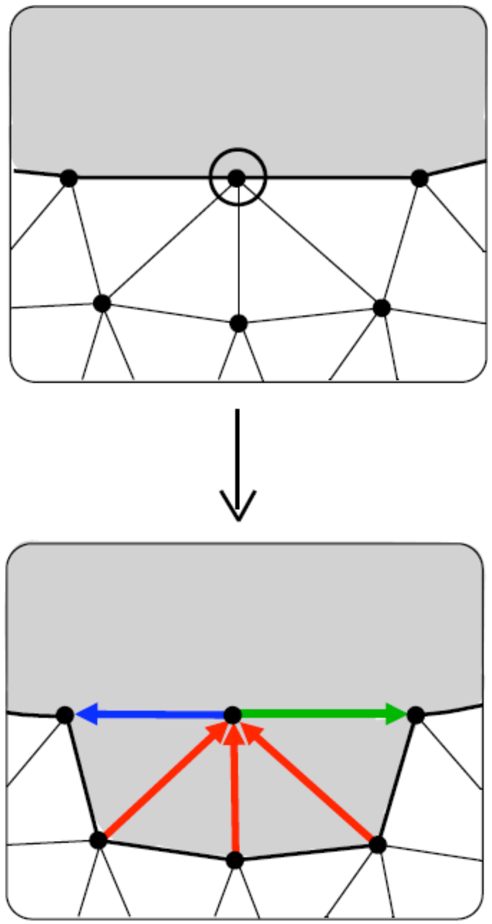

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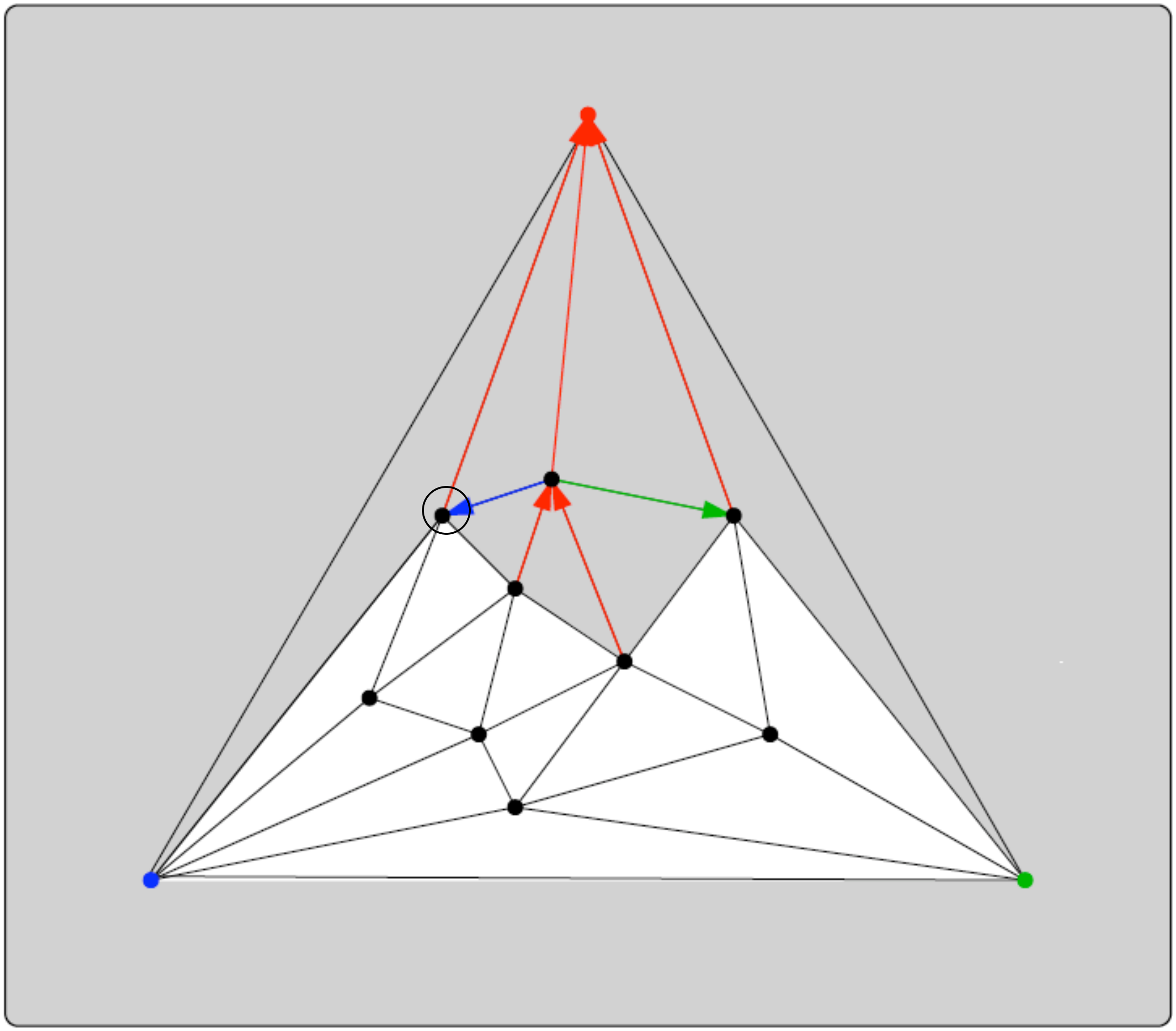

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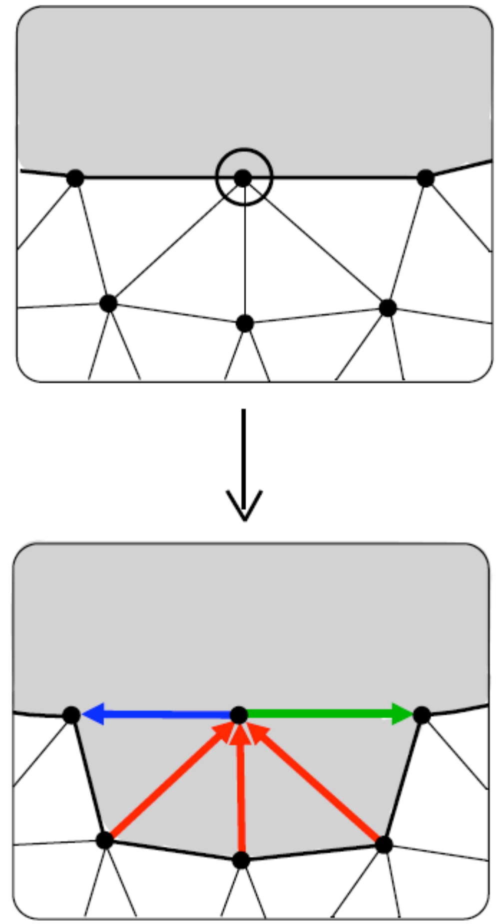

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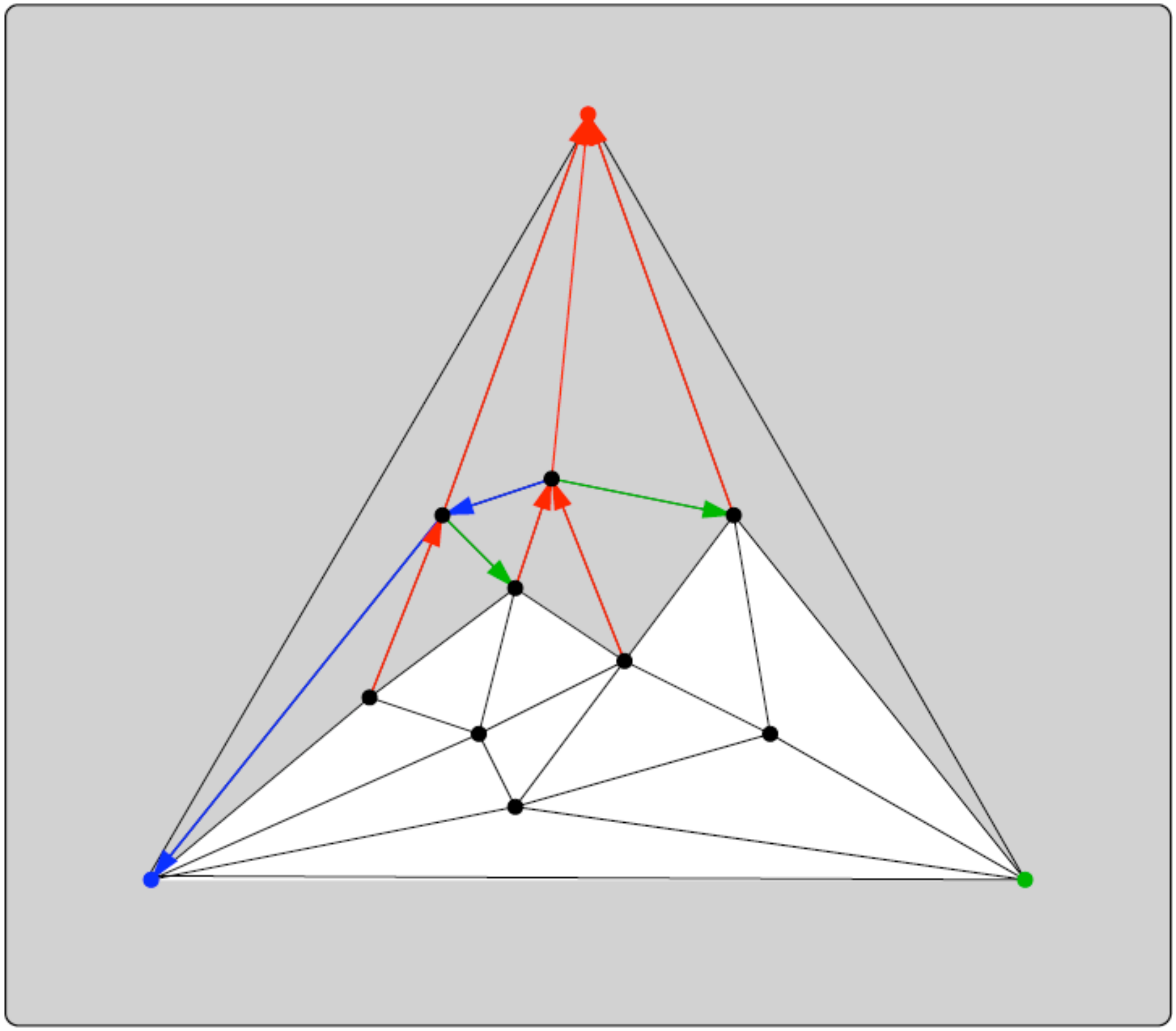

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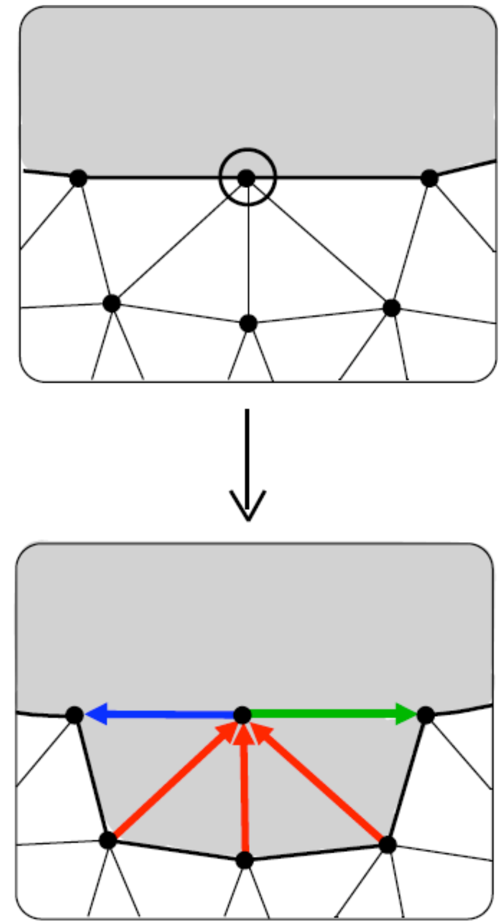

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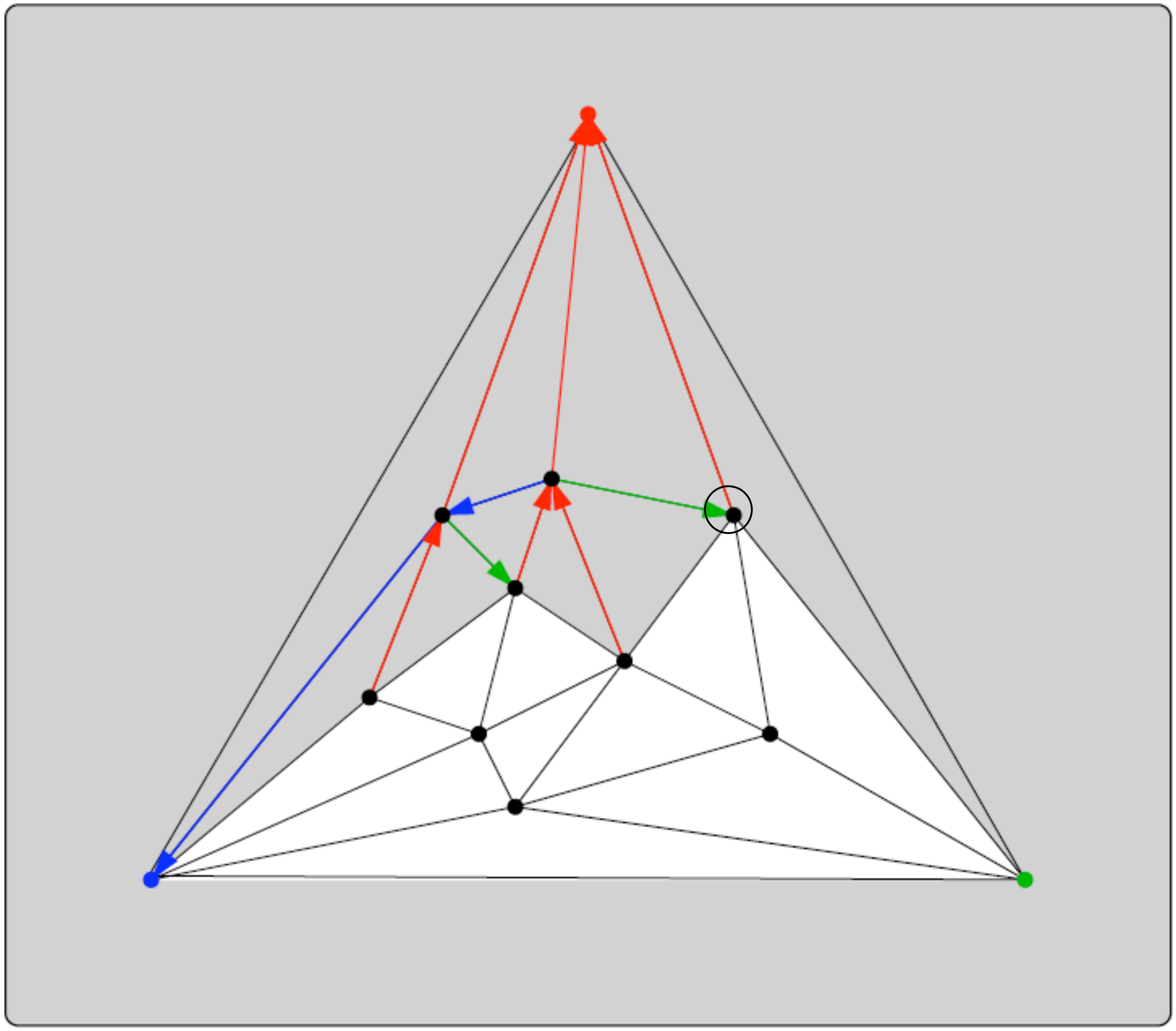

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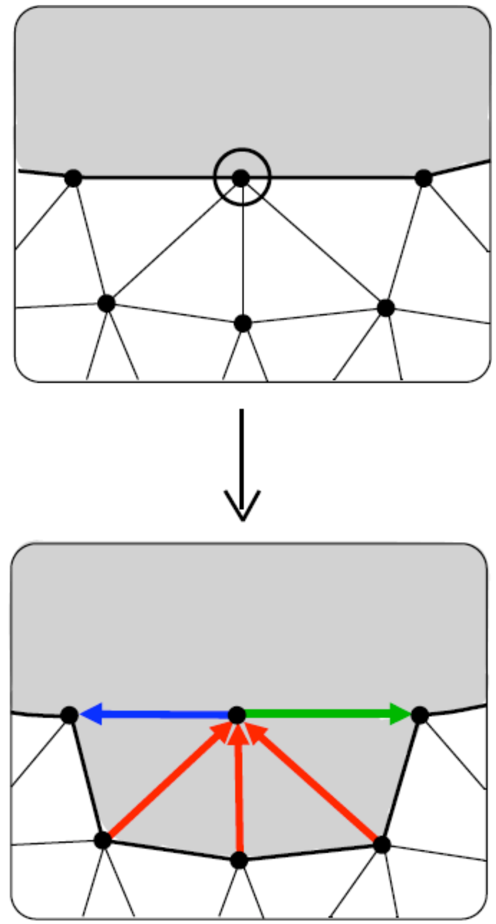

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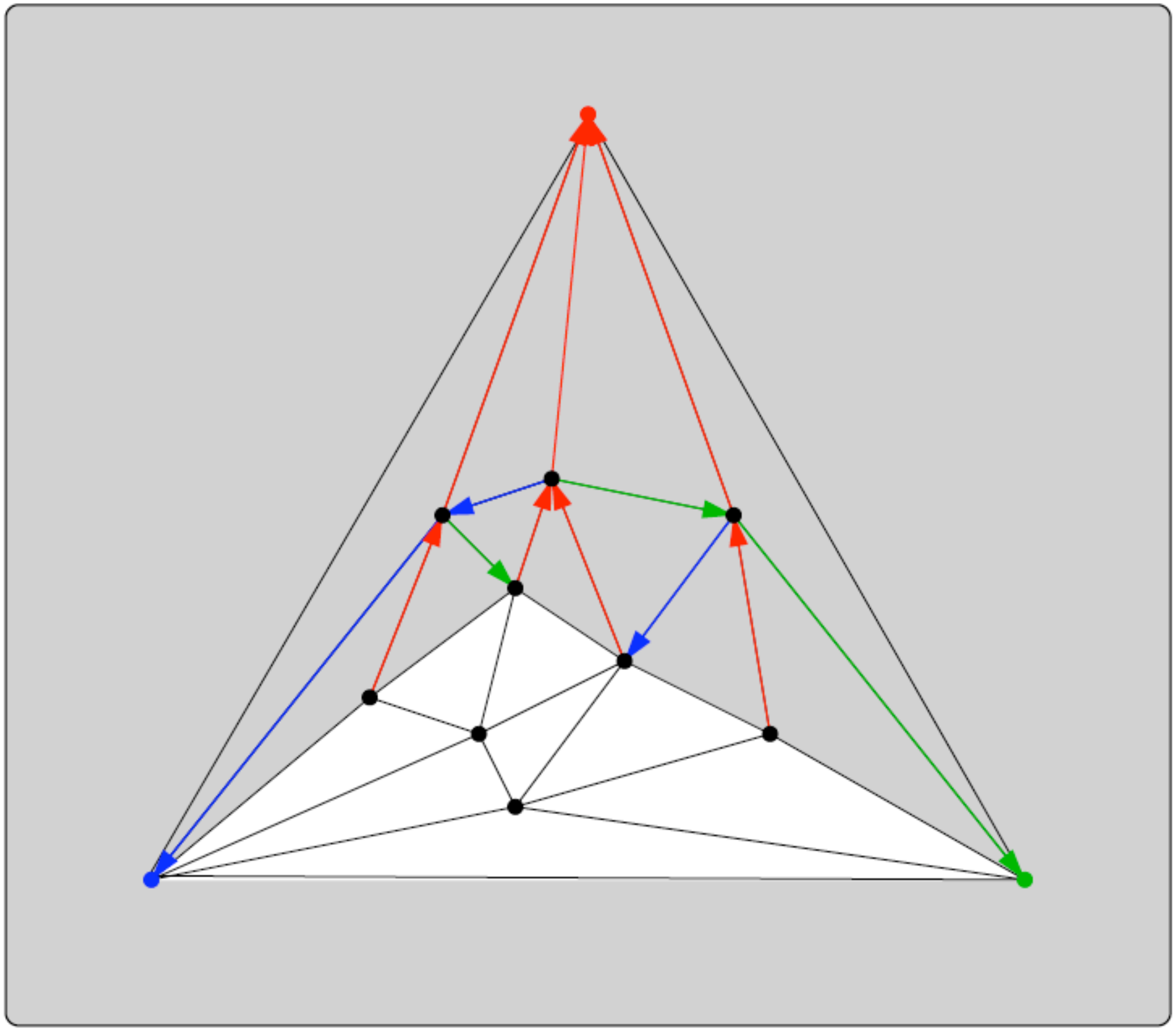

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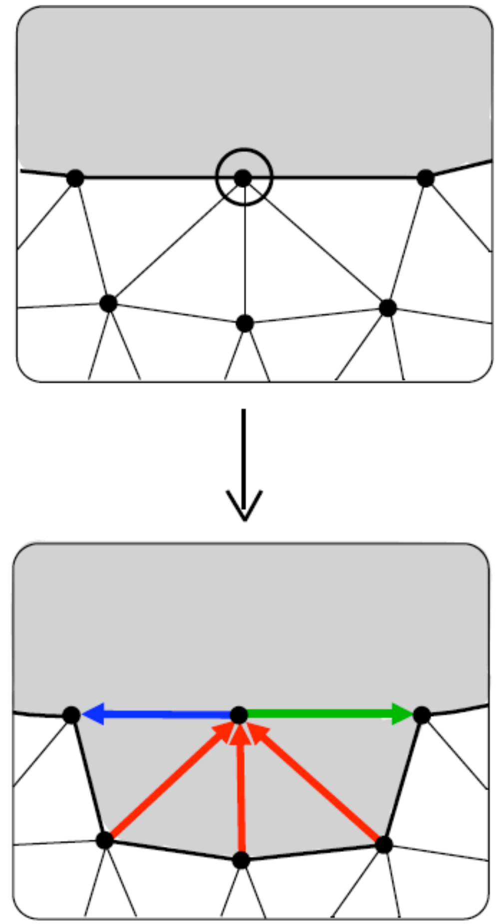

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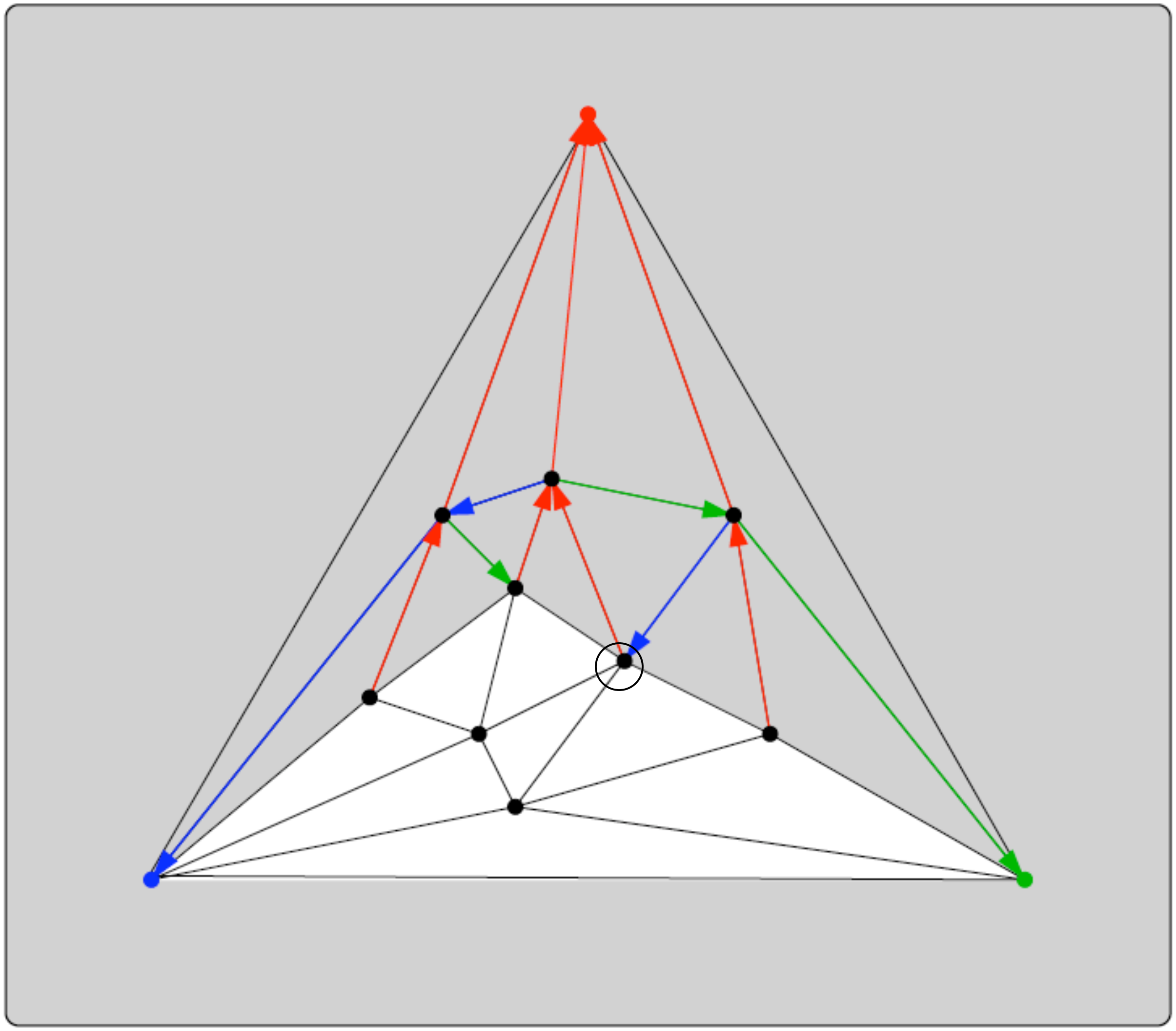

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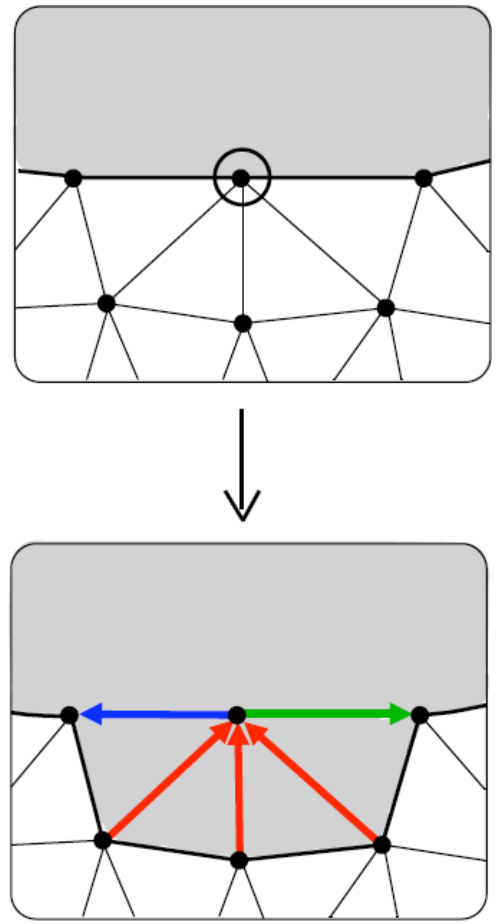

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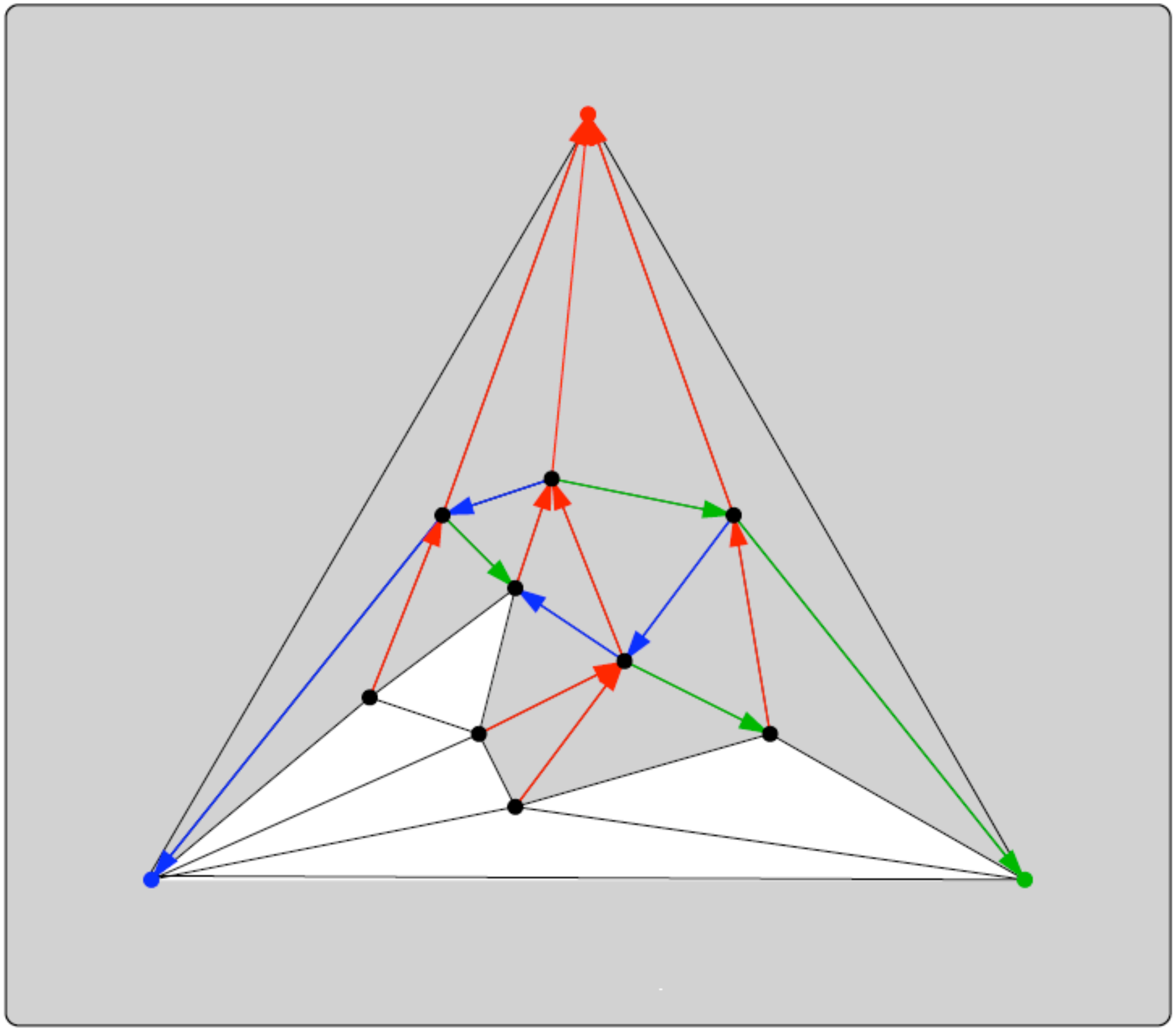

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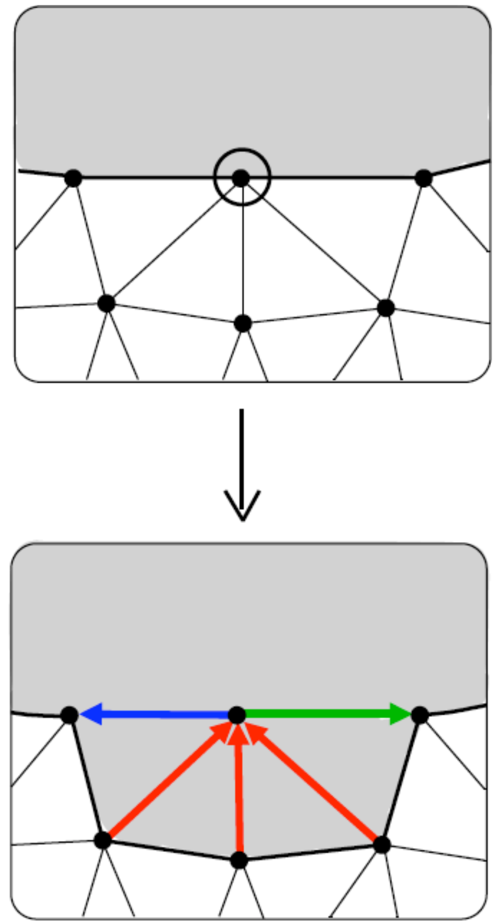

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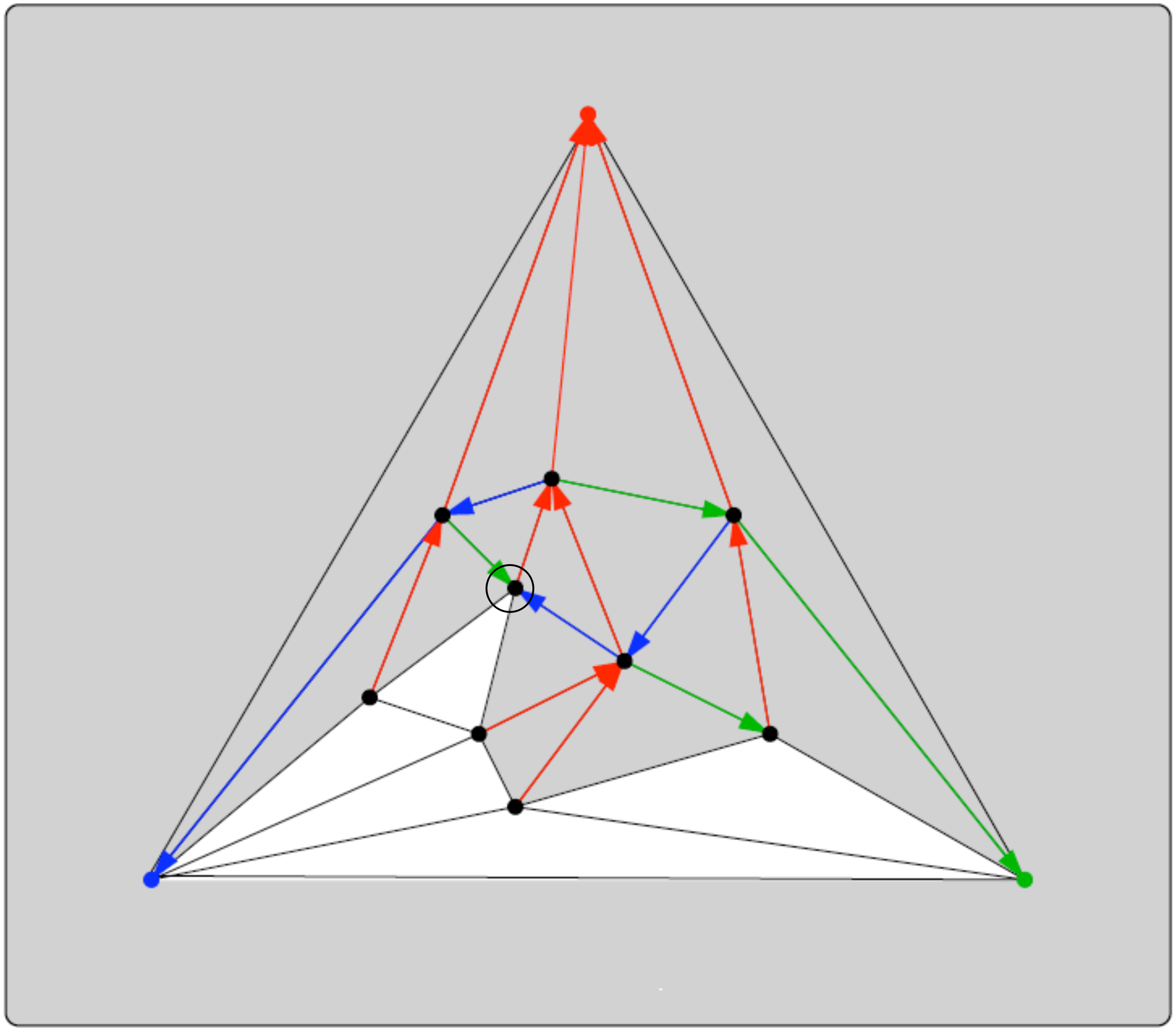

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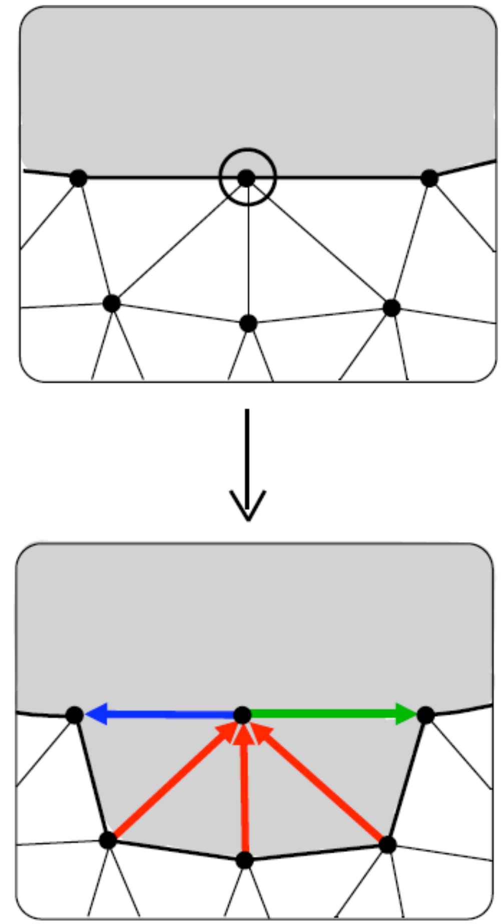

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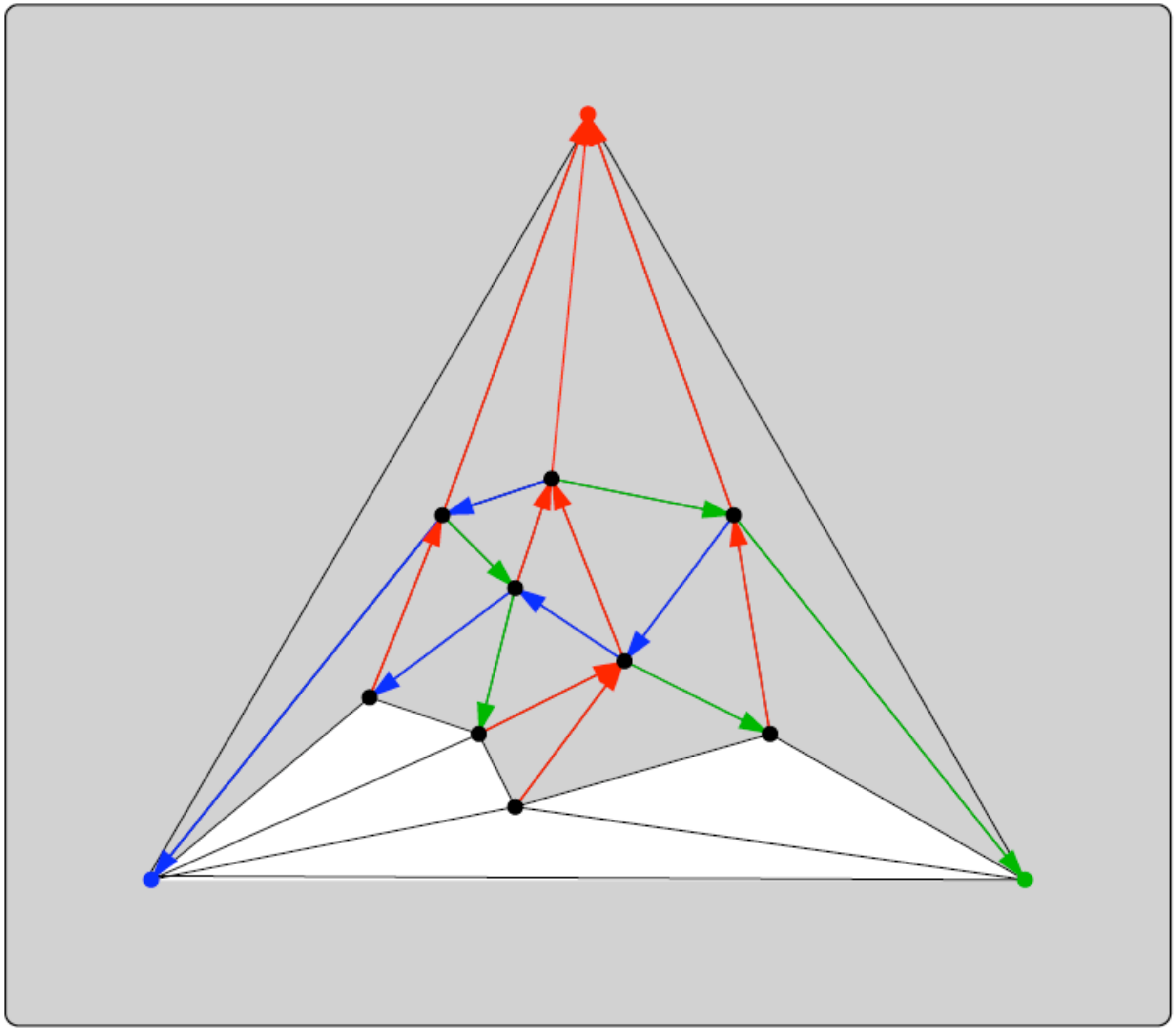

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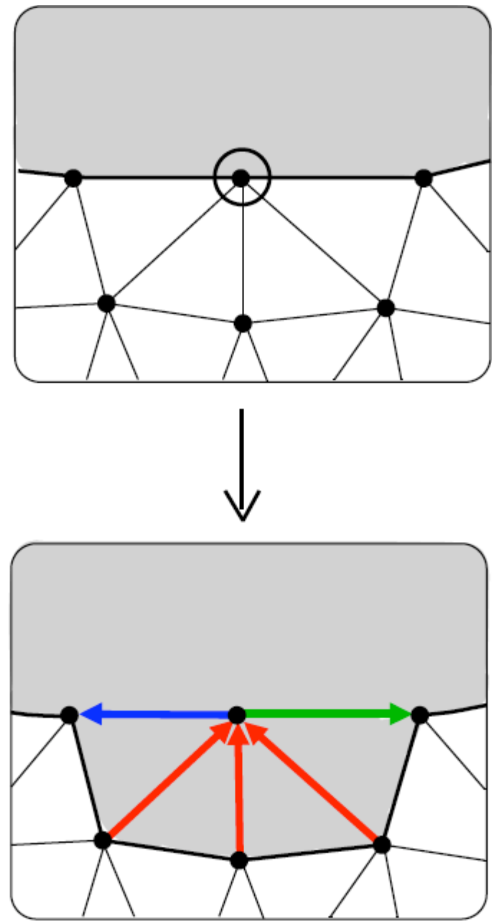

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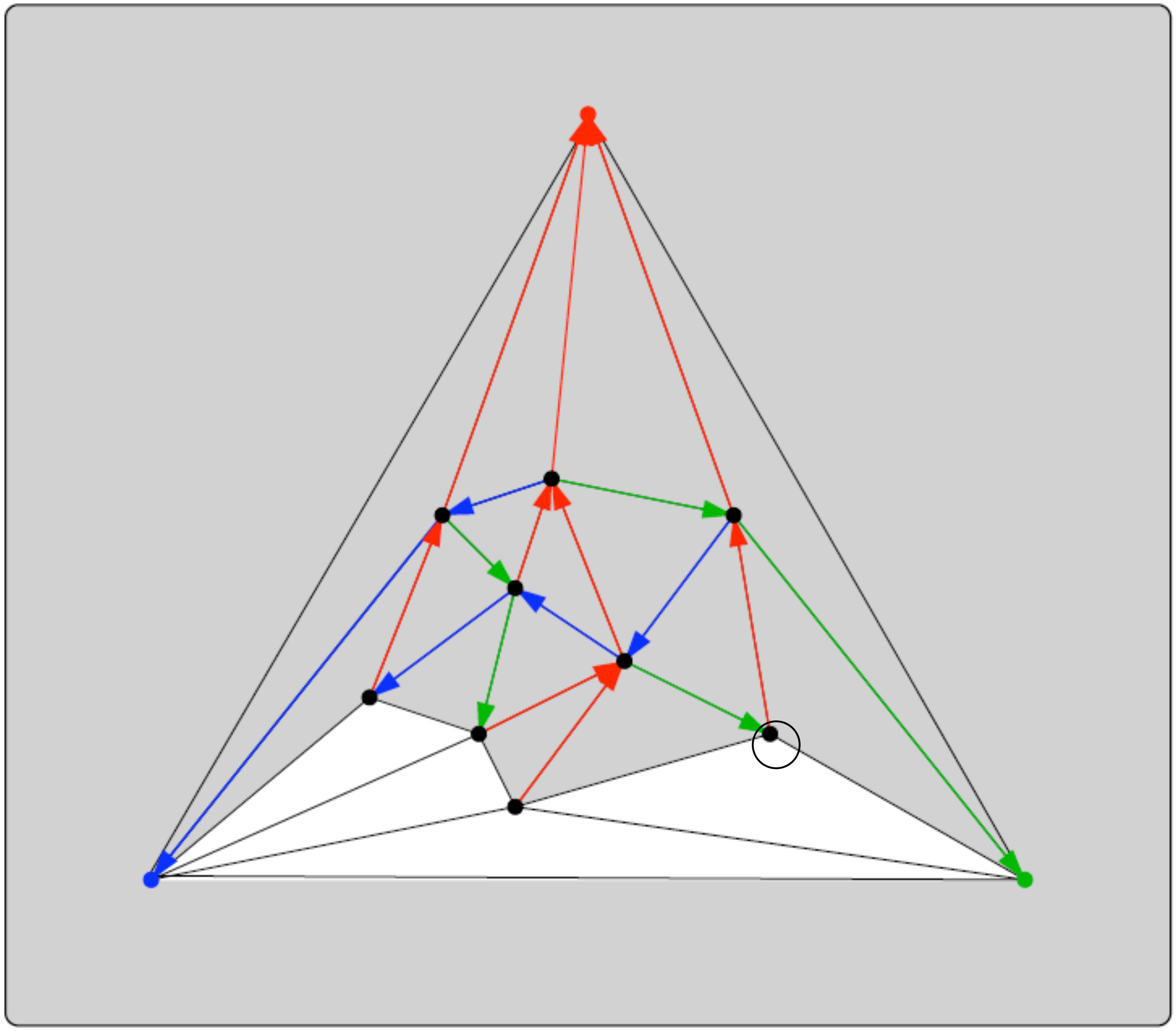

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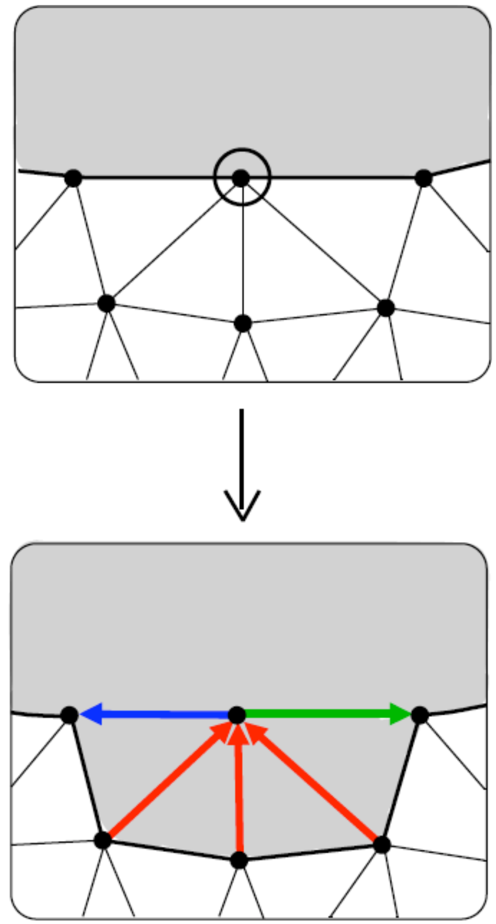

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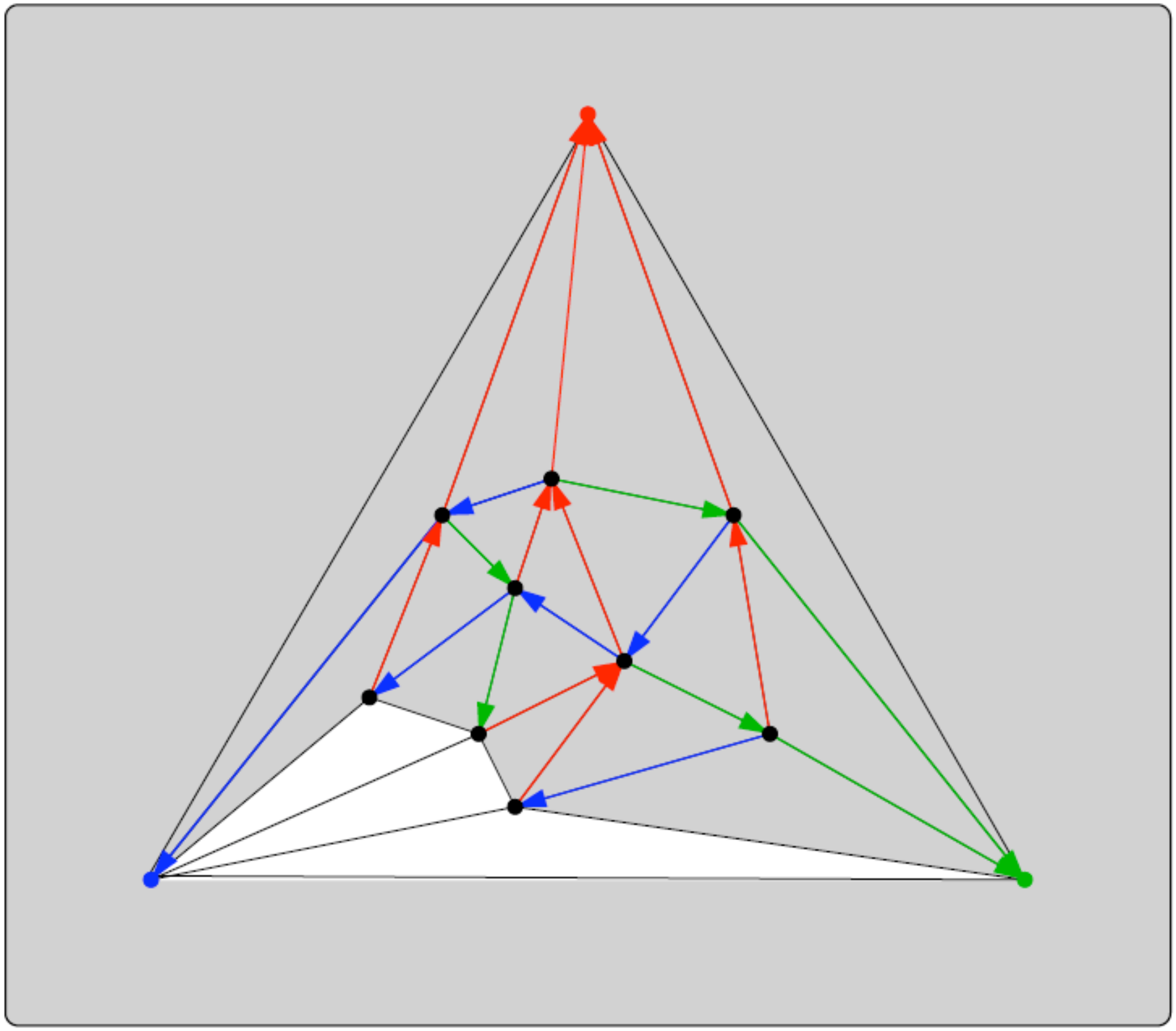

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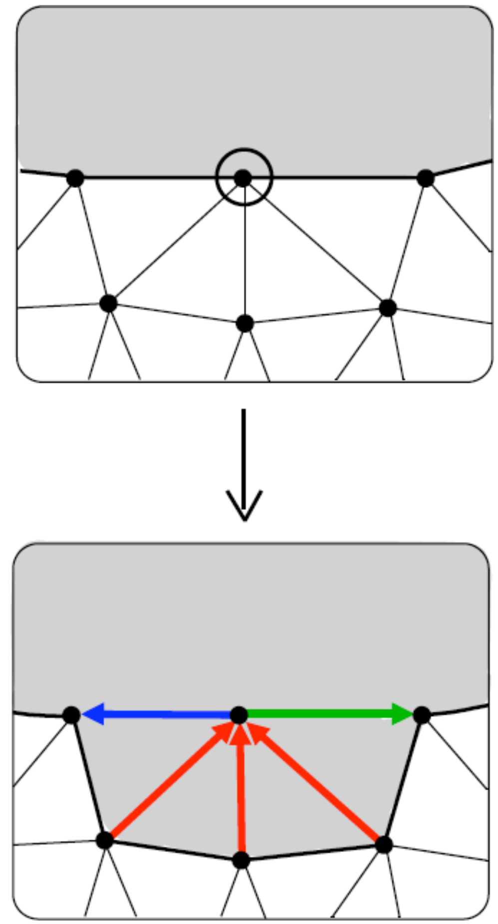

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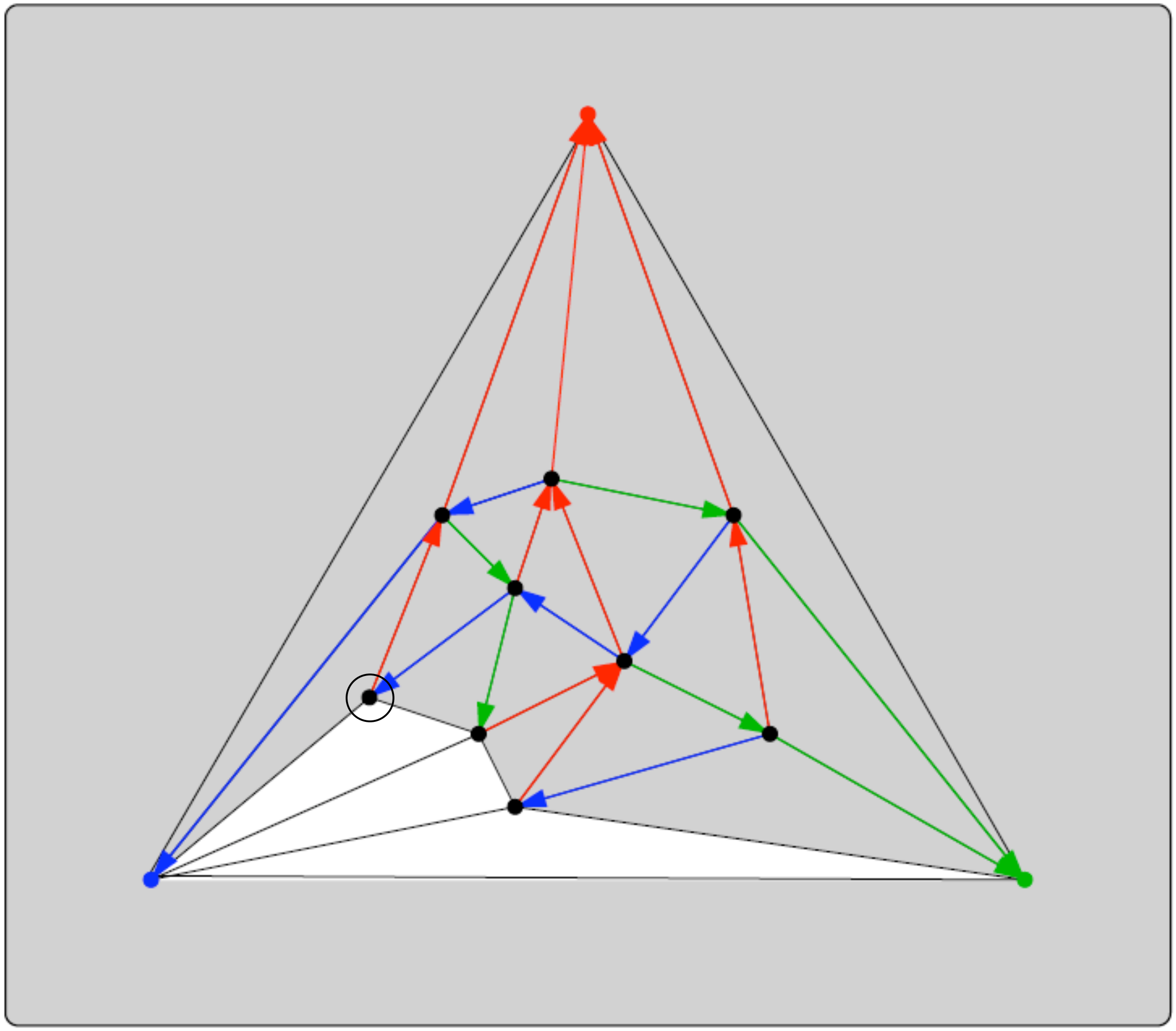

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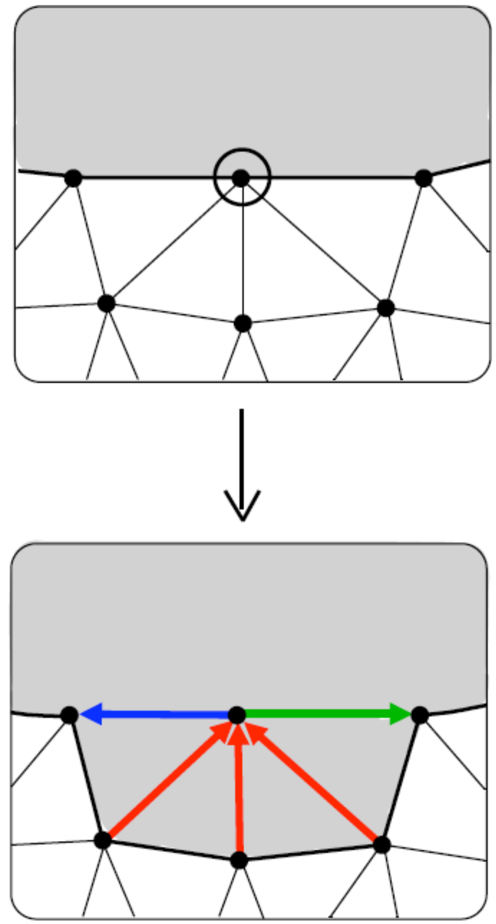

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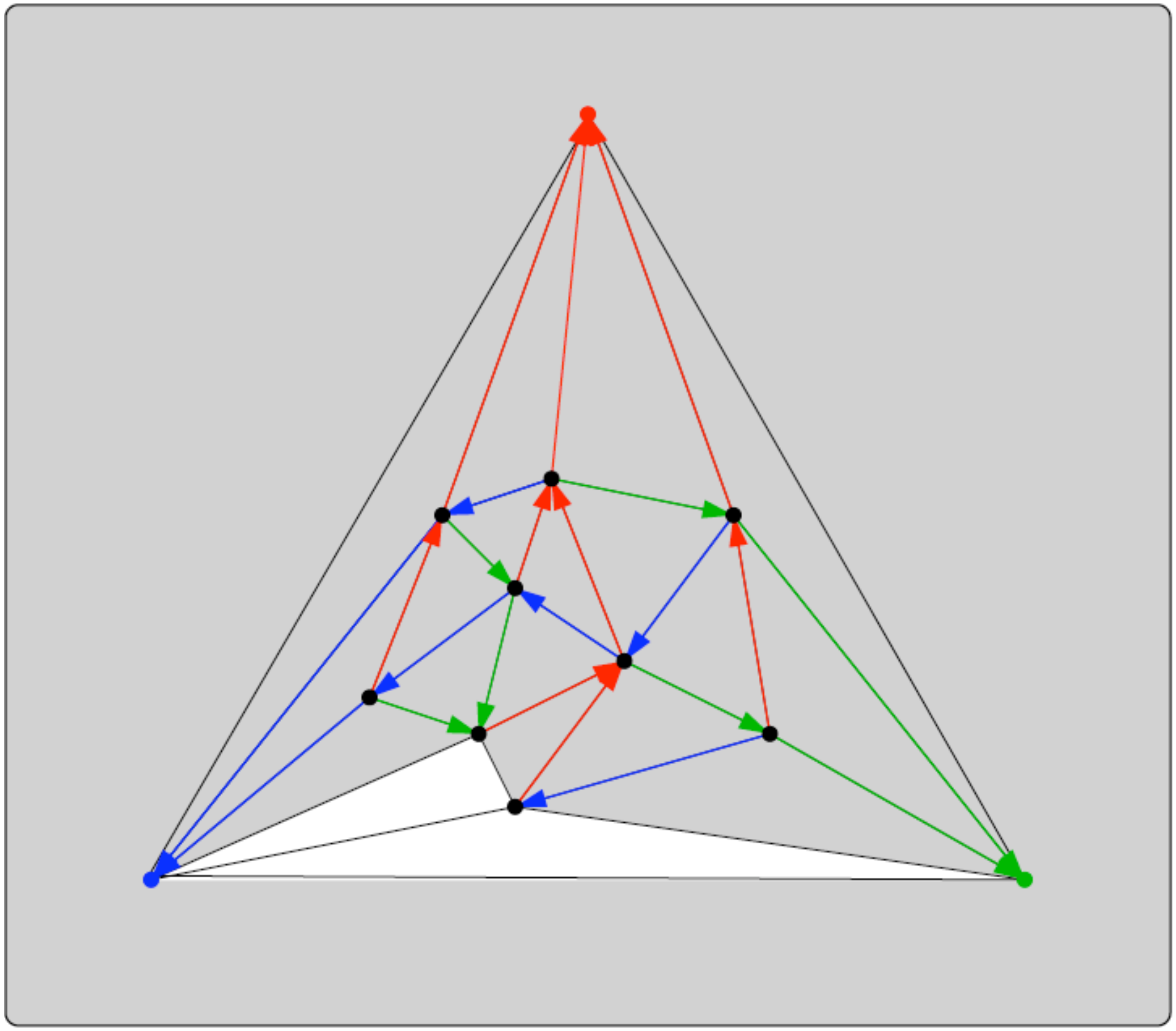

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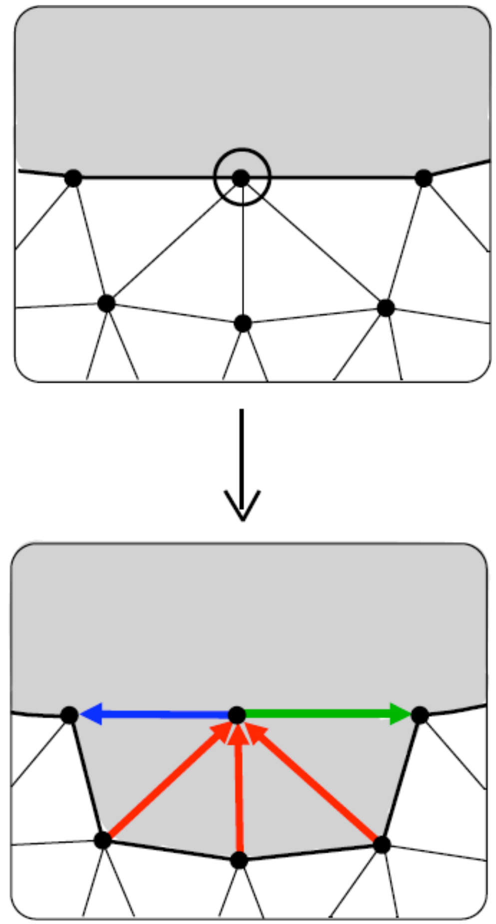

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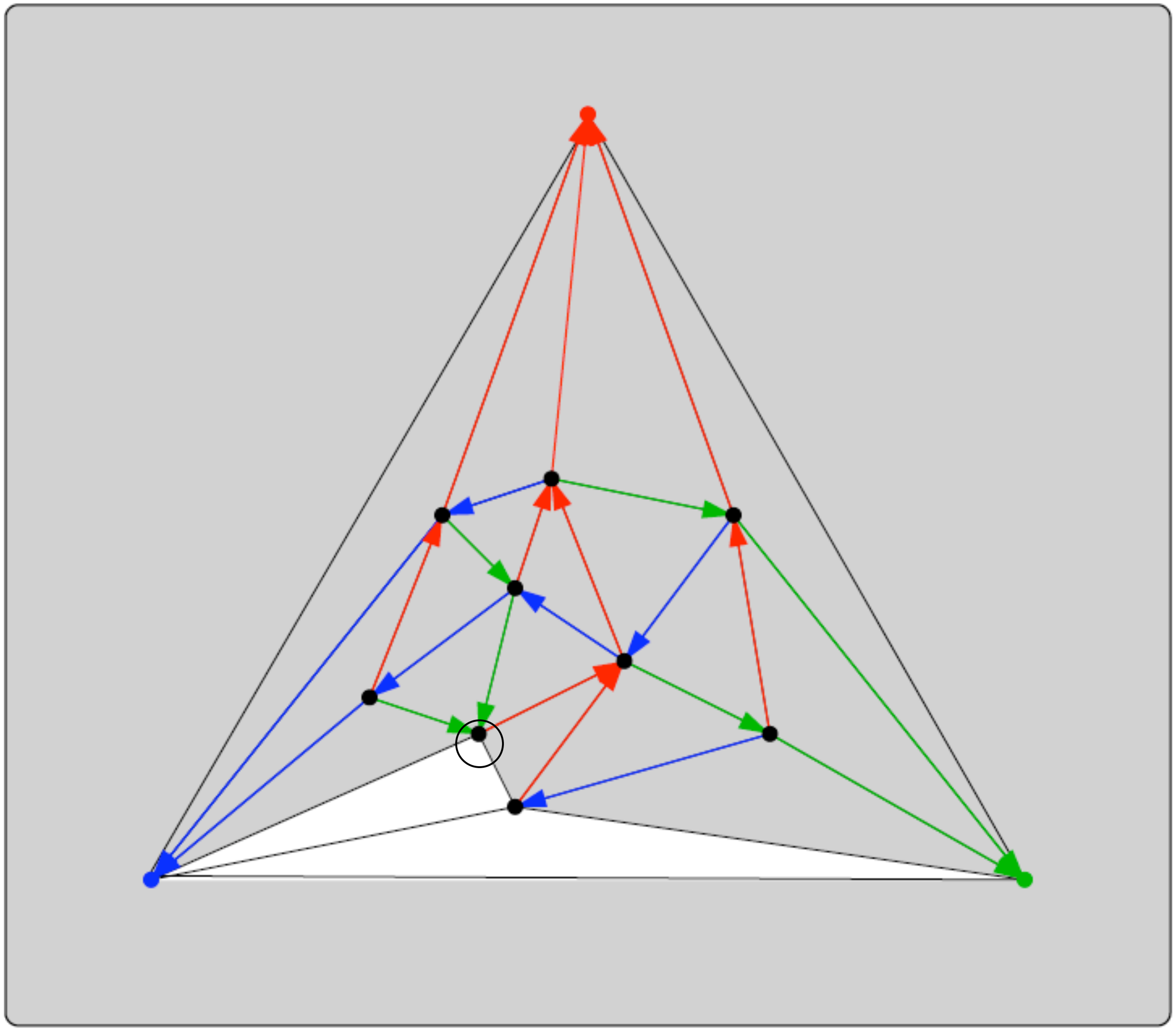

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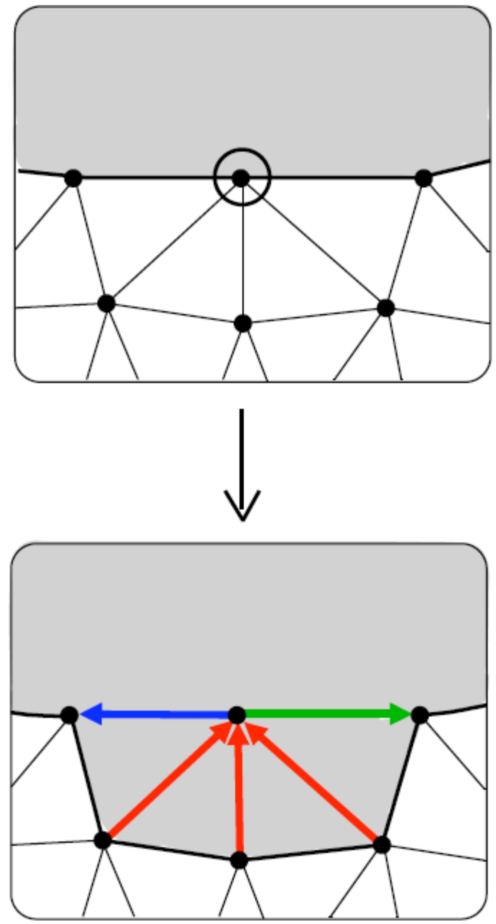

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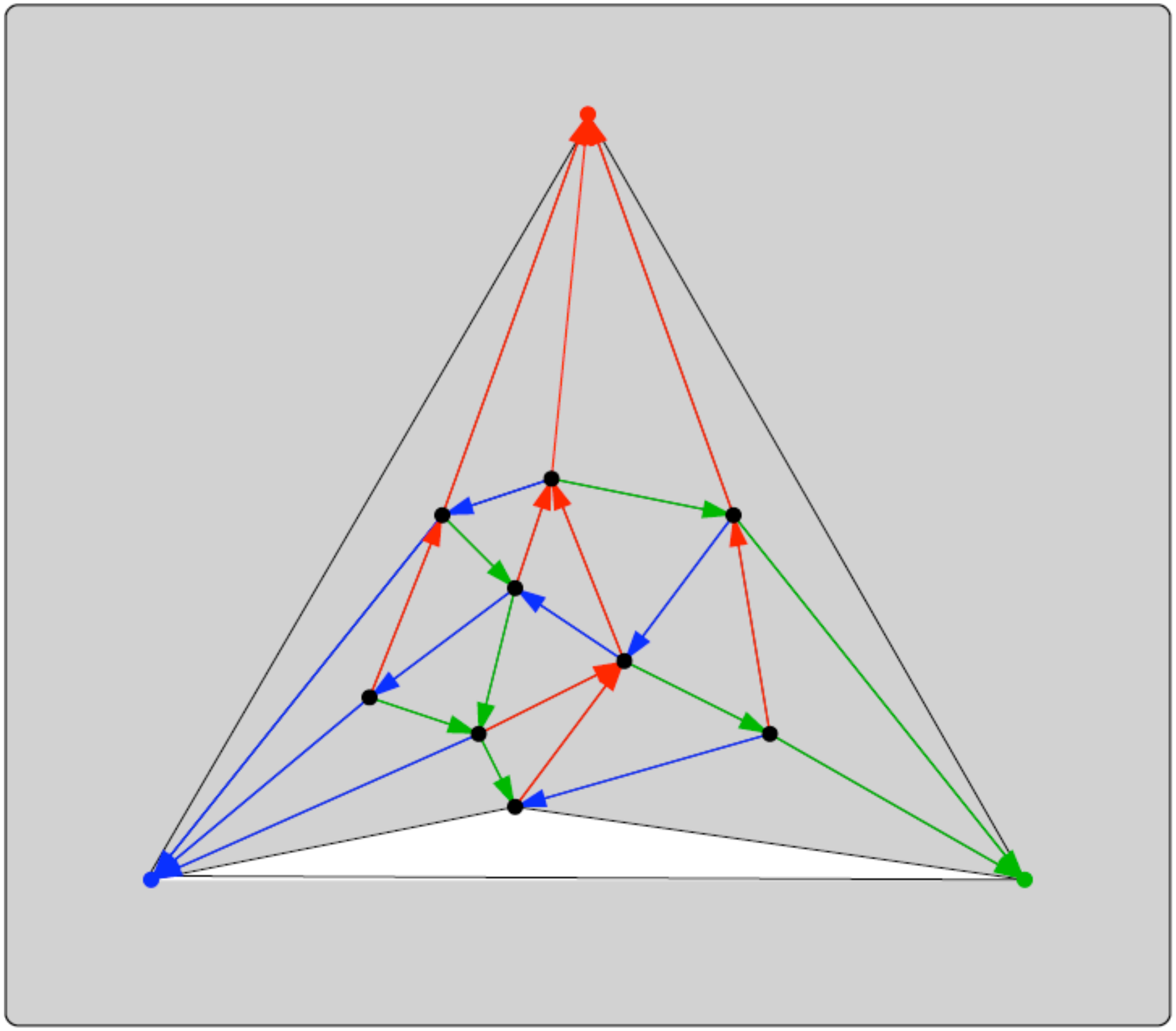

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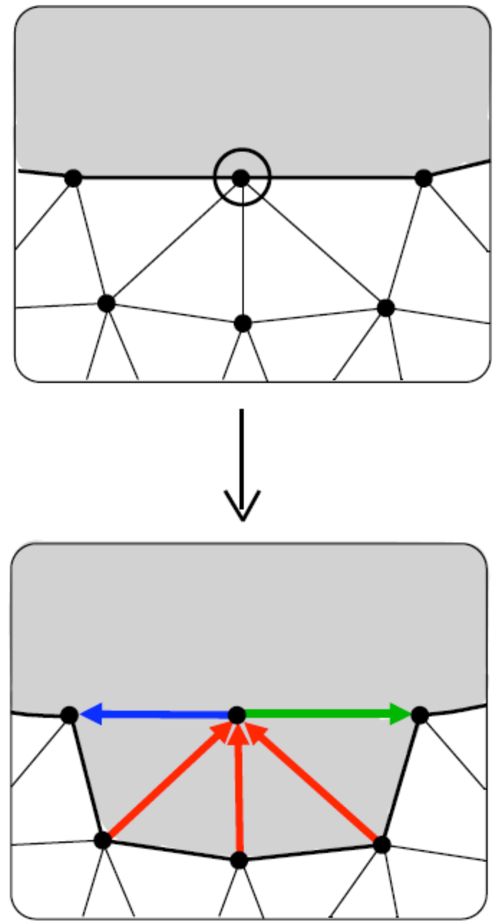

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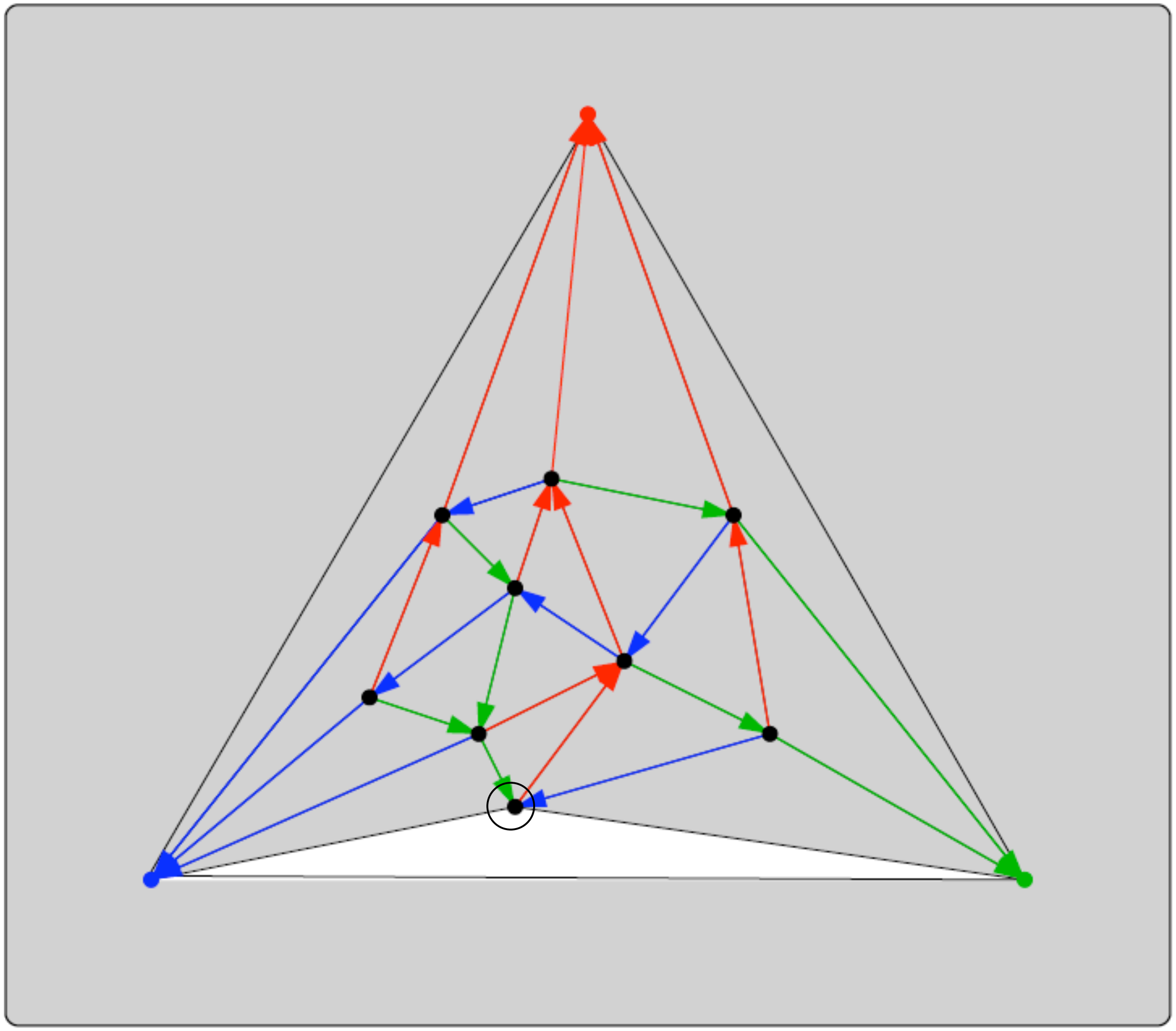

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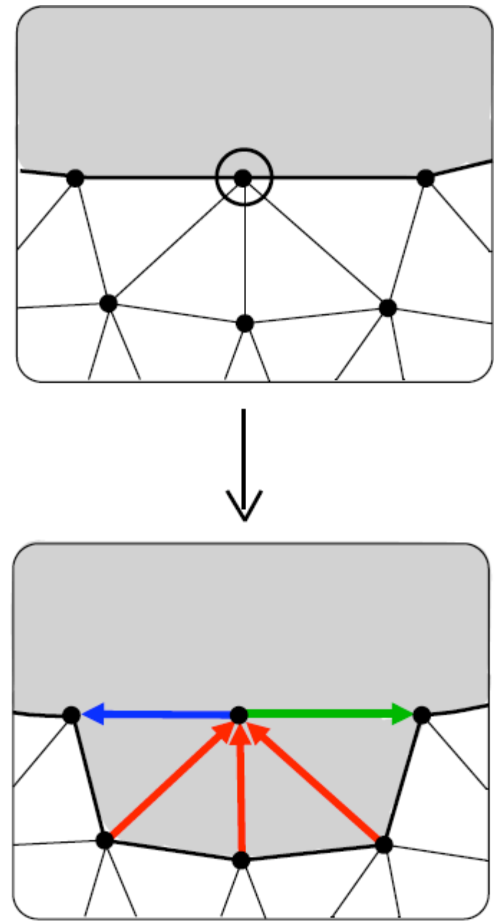

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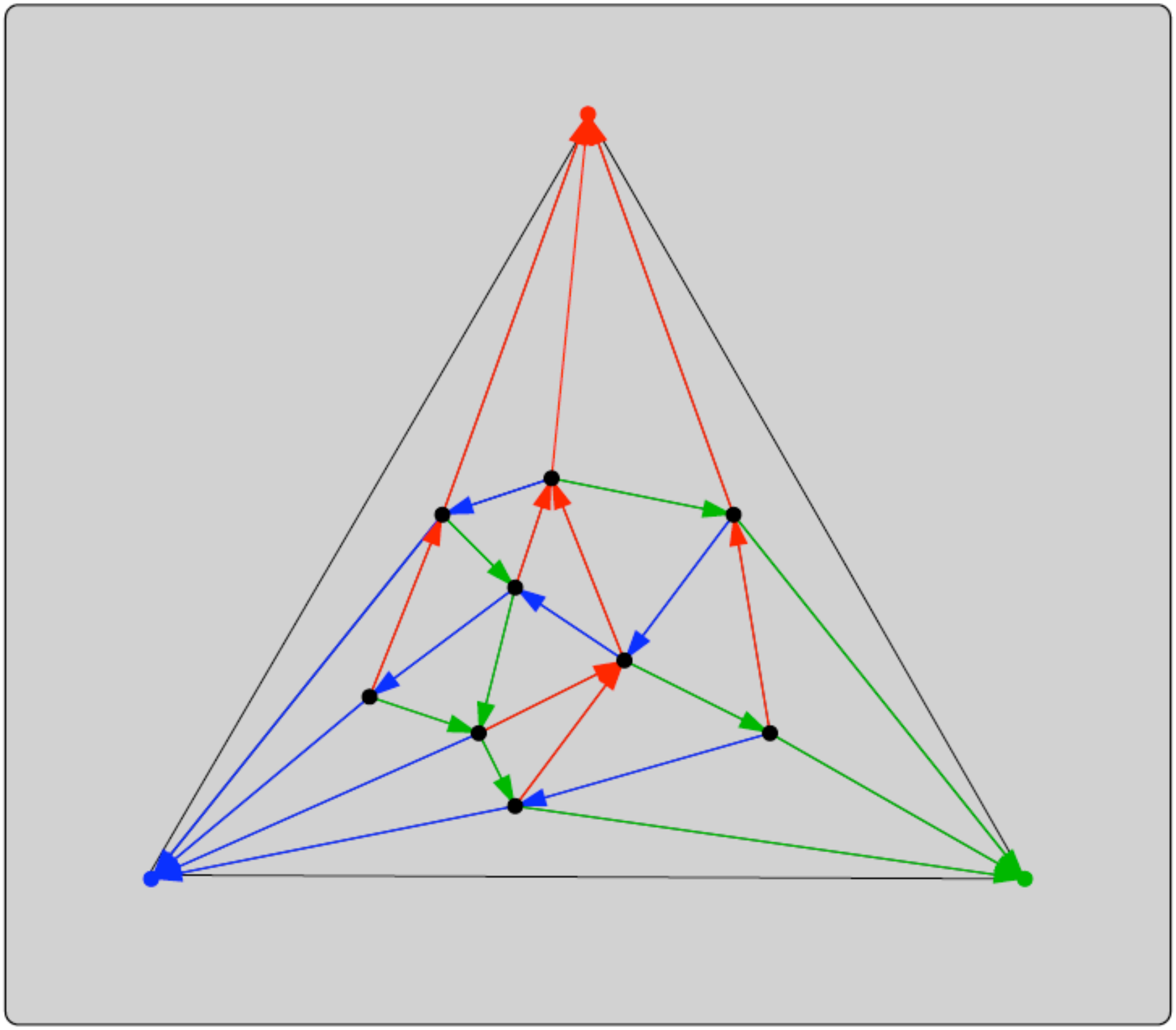

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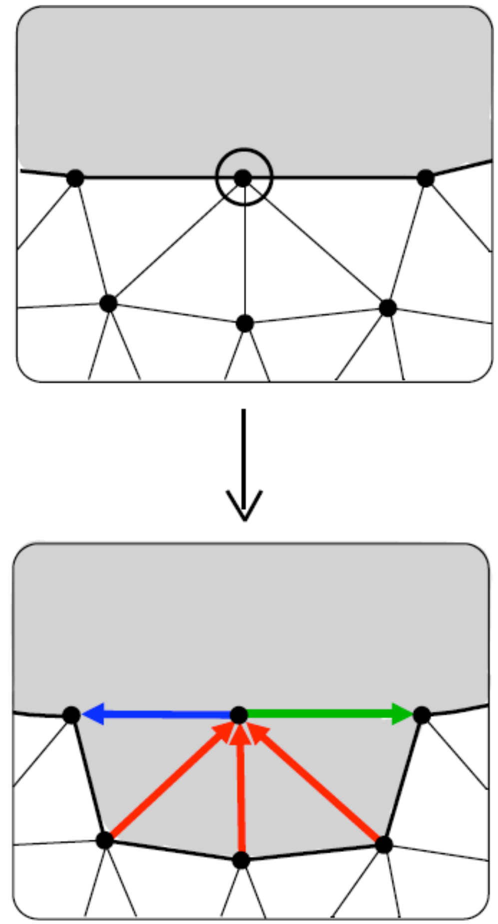
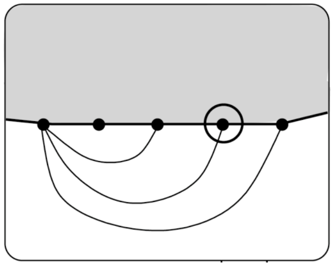
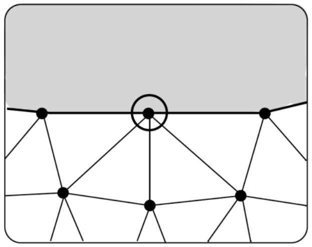


Result

- At each step, take care that the chosen vertex is not incident to a chord (nor to the bottom outer edge)

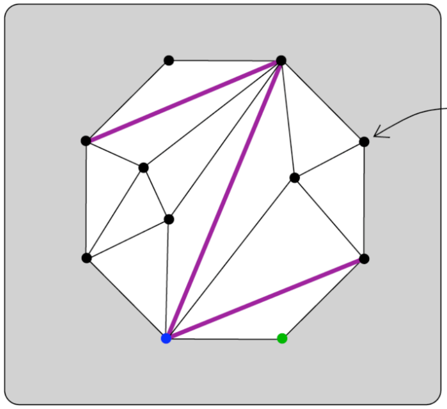


is forbidden



is accepted

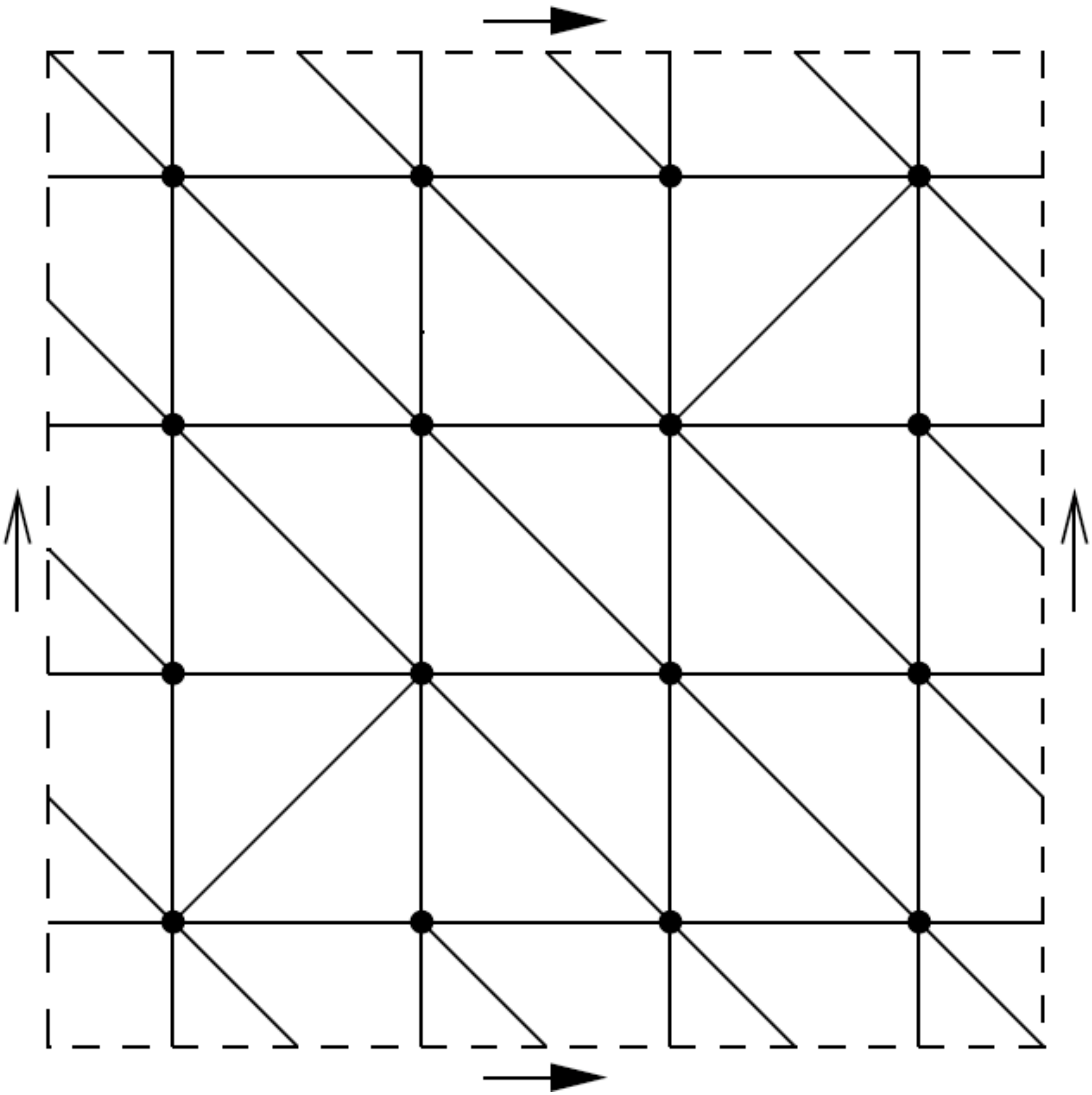
- There is always such a vertex



admissible vertex

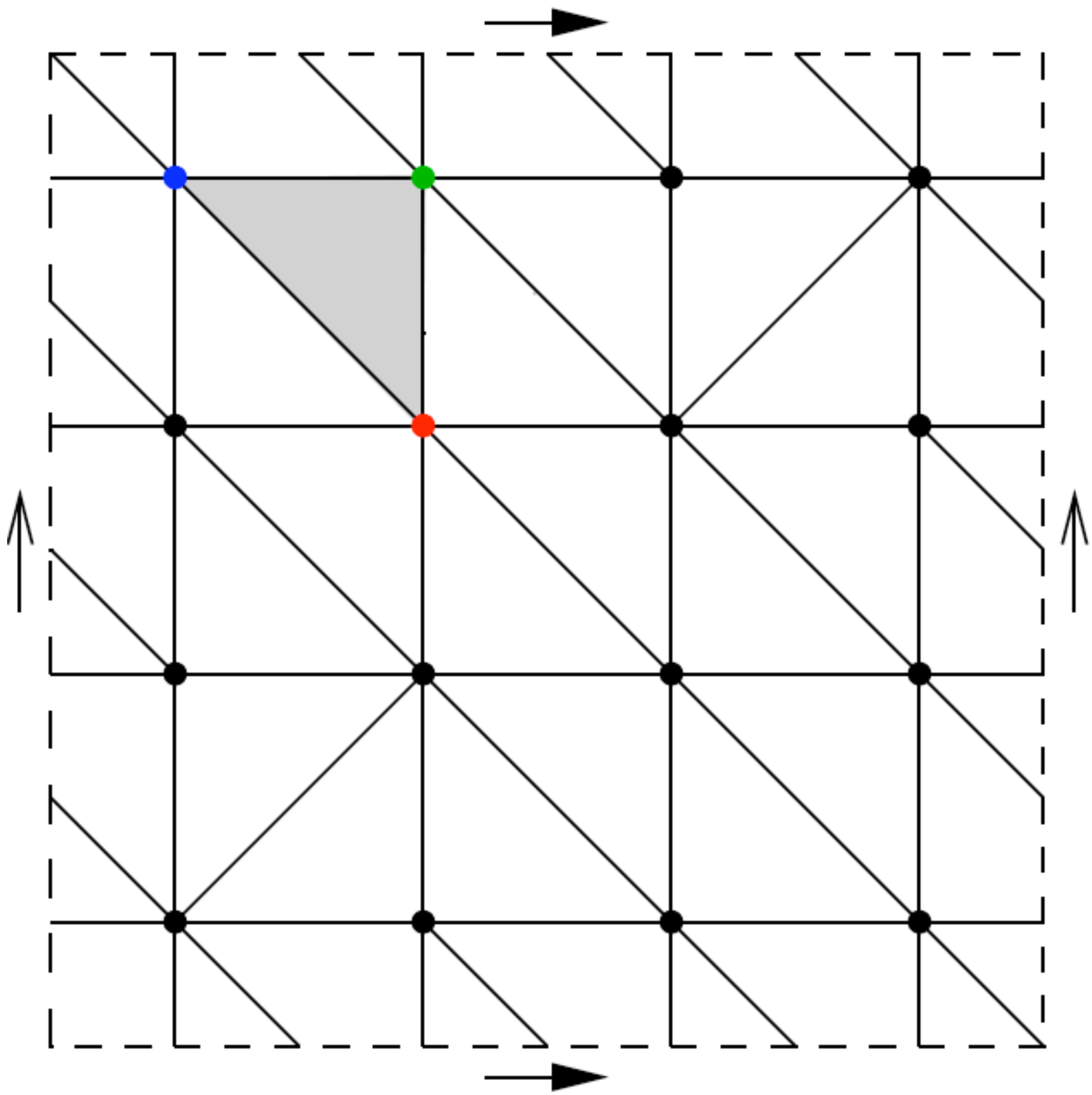
- Hence the algorithm terminates, it outputs a Schnyder wood [Schnyder'89, Brehm'03]

Triangulations in higher genus



A triangulation of genus 1

Triangulations in higher genus



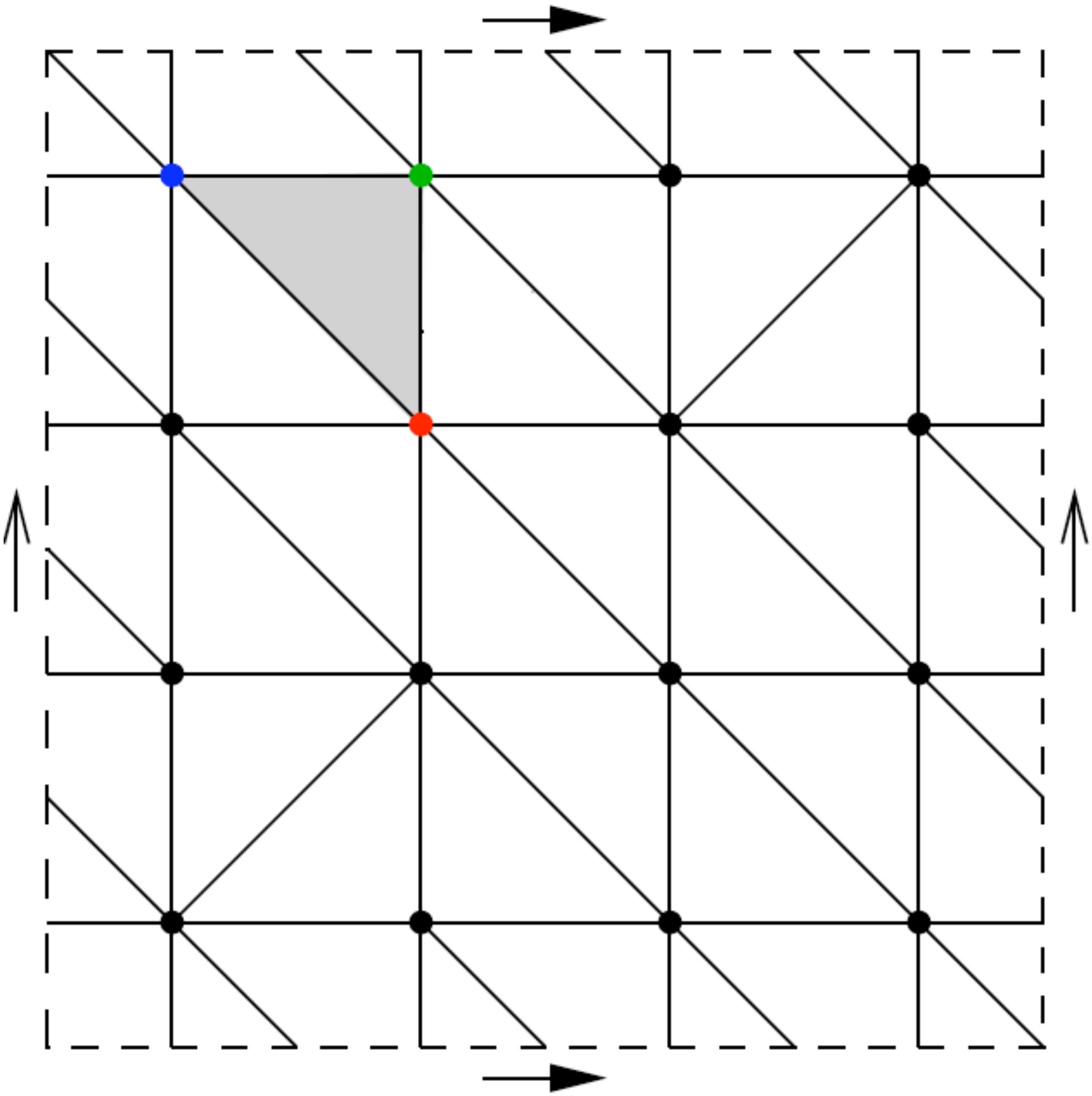
A triangulation
of genus 1,
with a root-face.

n inner vertices

↓

$3(n+2g)$ inner edges

Triangulations in higher genus



A triangulation
of genus 1,
with a root-face.

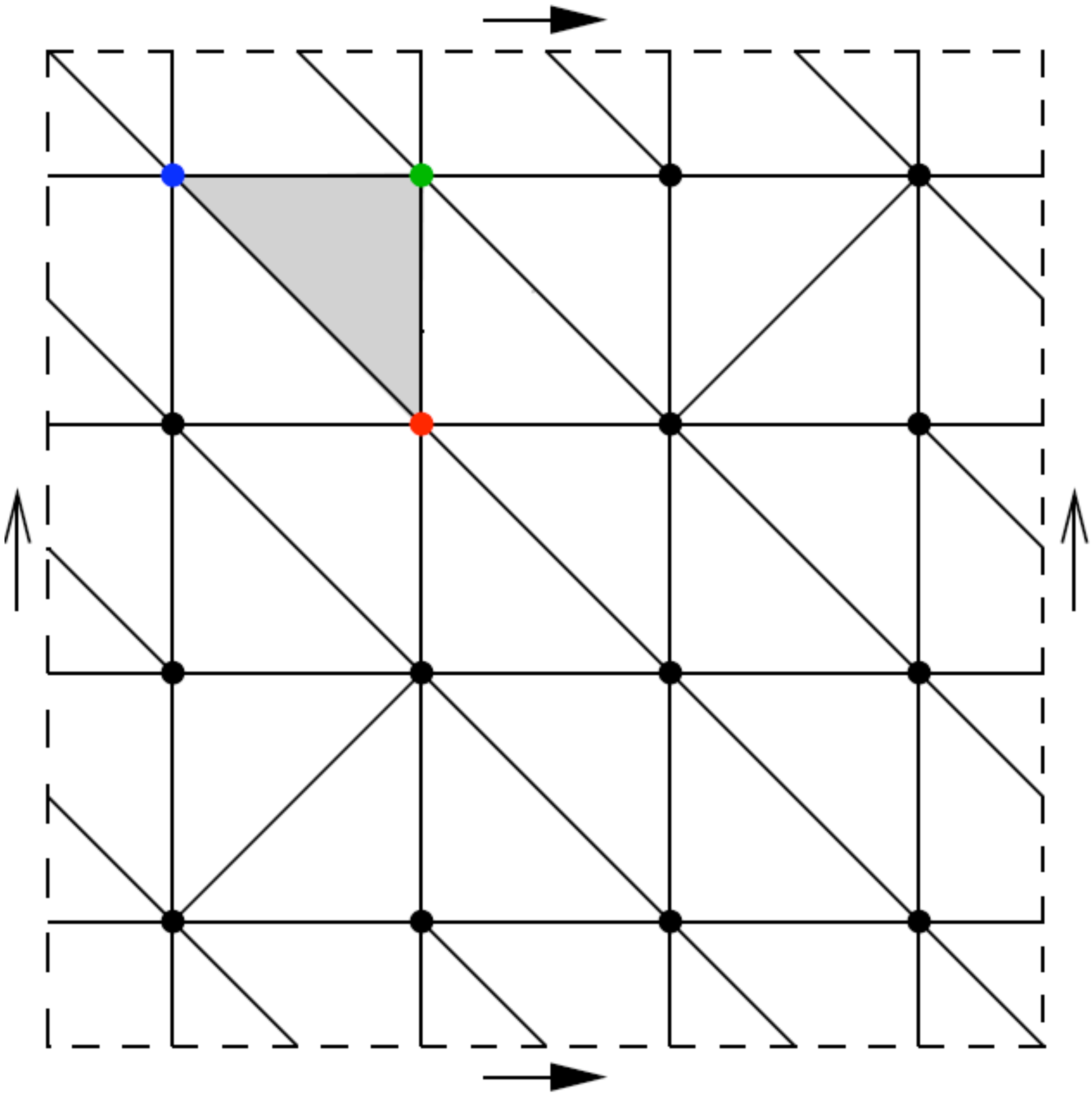
n inner vertices

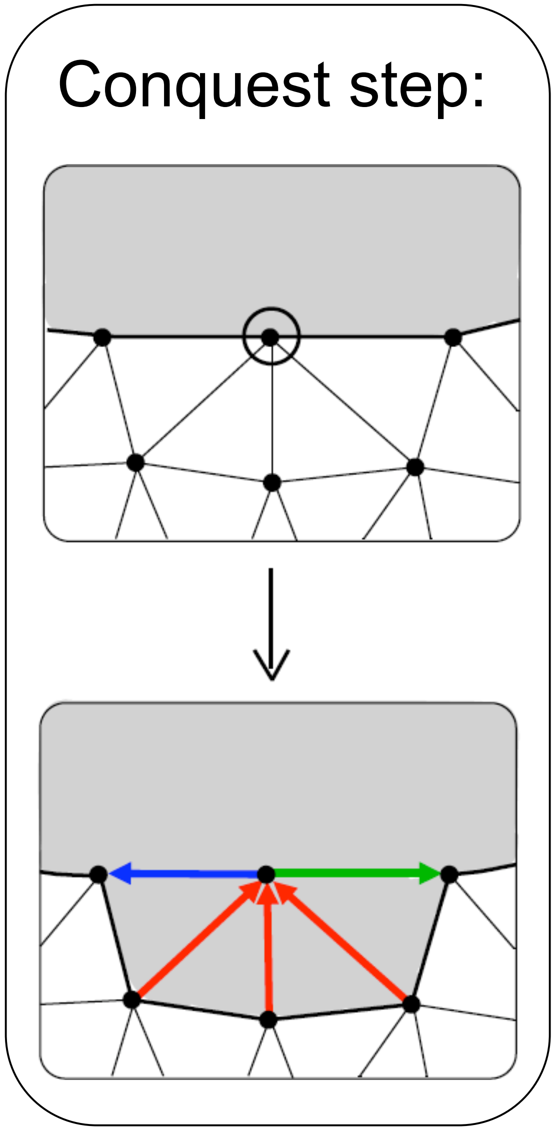
↓

3(n+2g) inner edges

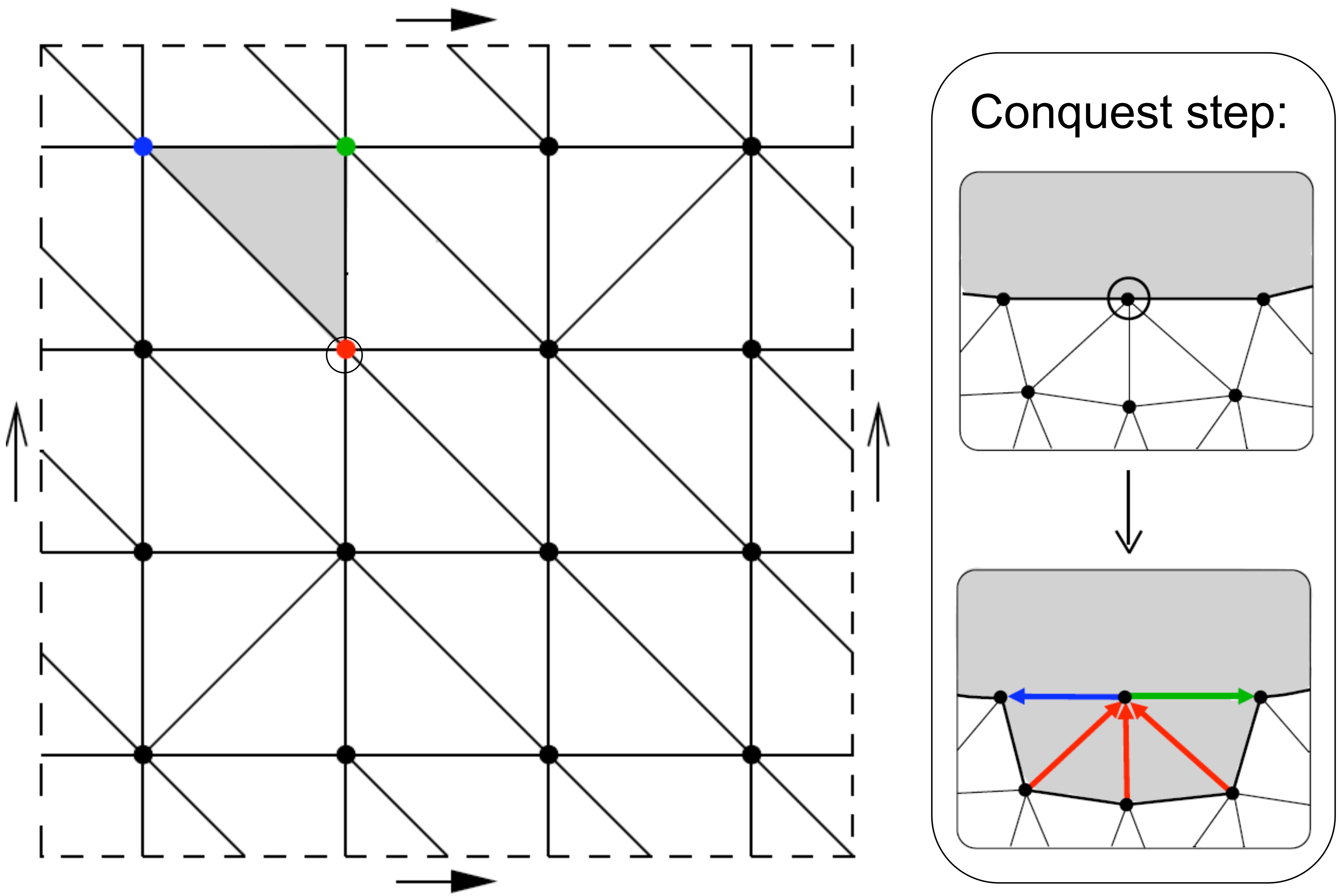
No hope to
have outdegree 3
everyhwhere

Conquest in higher genus

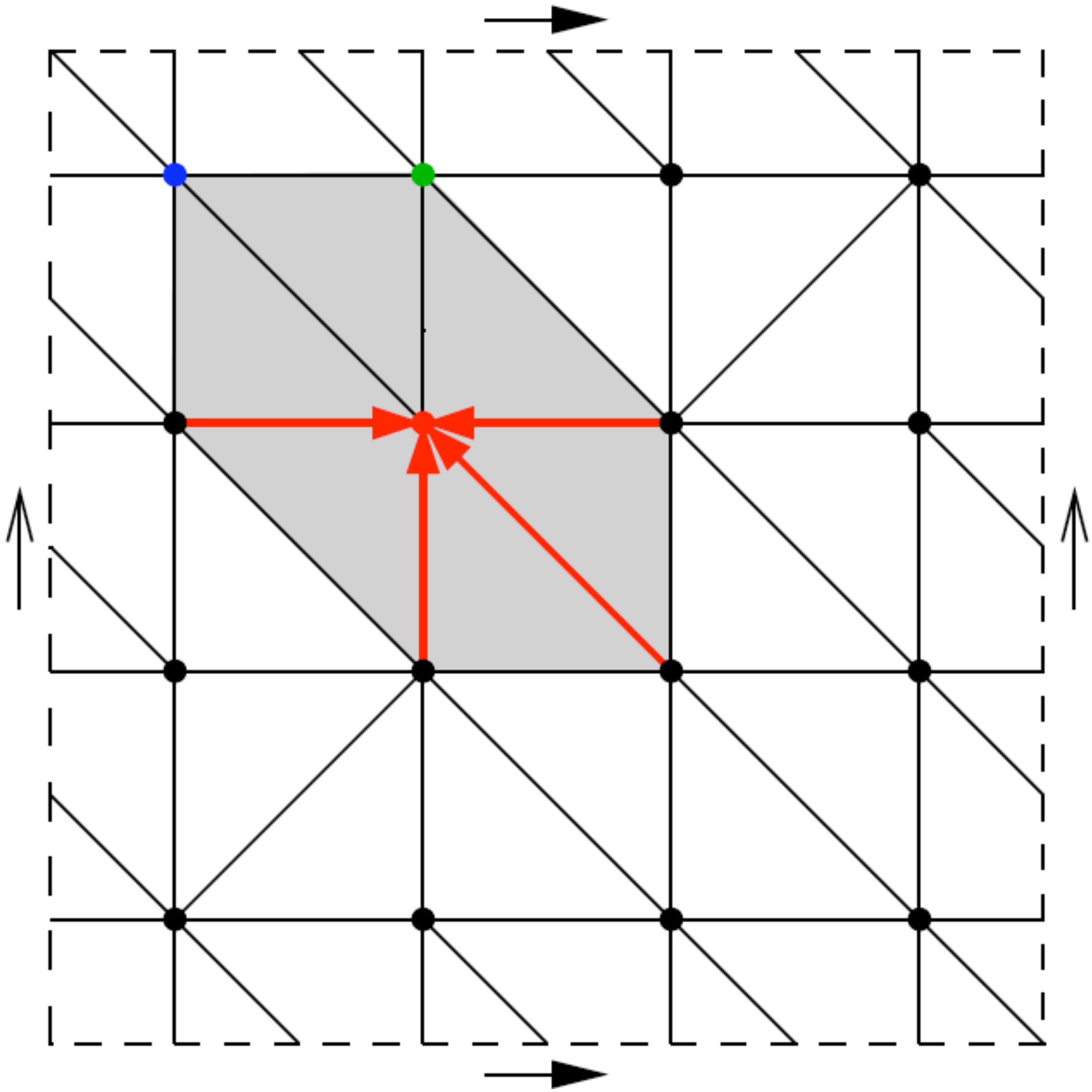


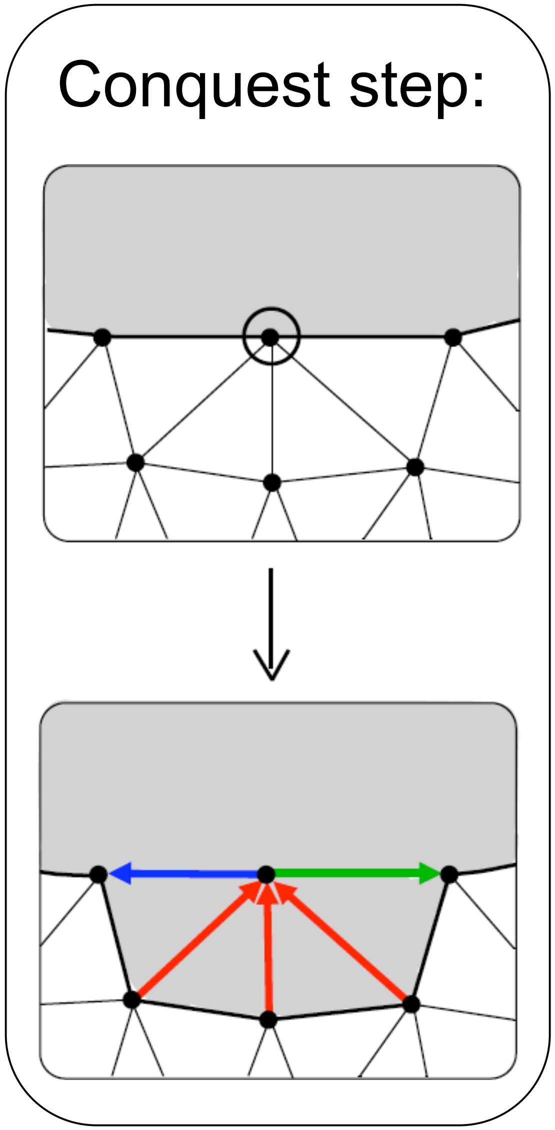


Conquest in higher genus

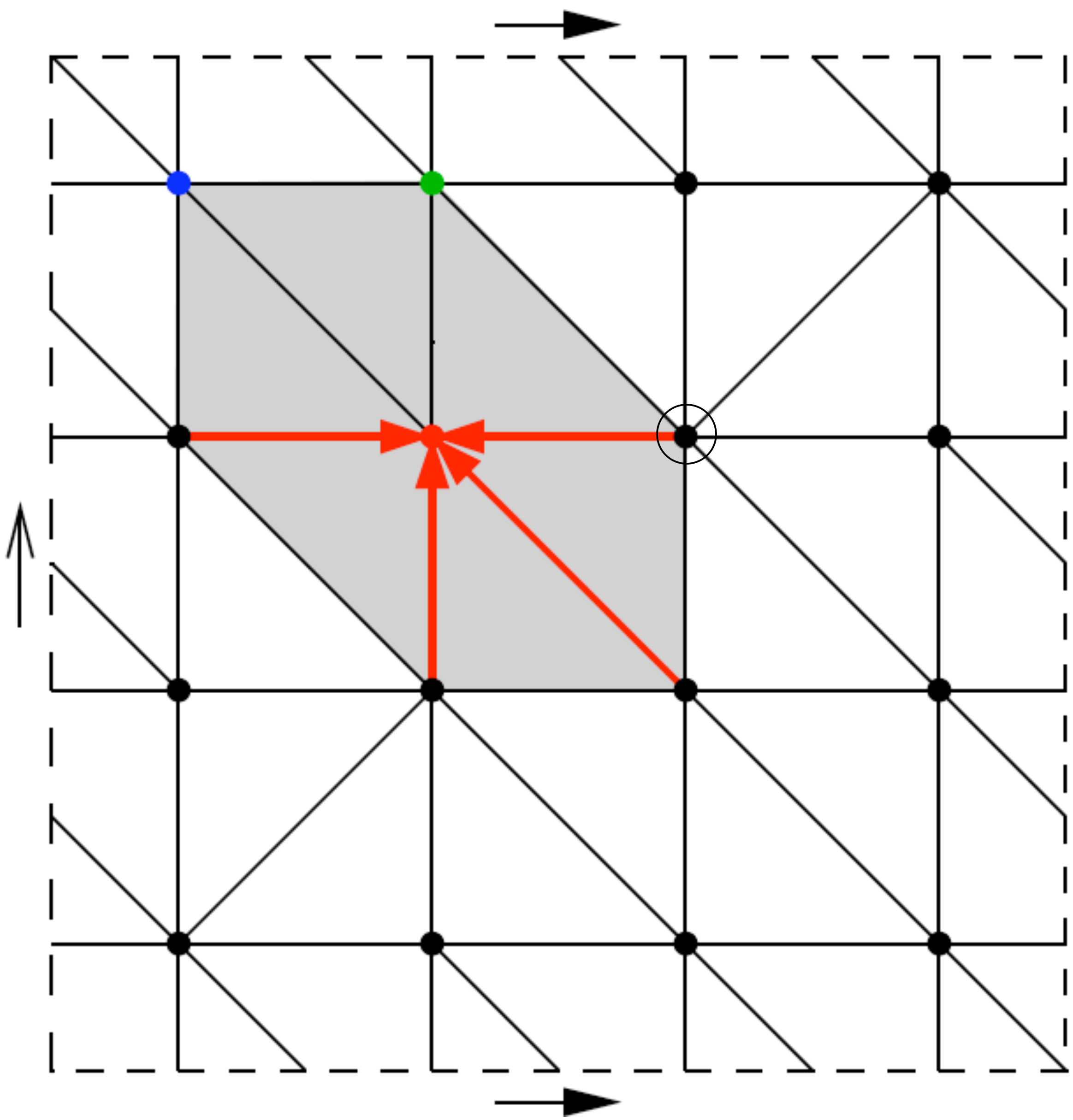


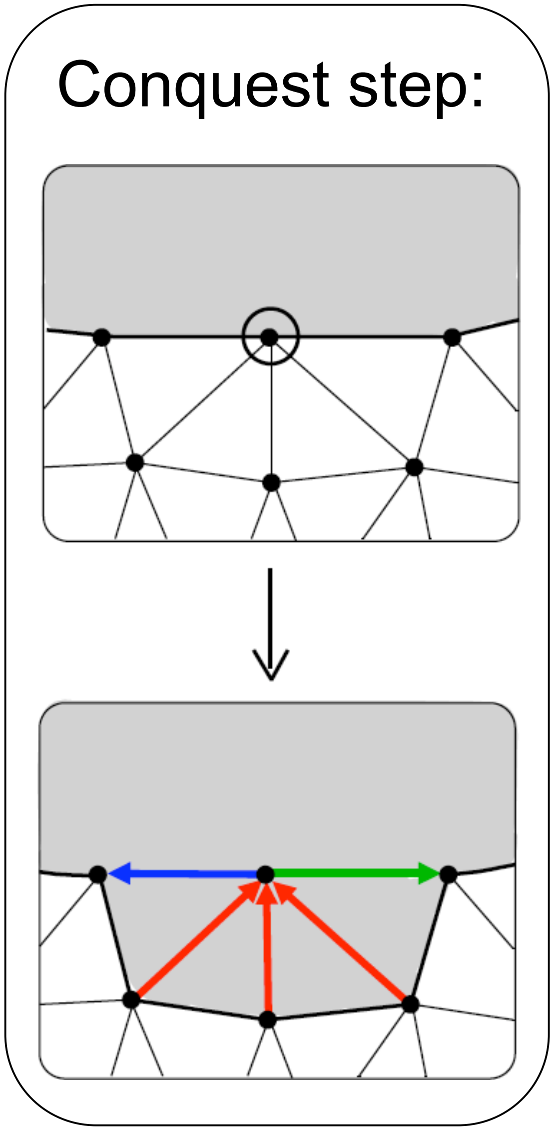
Conquest in higher genus



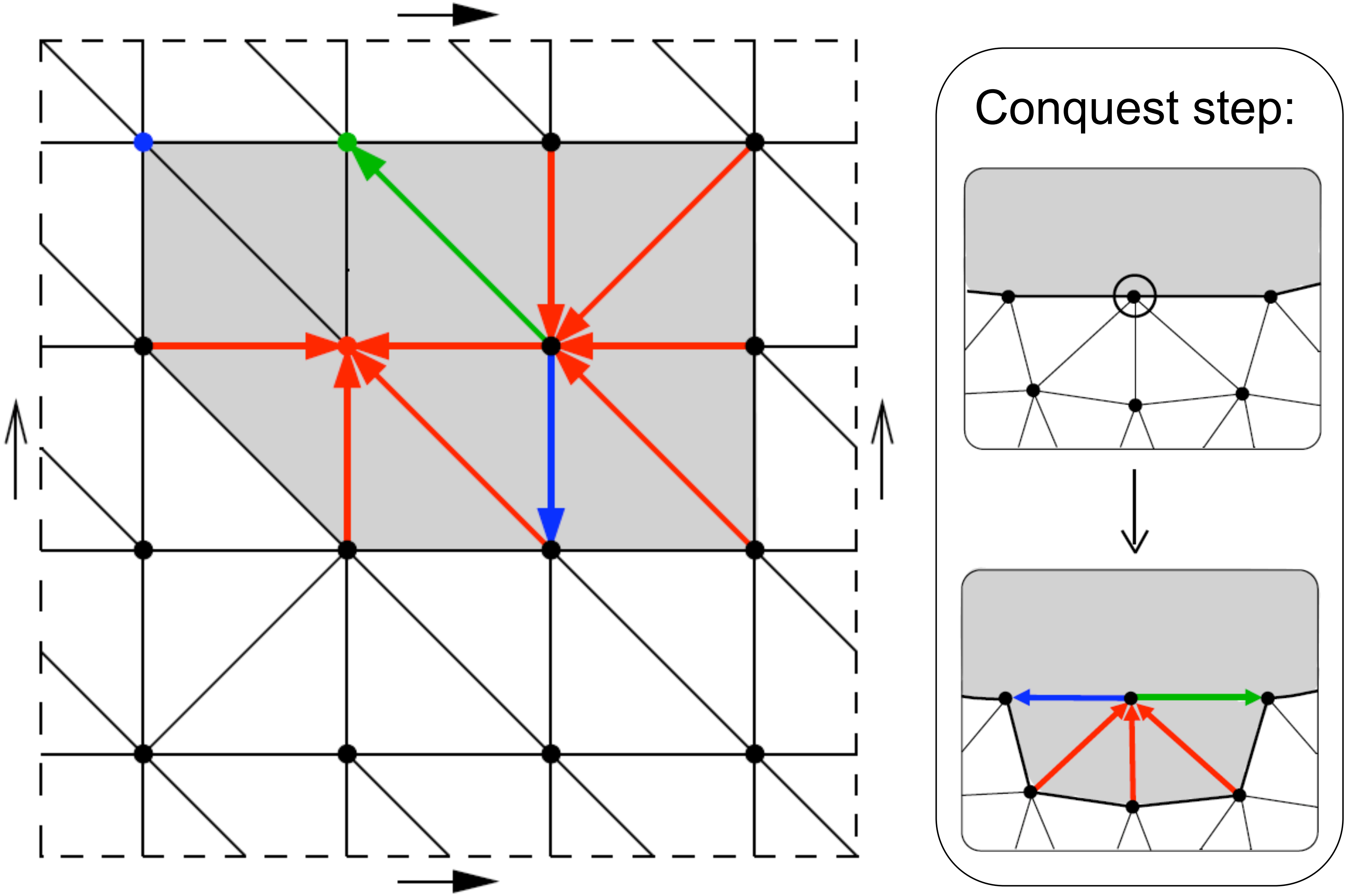


Conquest in higher genus

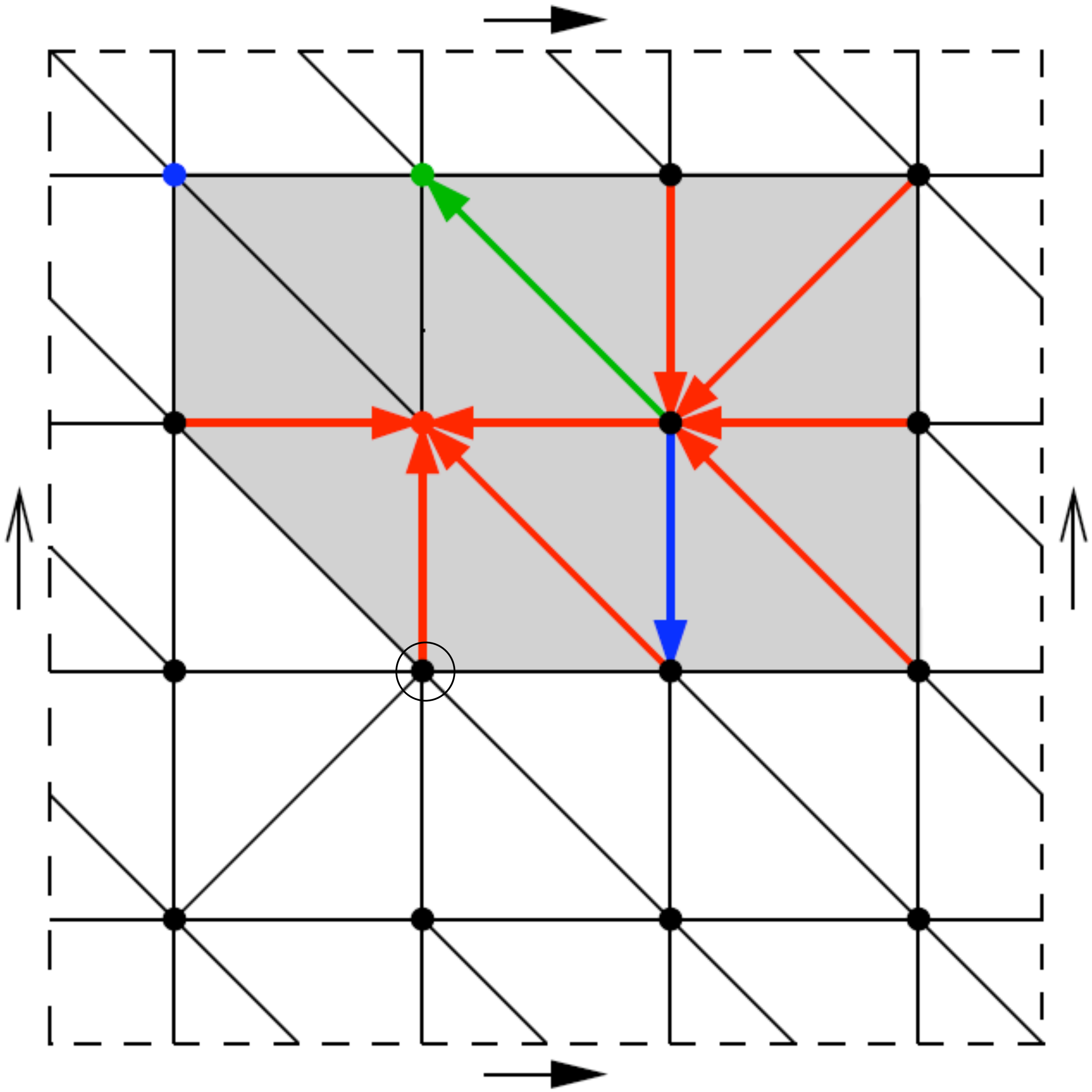


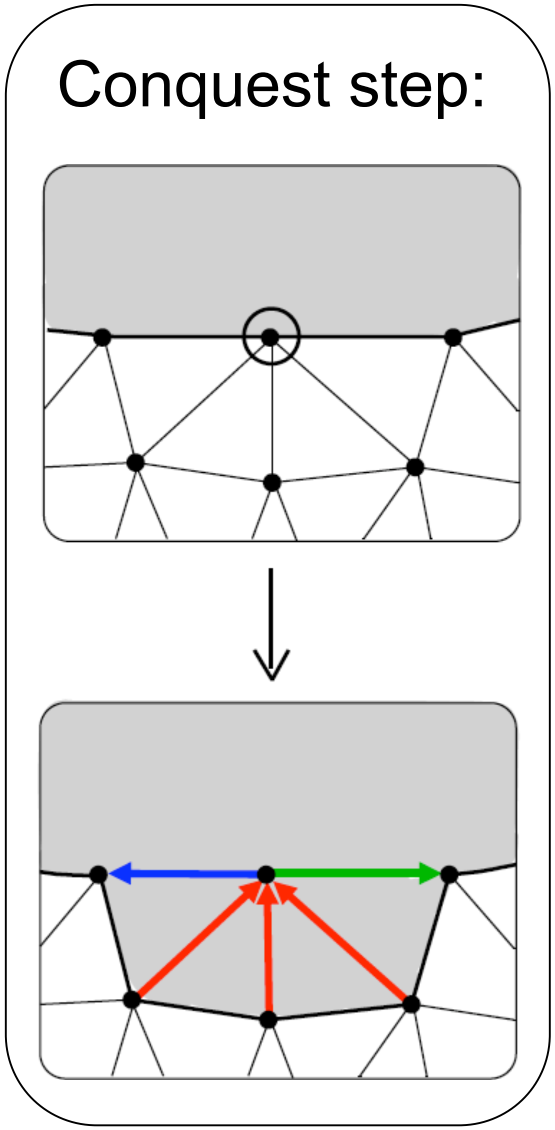


Conquest in higher genus

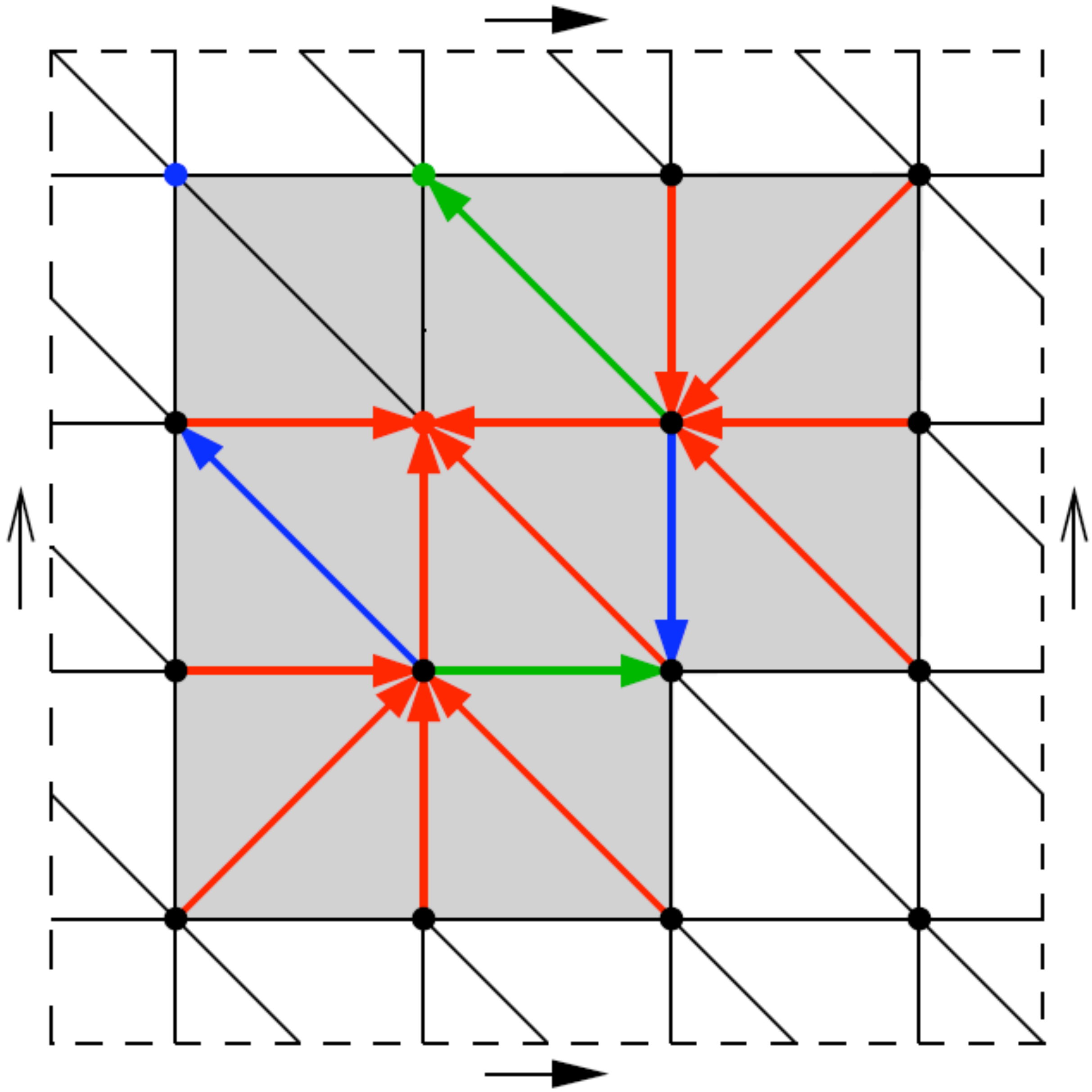


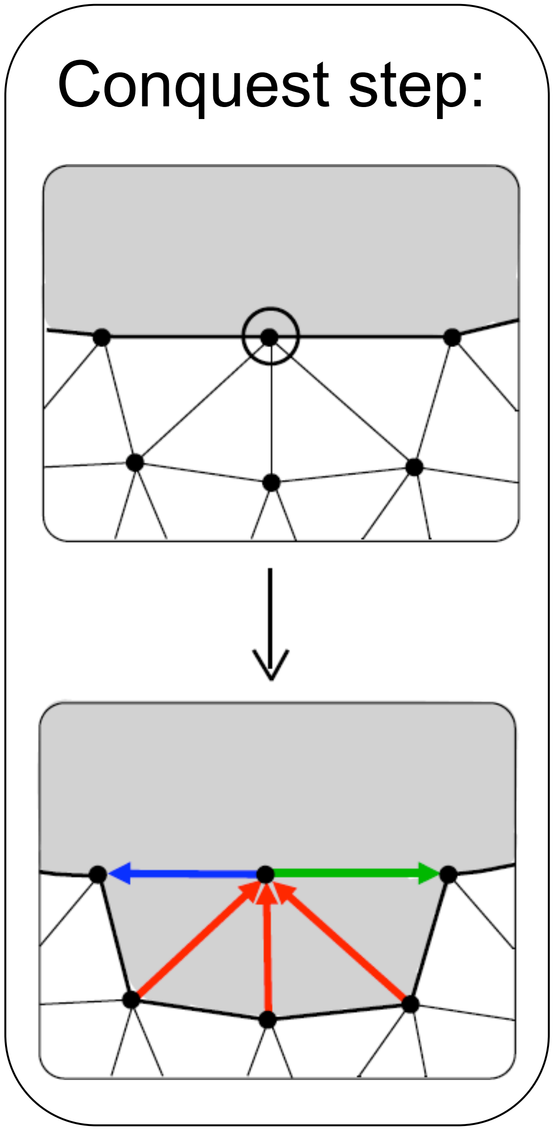
Conquest in higher genus



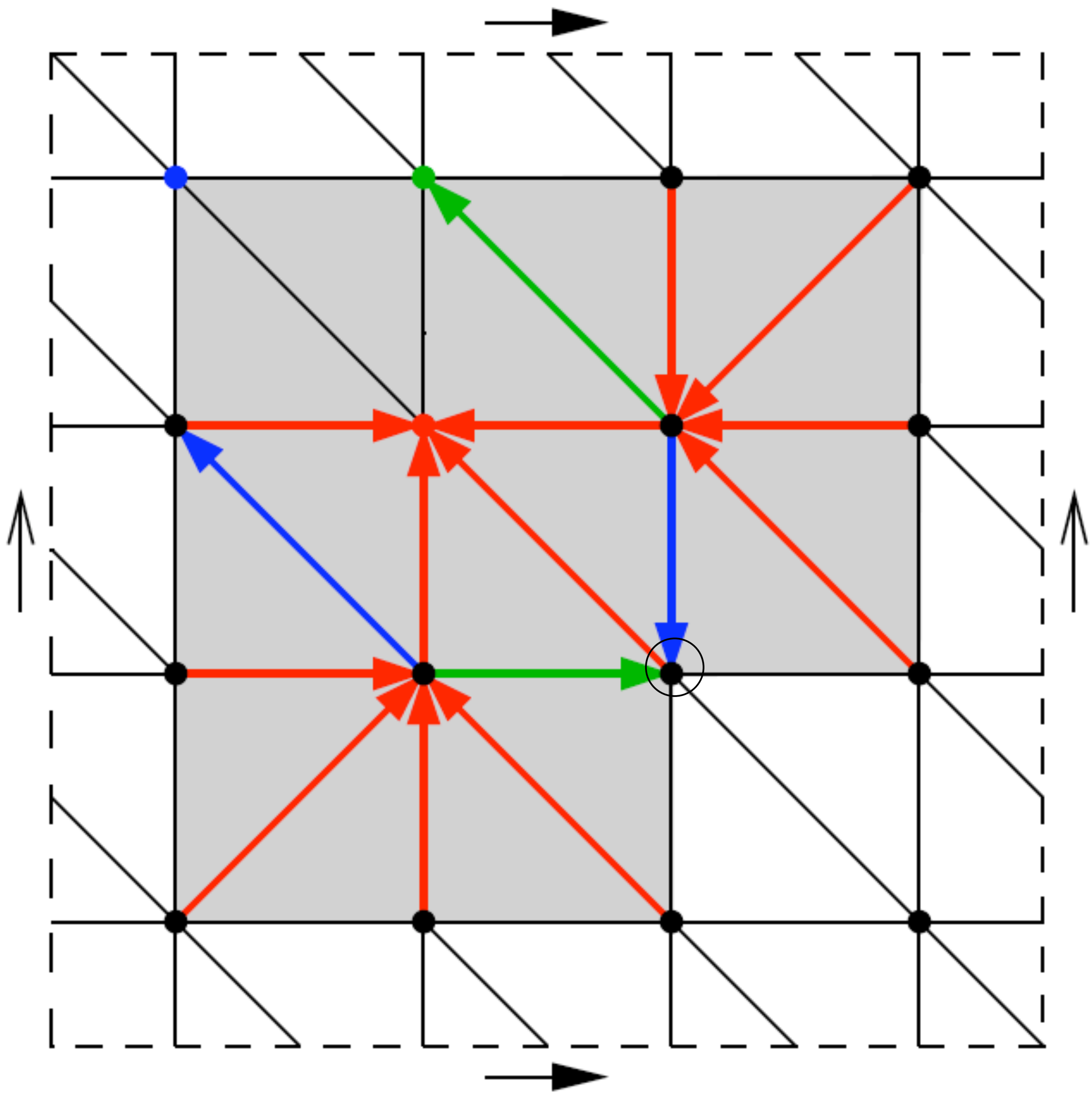


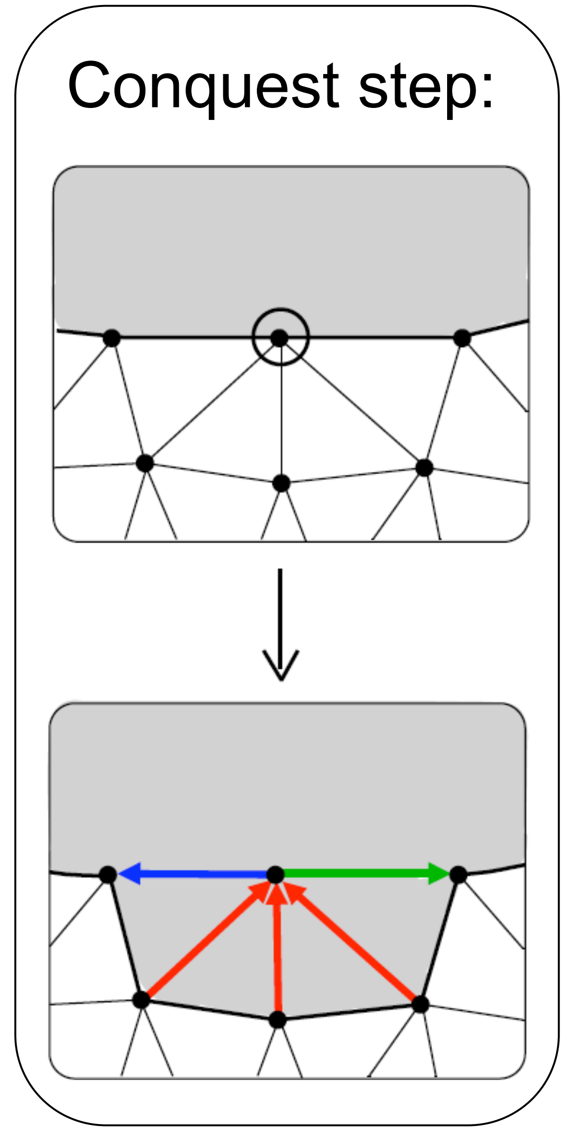
Conquest in higher genus



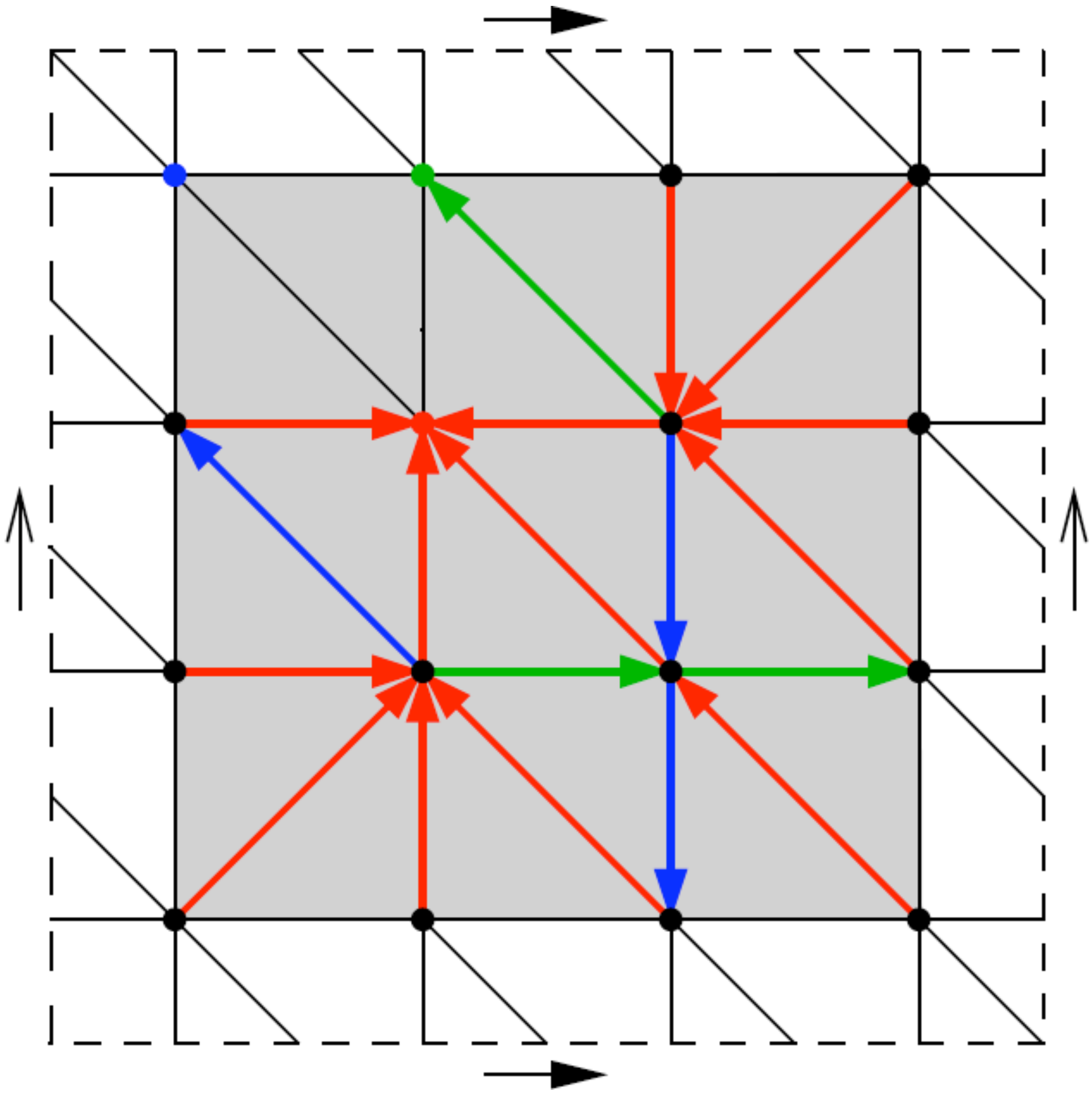


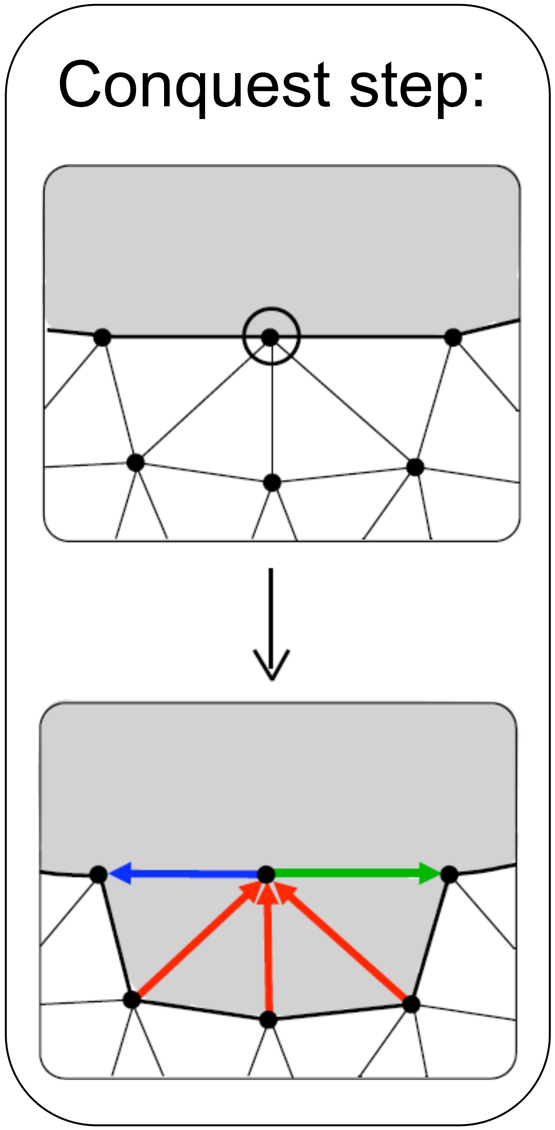
Conquest in higher genus



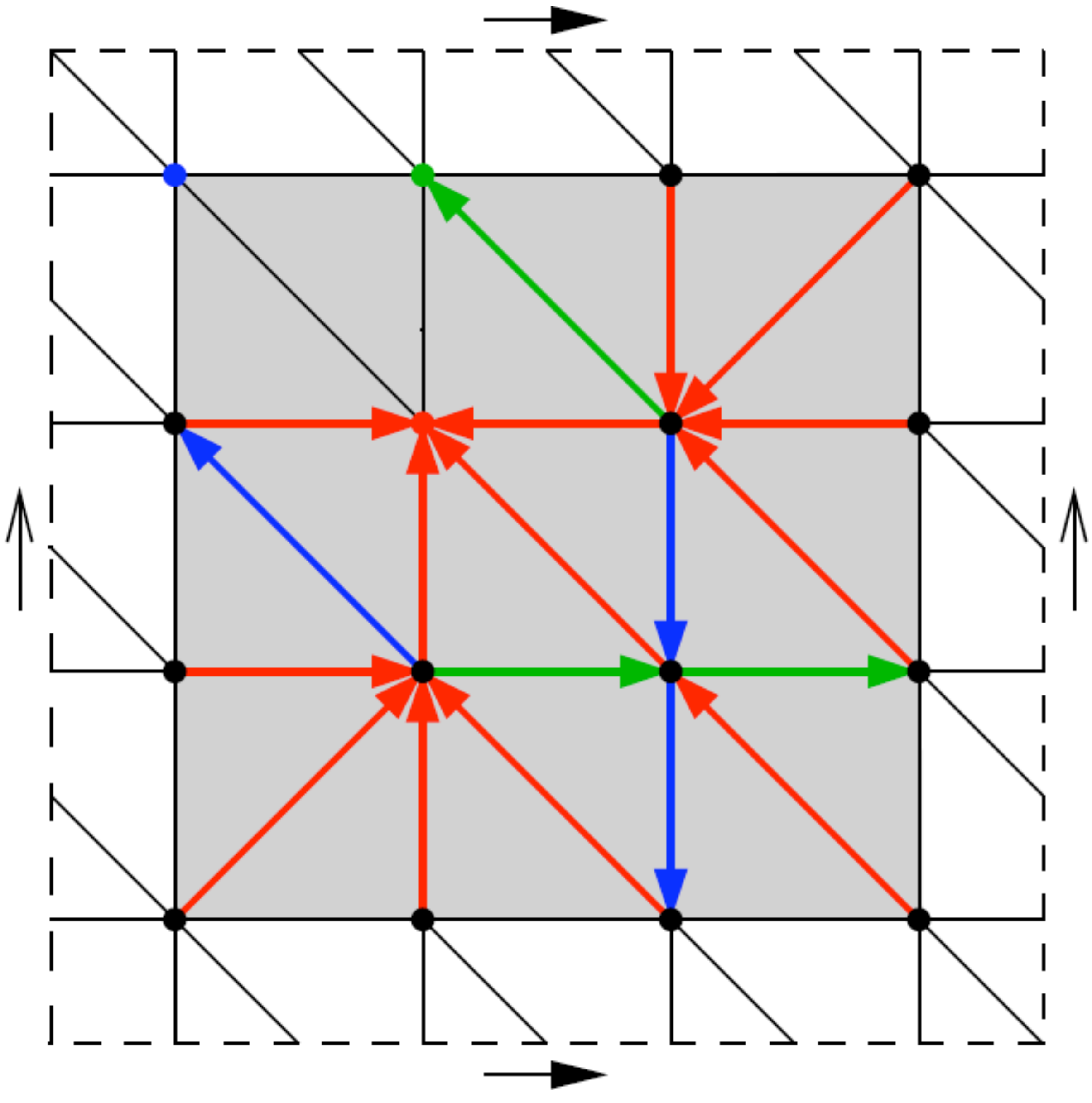


Conquest in higher genus



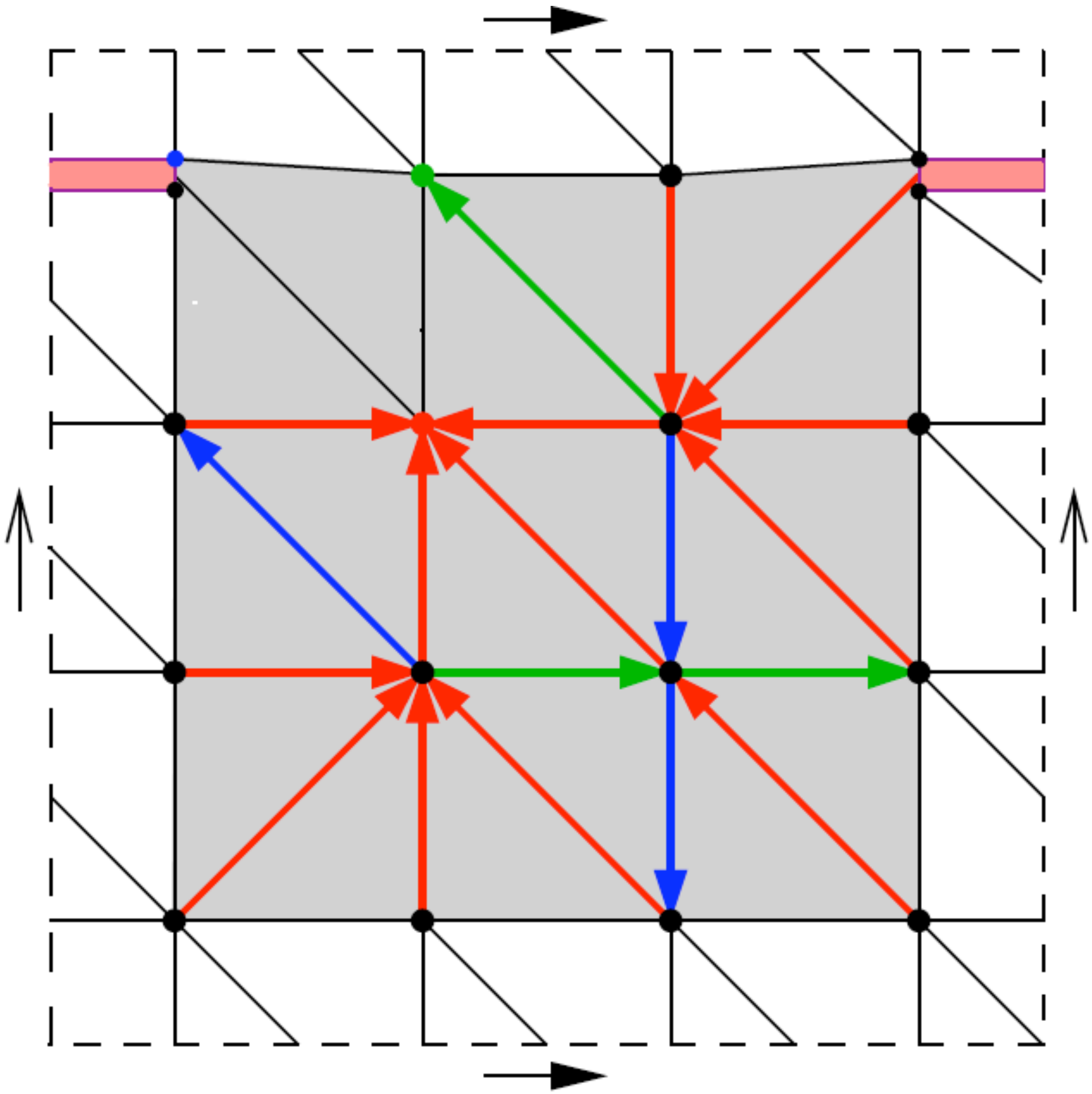


Conquest in higher genus



Can not extend conquered area C

Conquest in higher genus

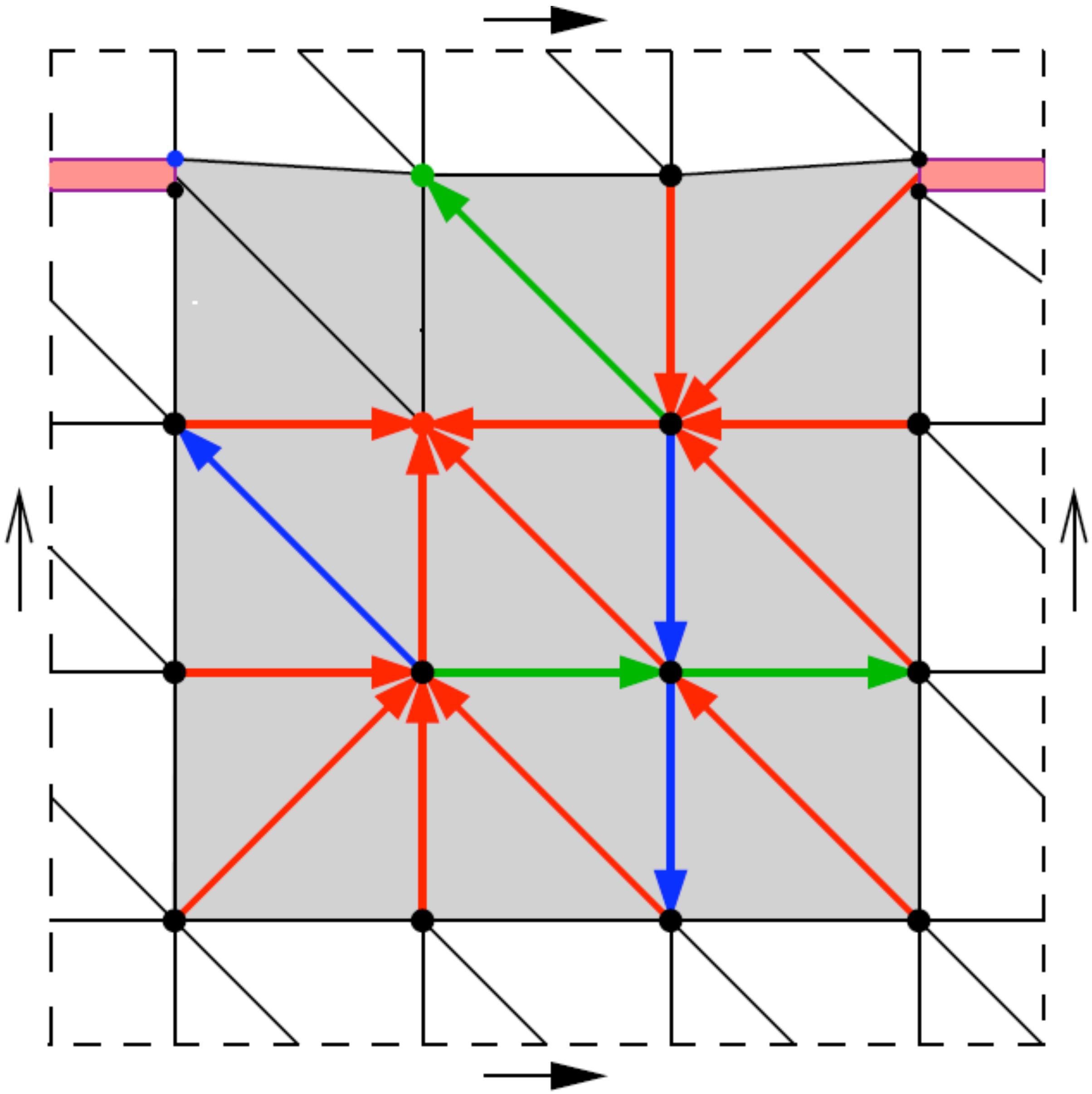


Can not extend conquered area *C*

Special step:

- choose chord *e*
- make it fat
- add it to *C*

Conquest in higher genus



Can not extend conquered area *C*

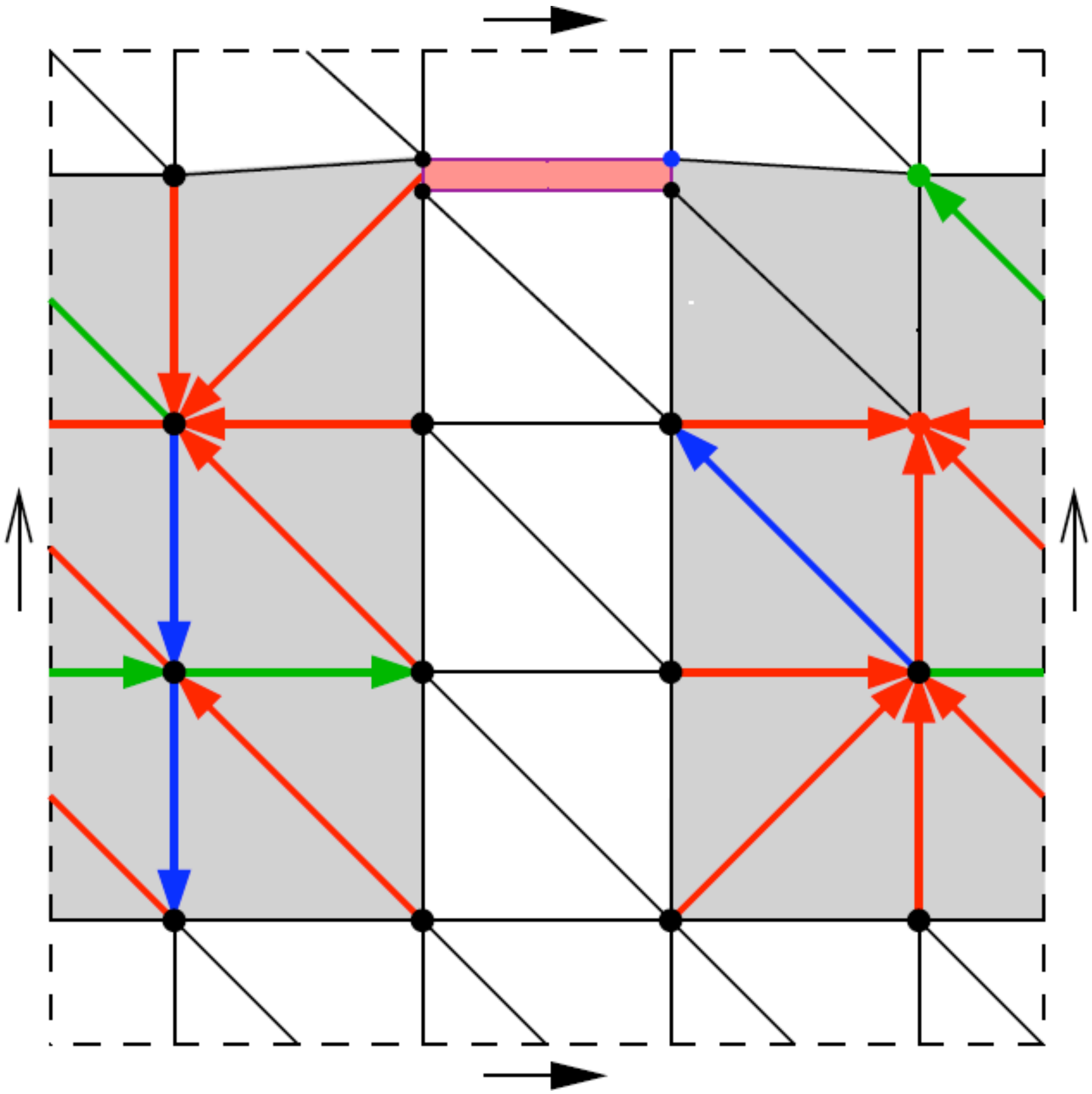
Special step:

- choose chord *e*
- make it fat
- add it to *C*

C : disk->cylinder

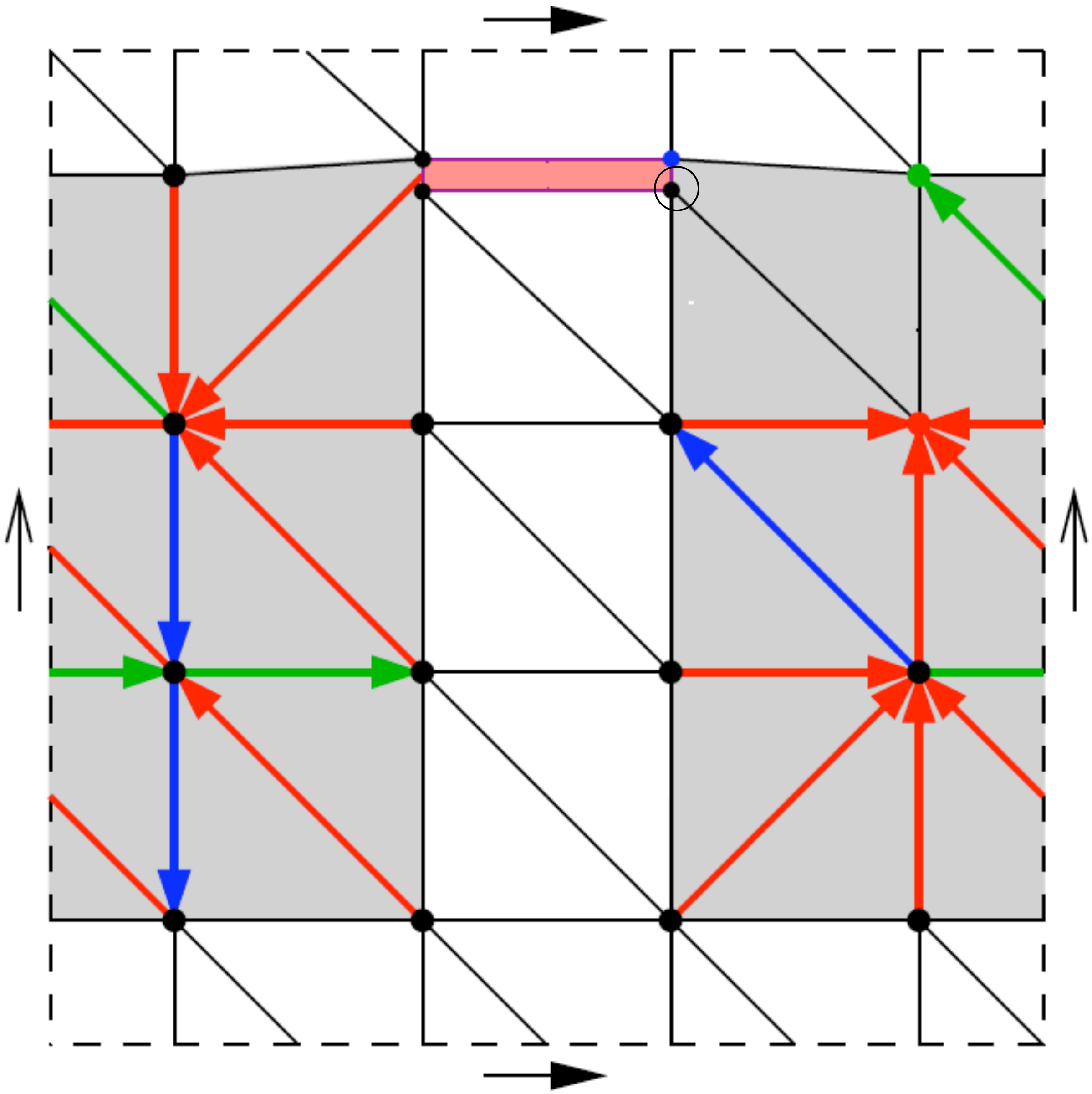
$S_g \backslash C$: torus->cylinder

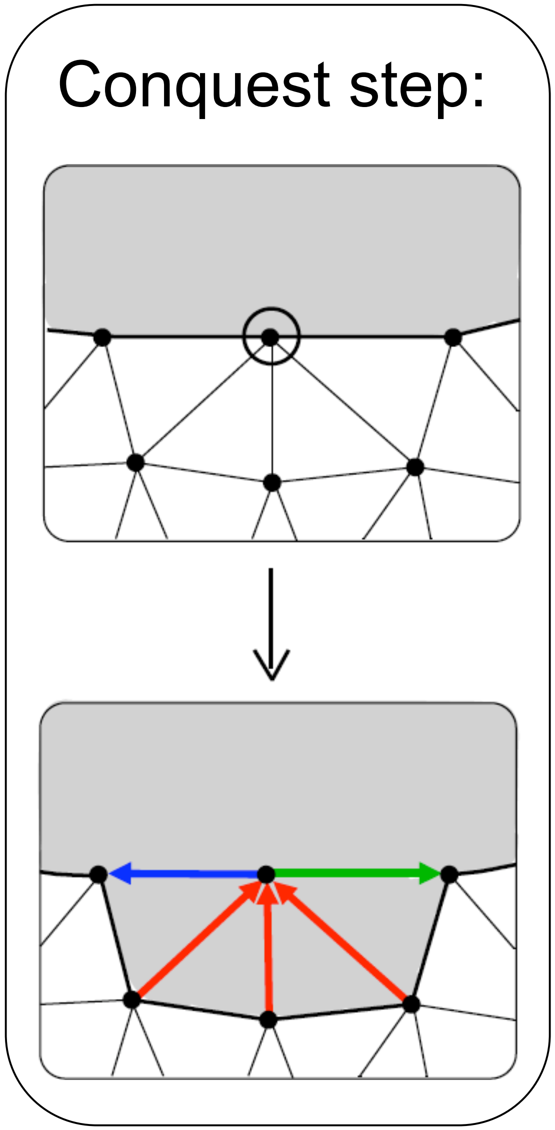
Conquest in higher genus



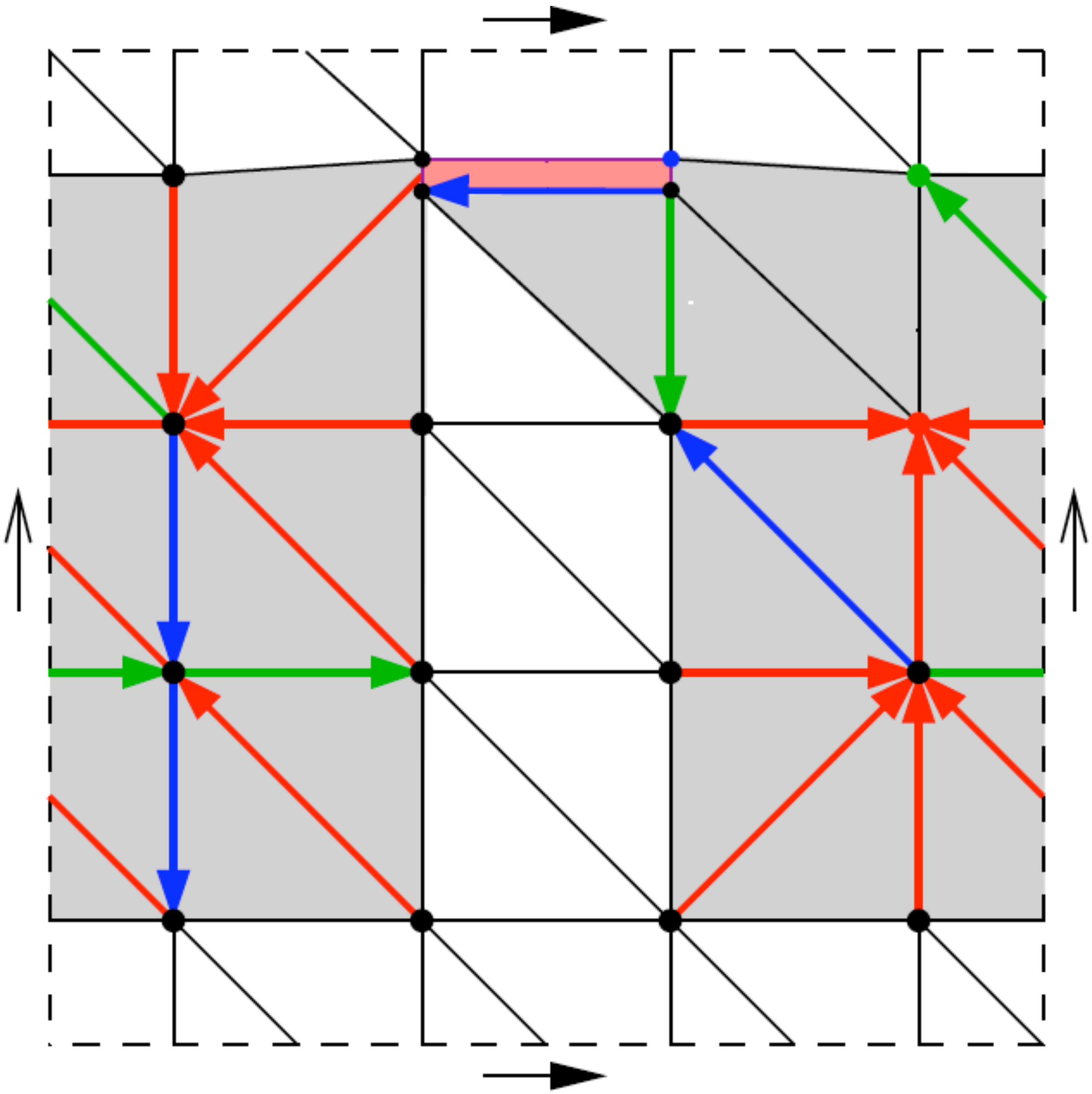
Continue !

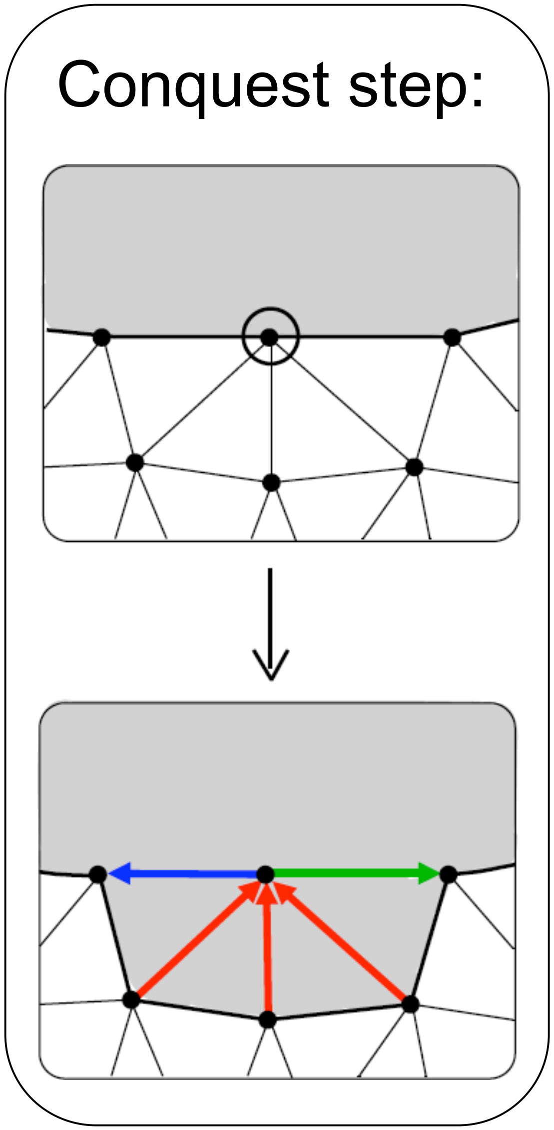
Conquest in higher genus



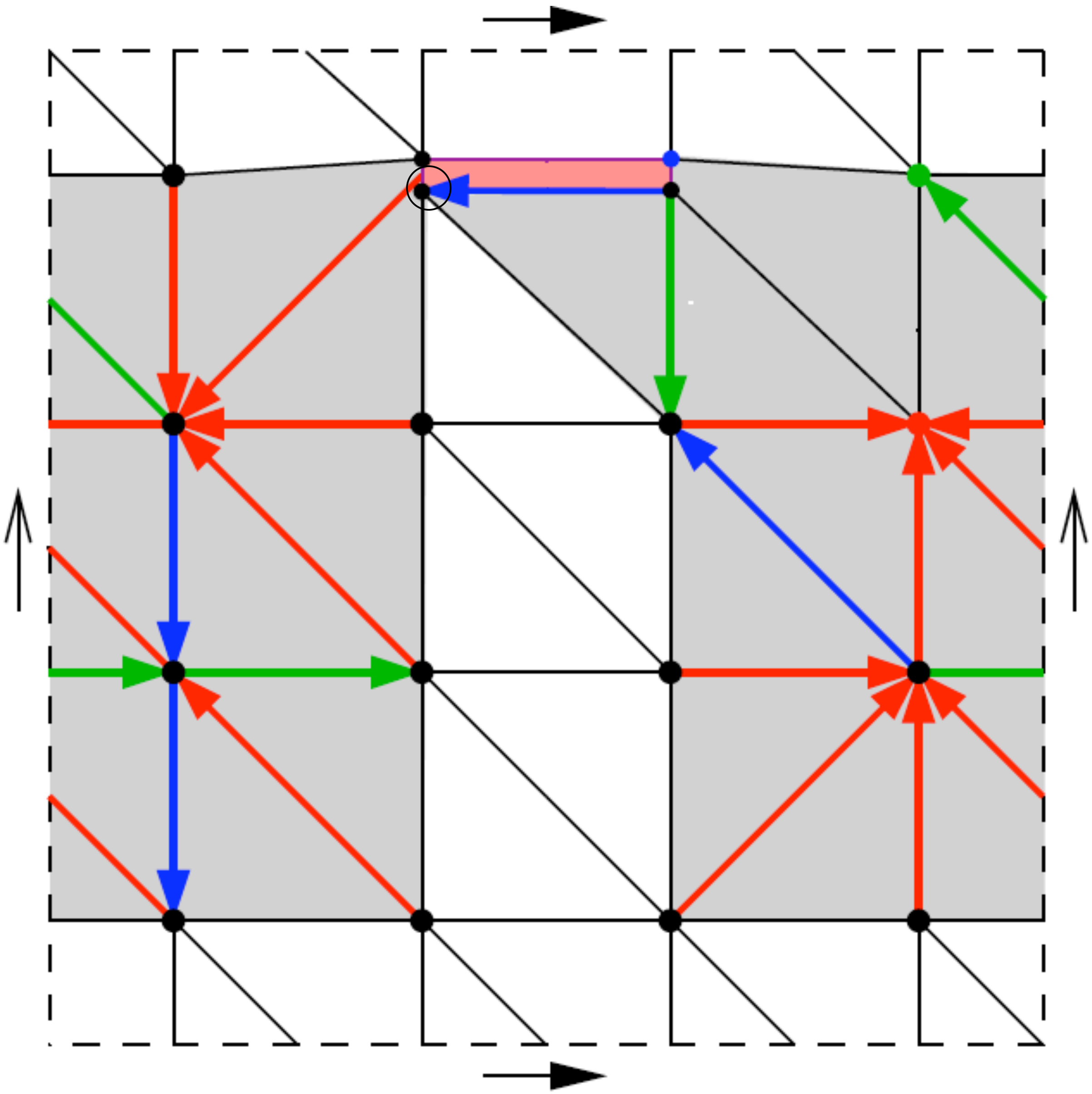


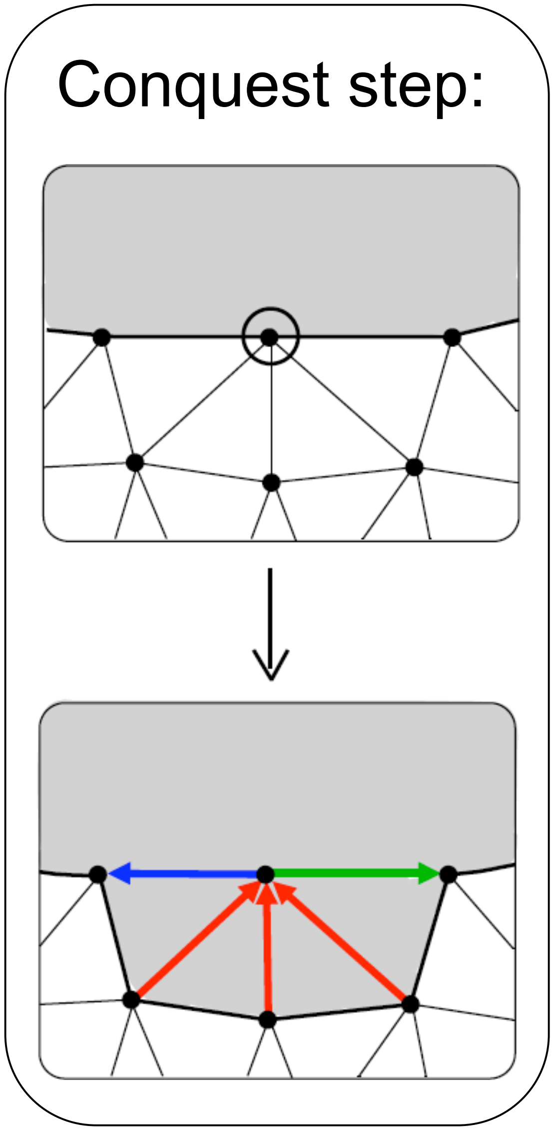
Conquest in higher genus



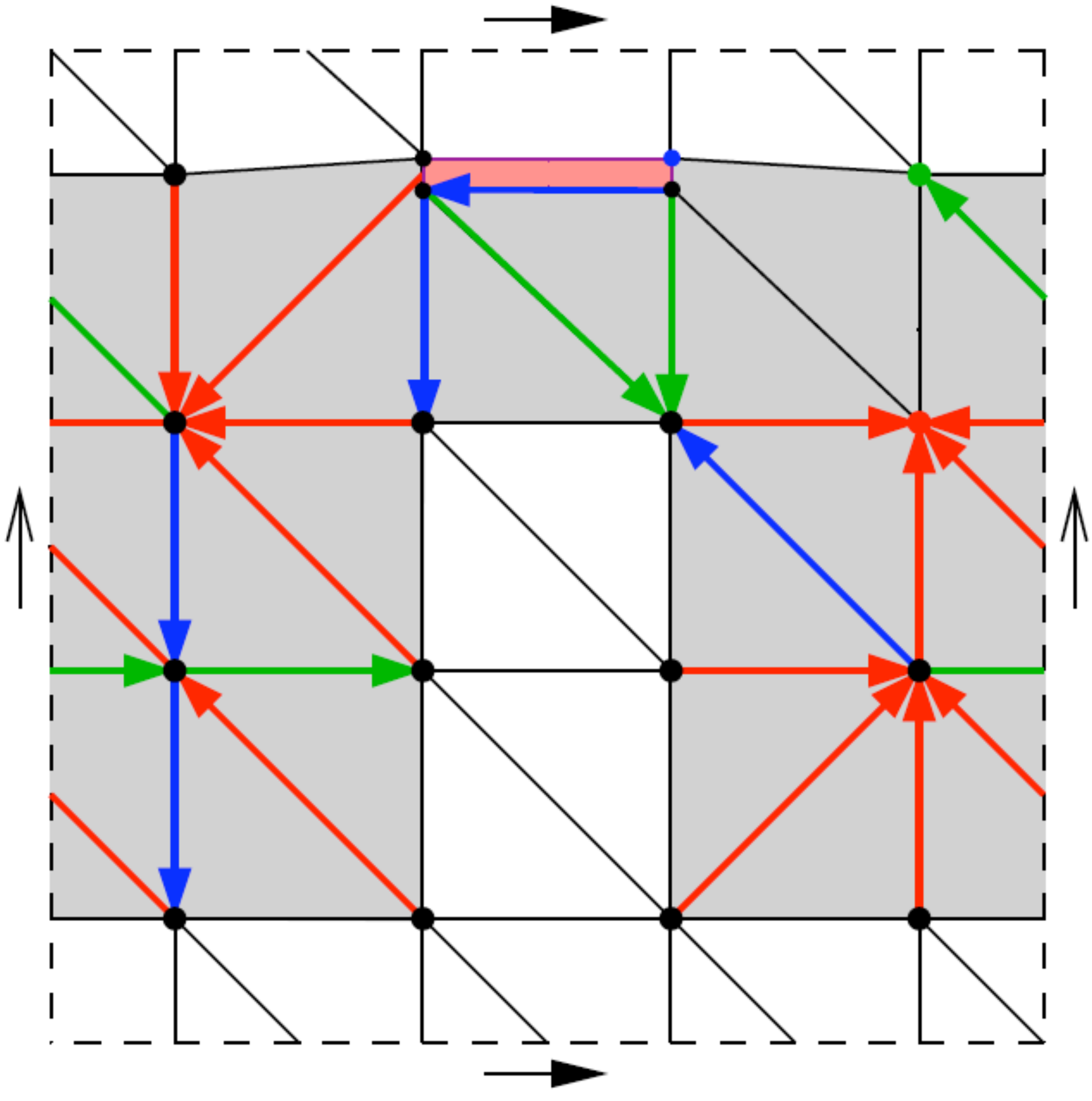


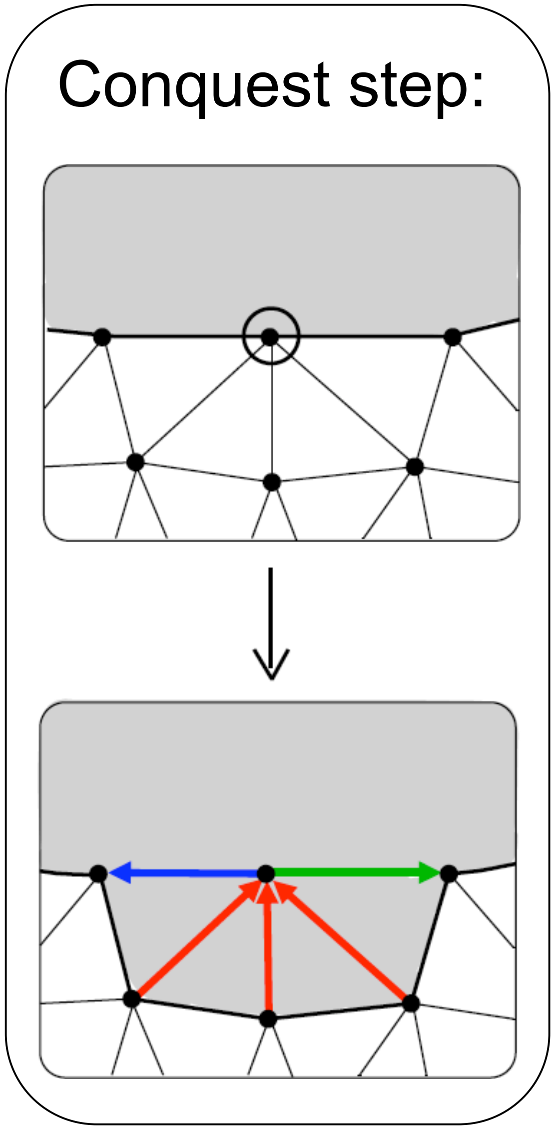
Conquest in higher genus



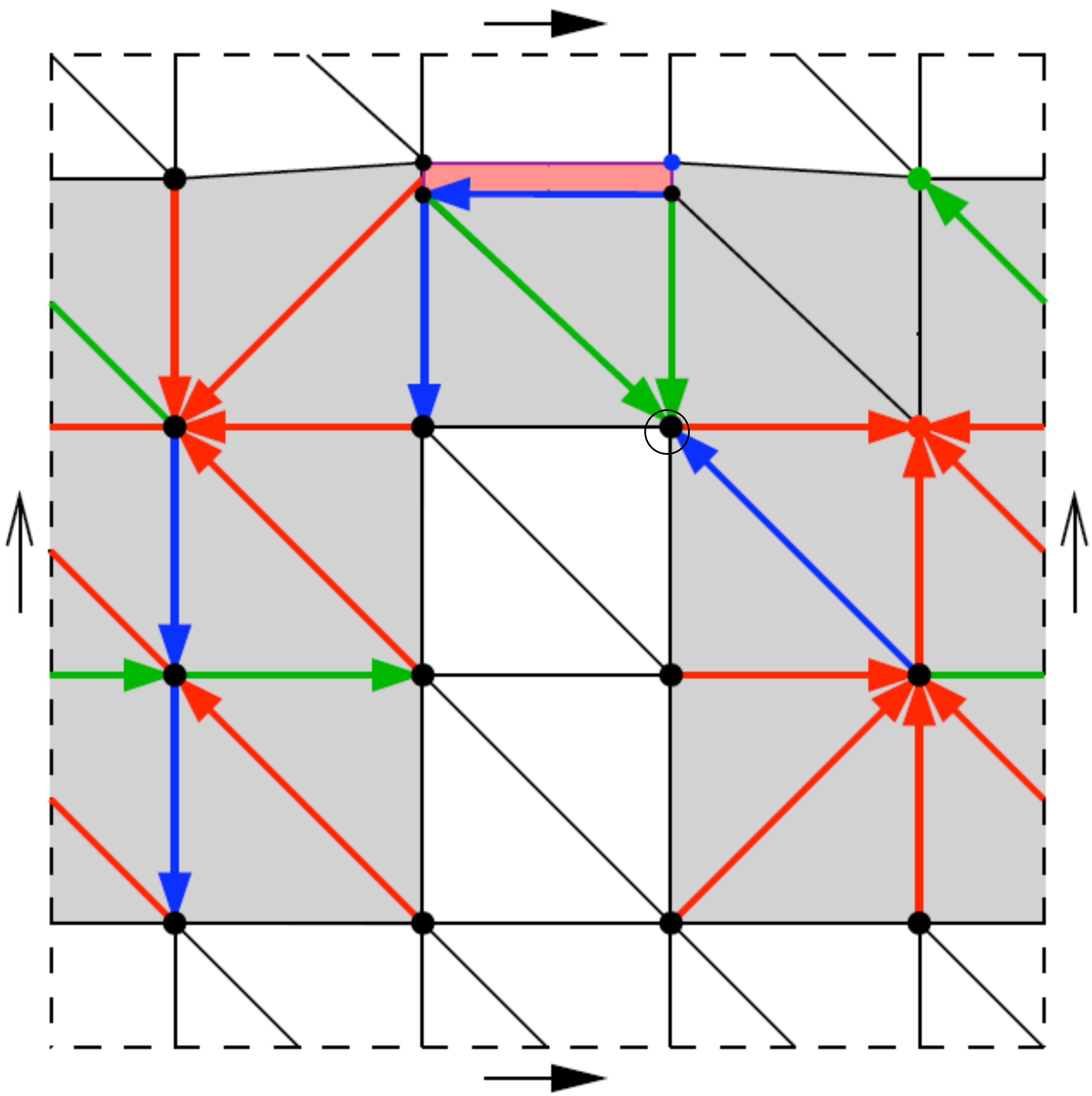


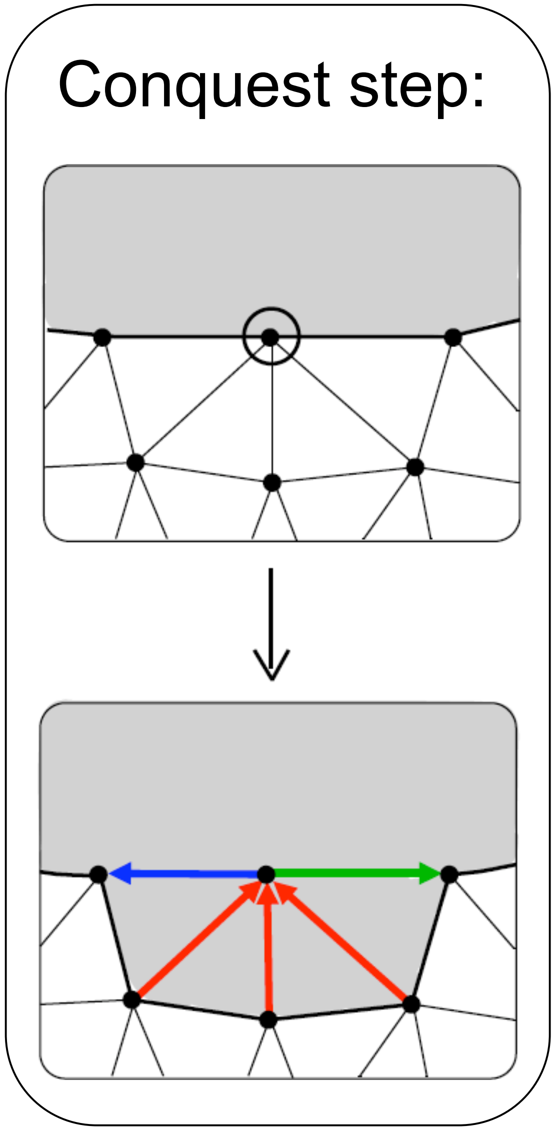
Conquest in higher genus



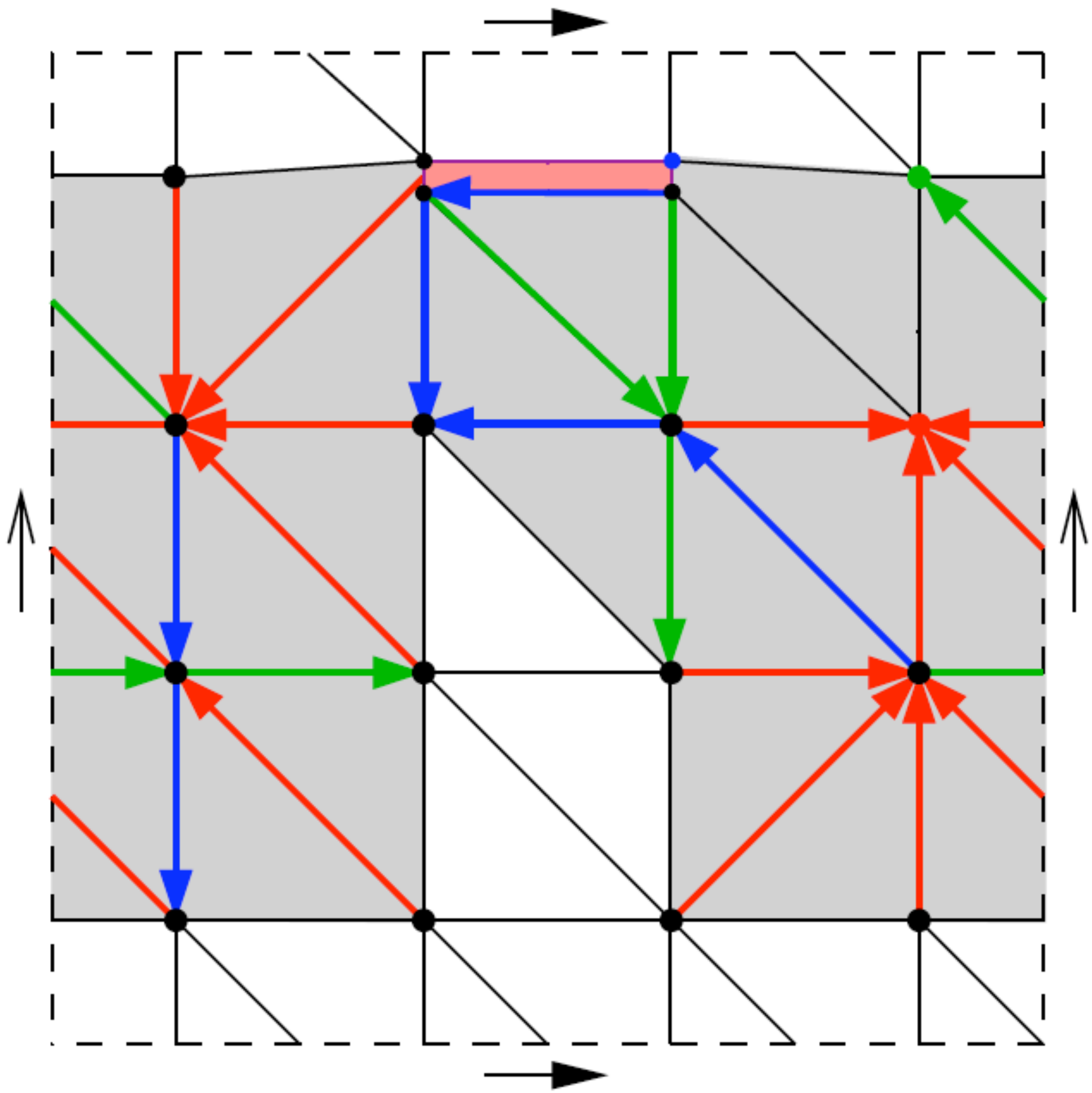


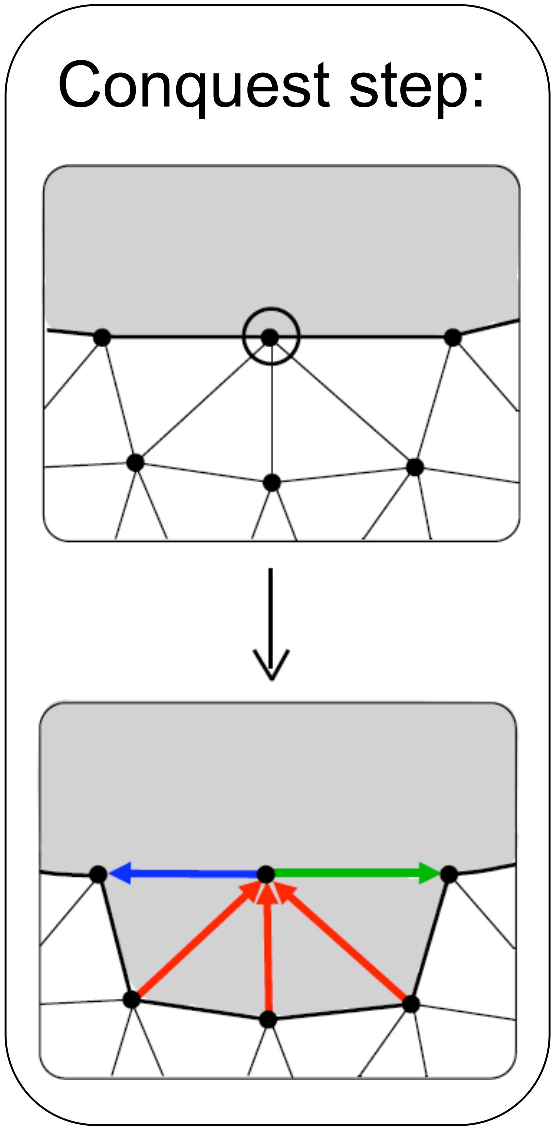
Conquest in higher genus



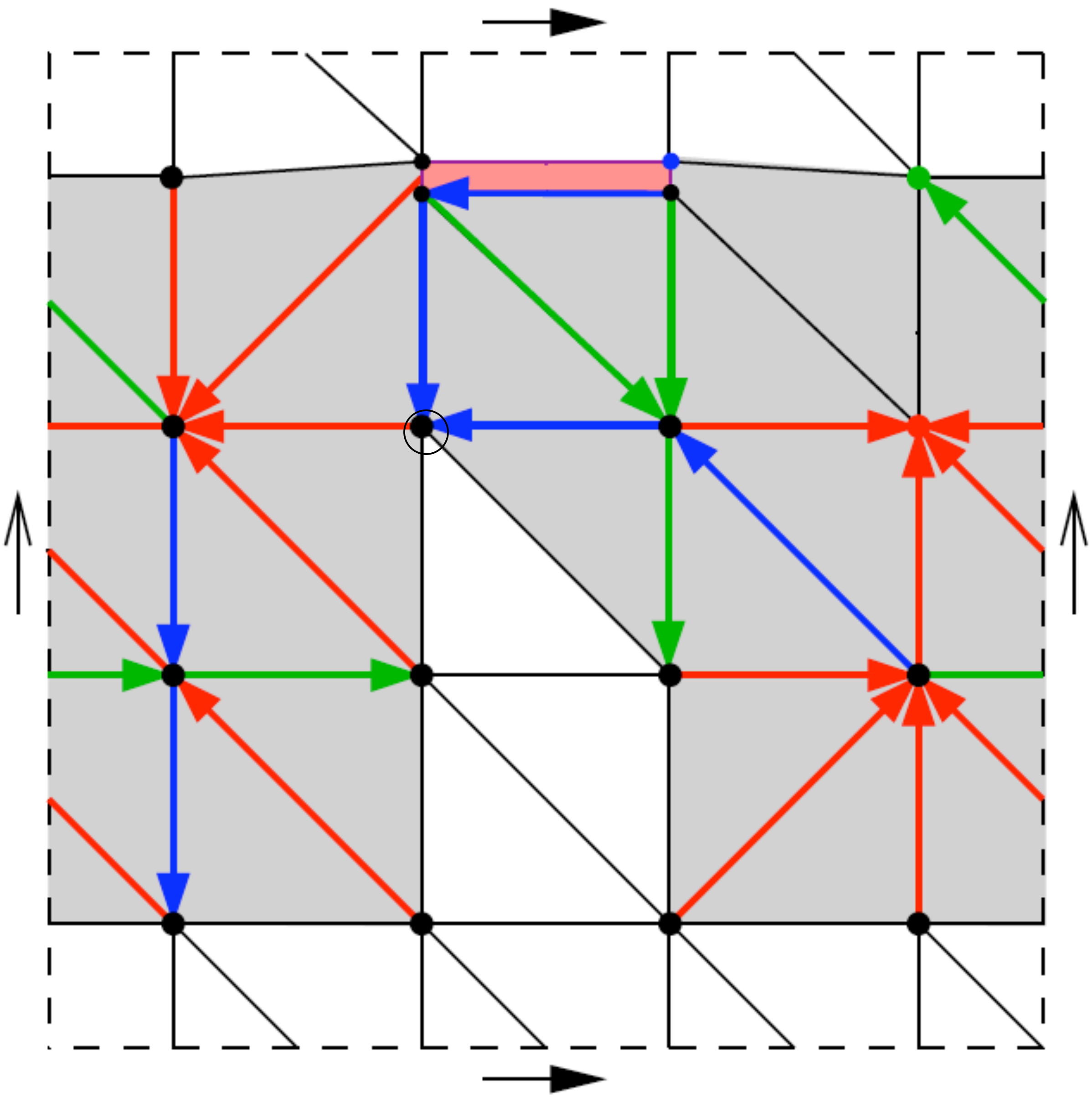


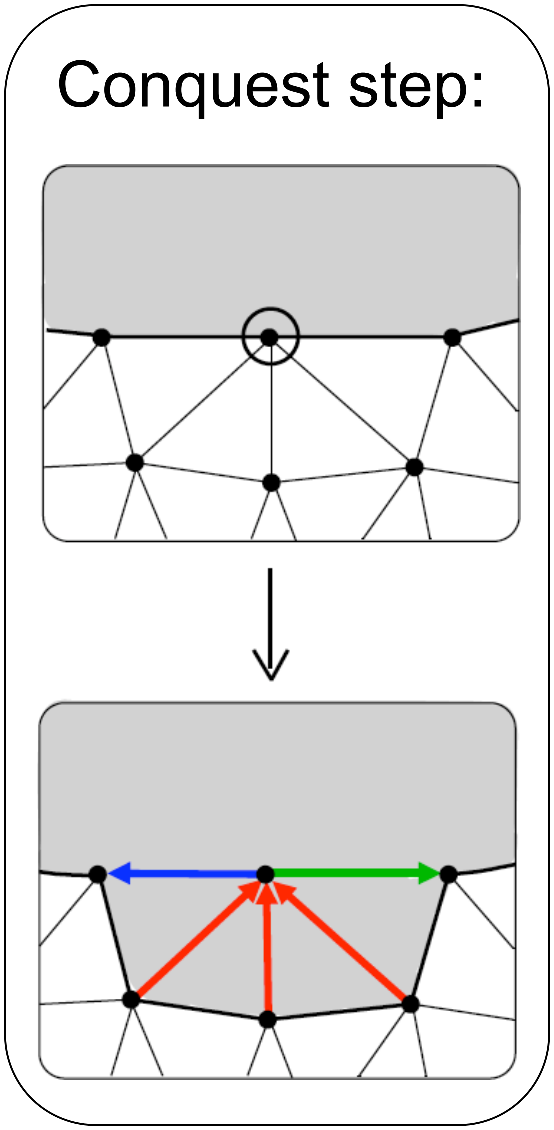
Conquest in higher genus



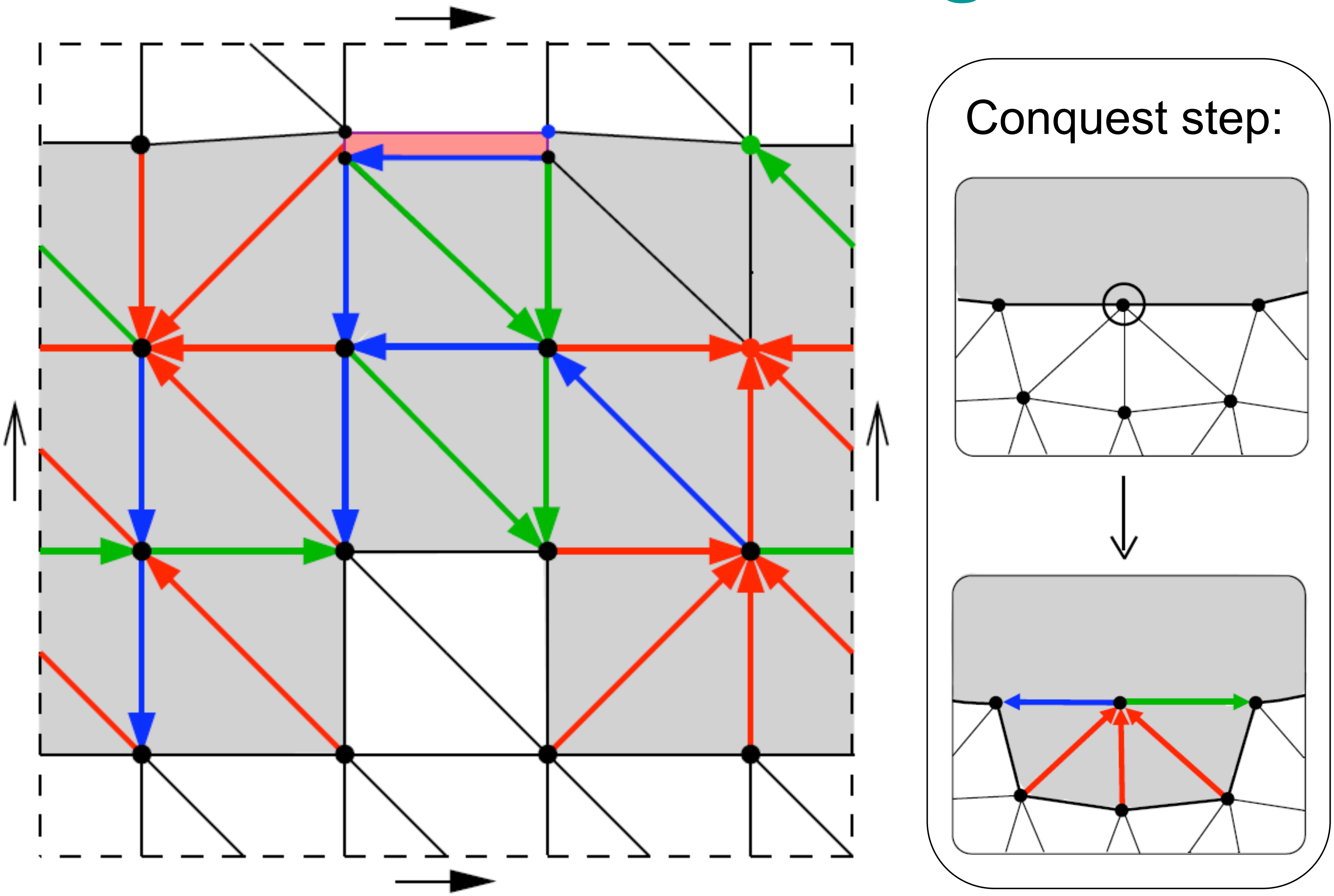


Conquest in higher genus

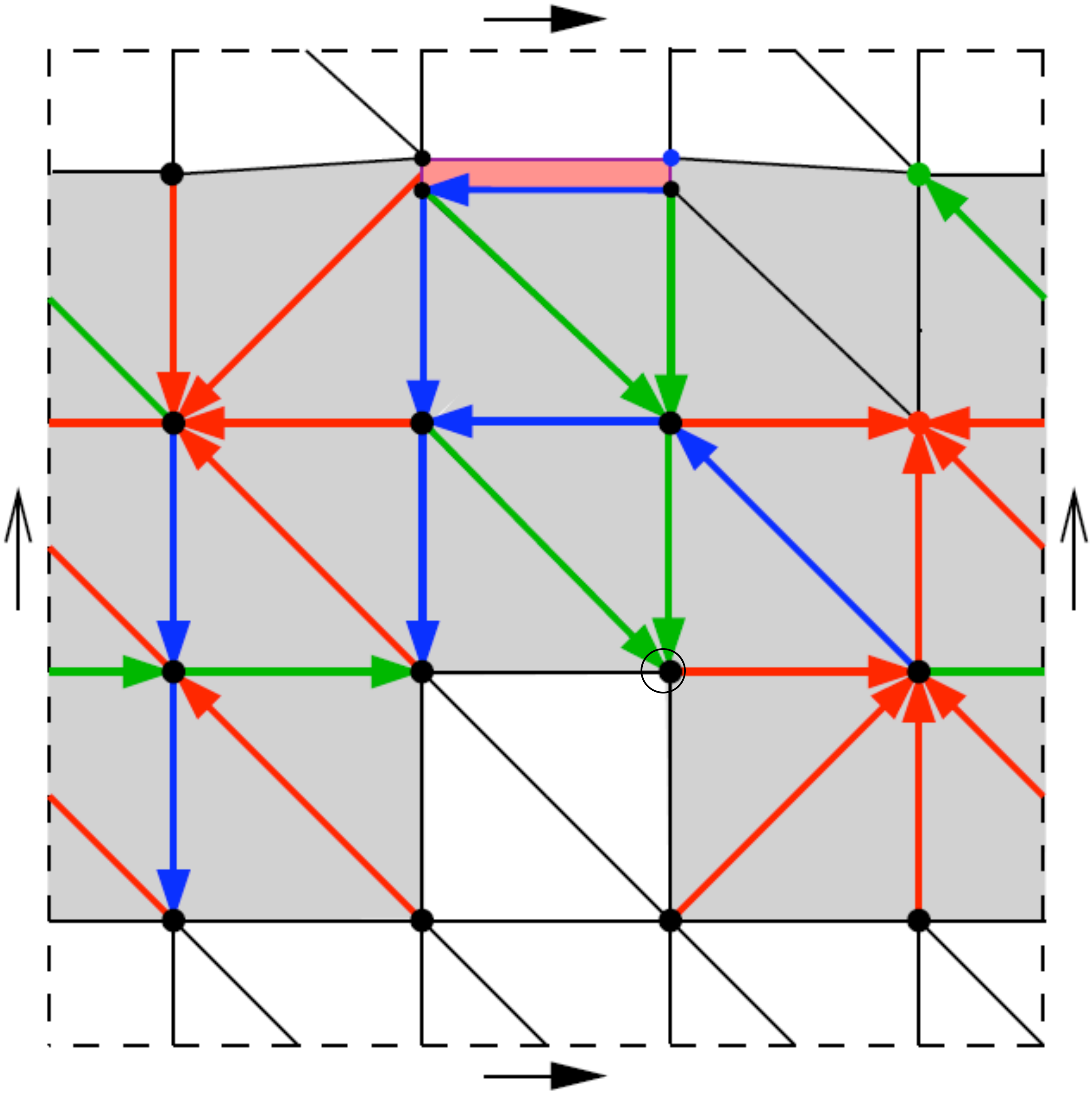


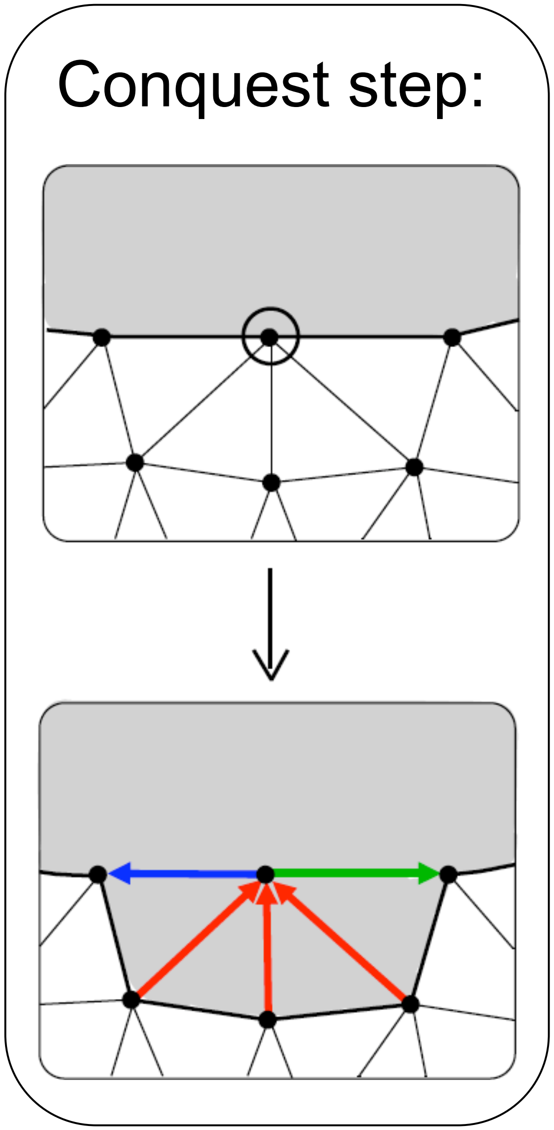


Conquest in higher genus

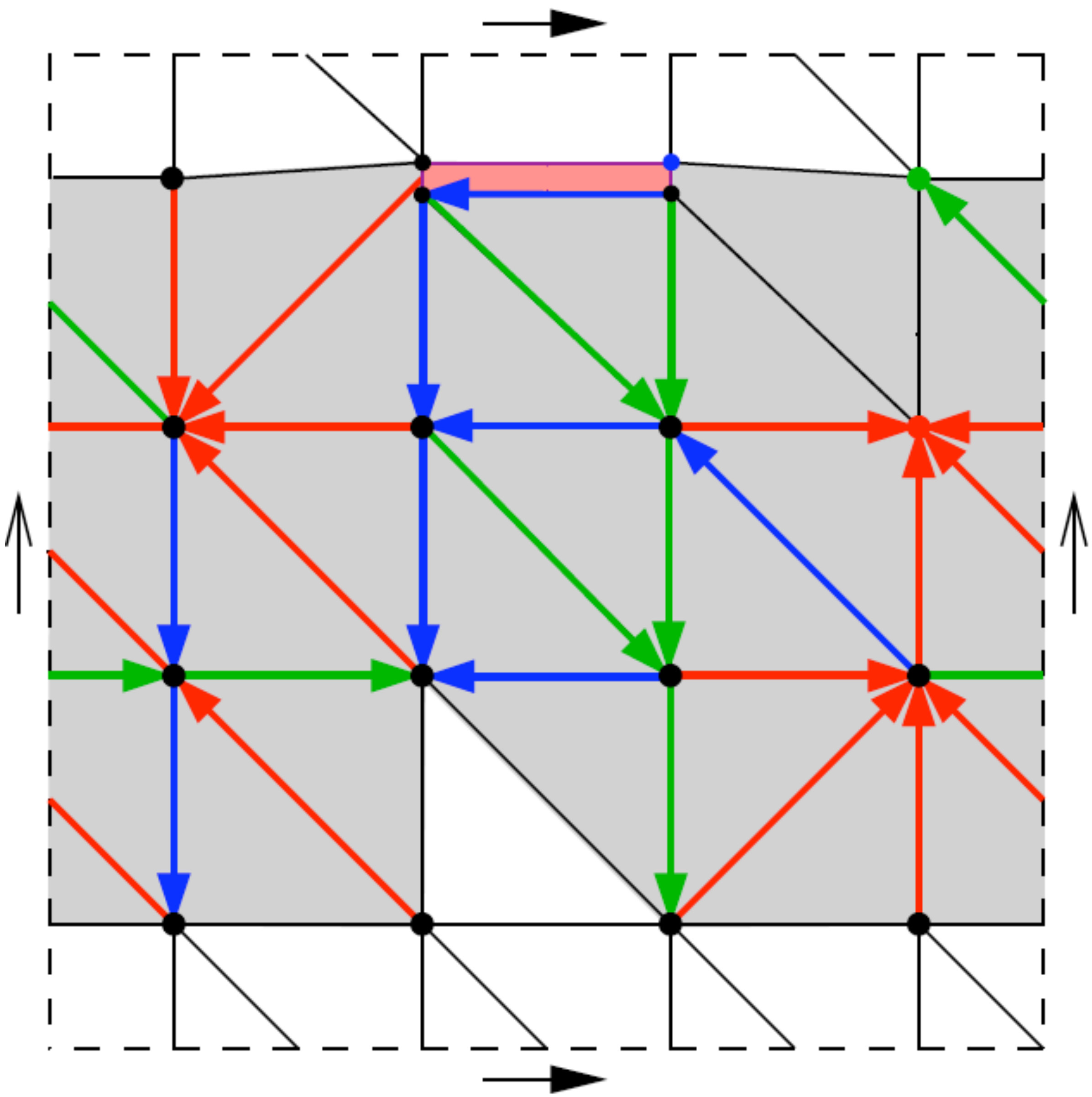


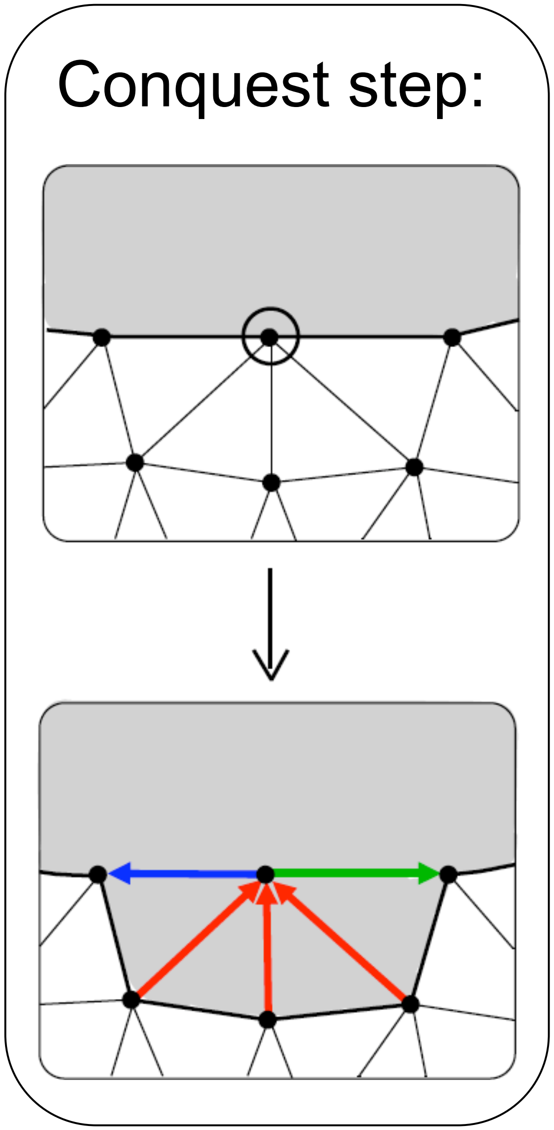
Conquest in higher genus



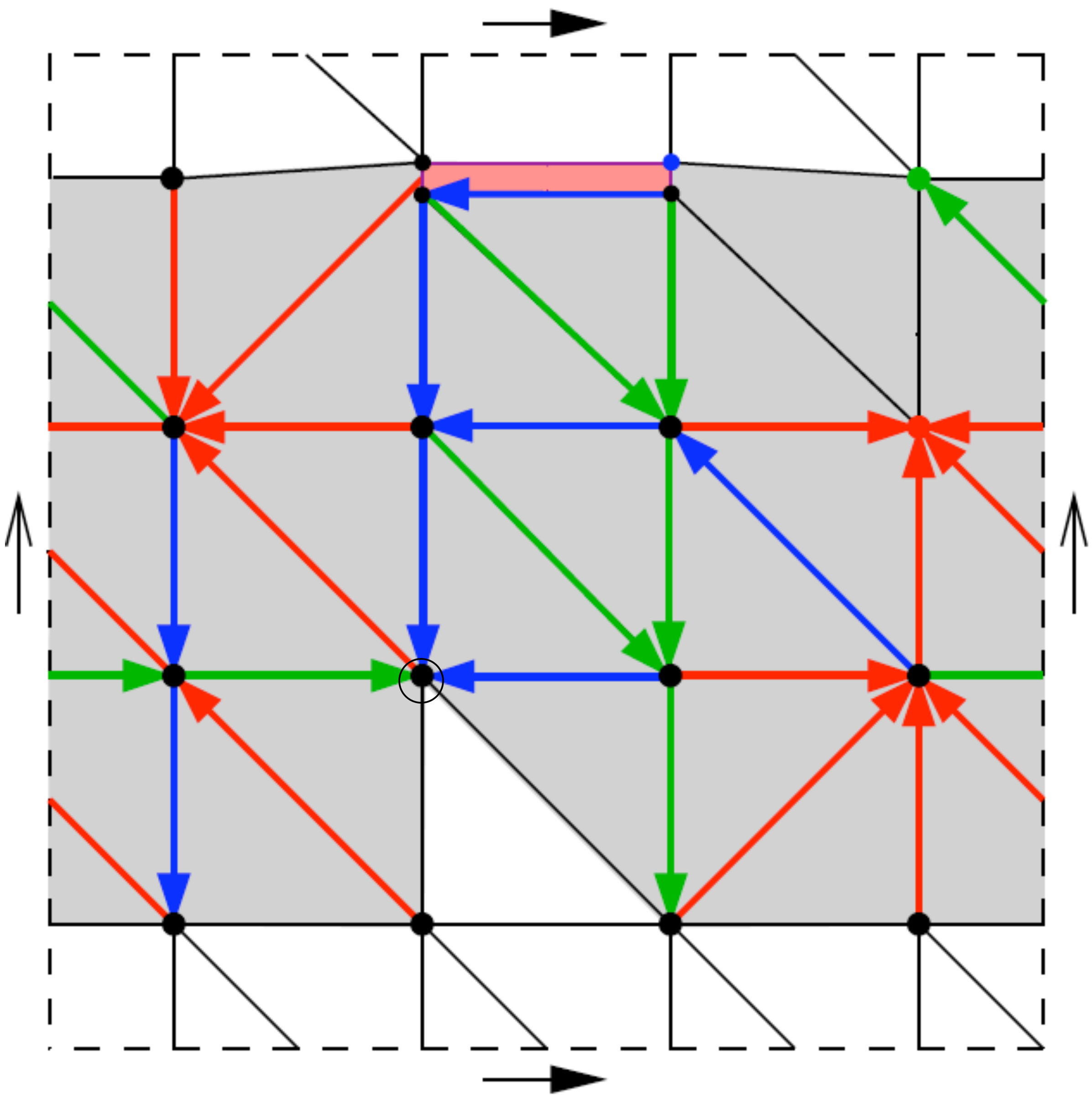


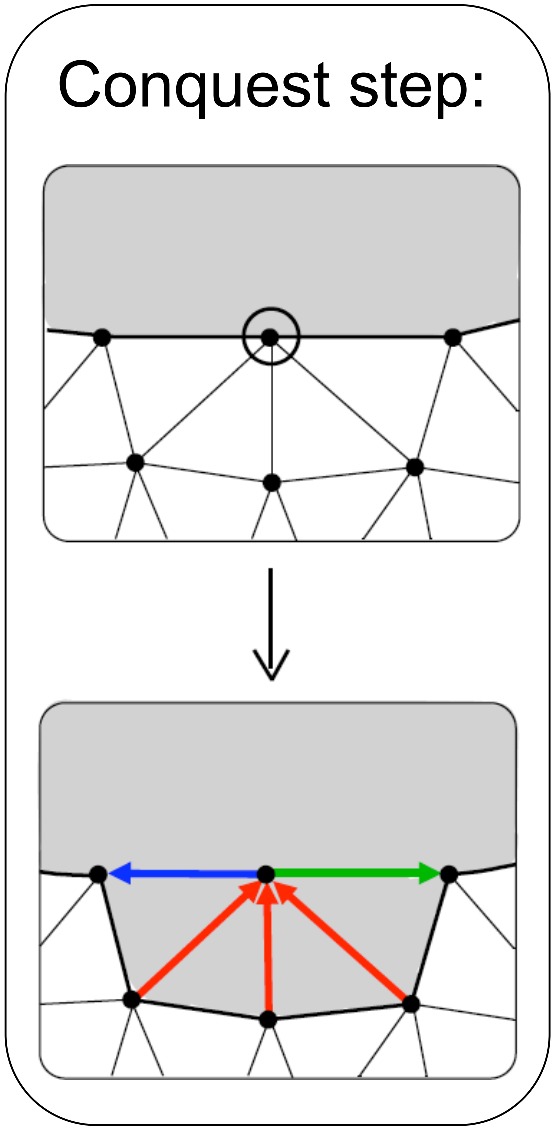
Conquest in higher genus



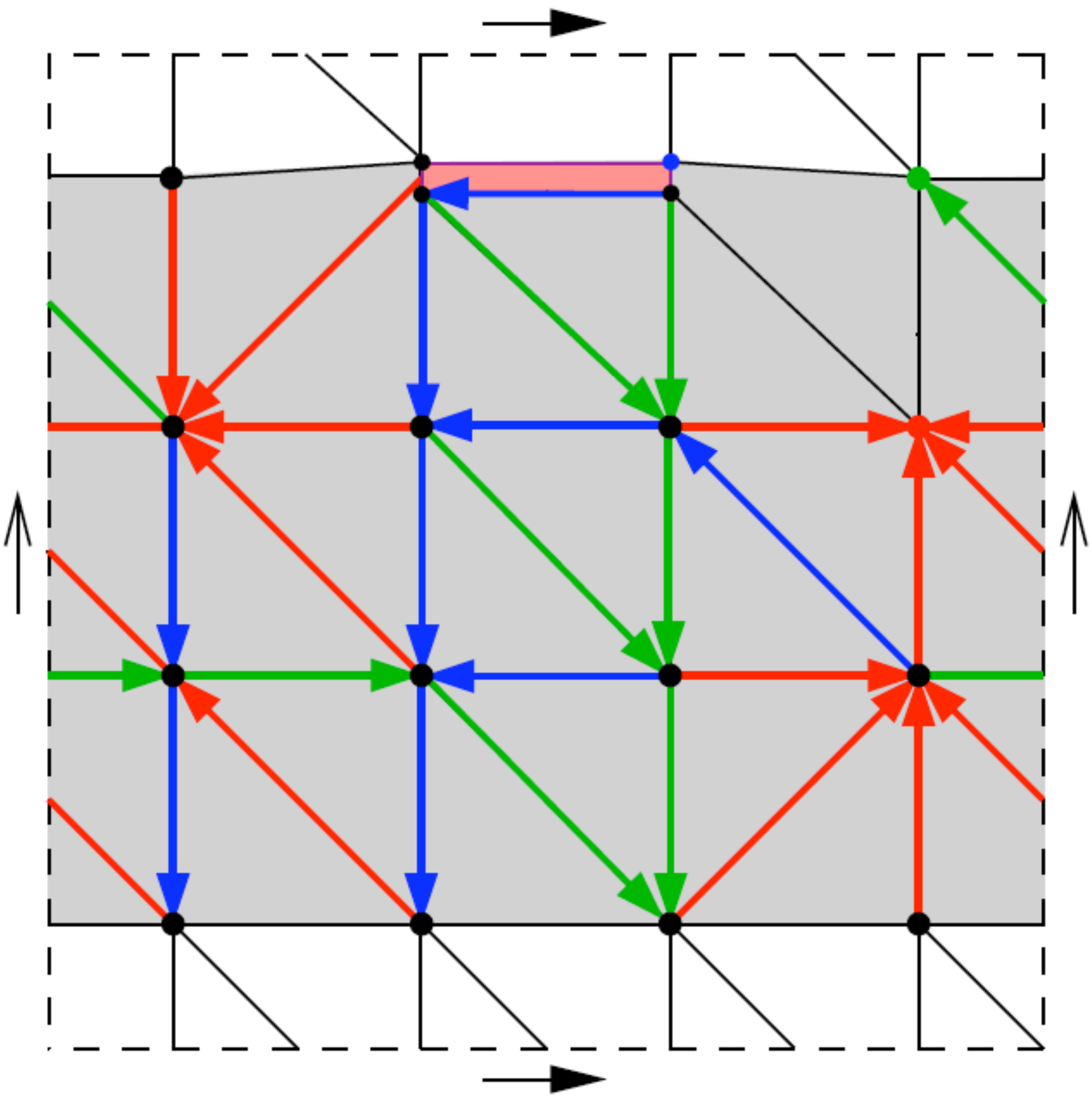


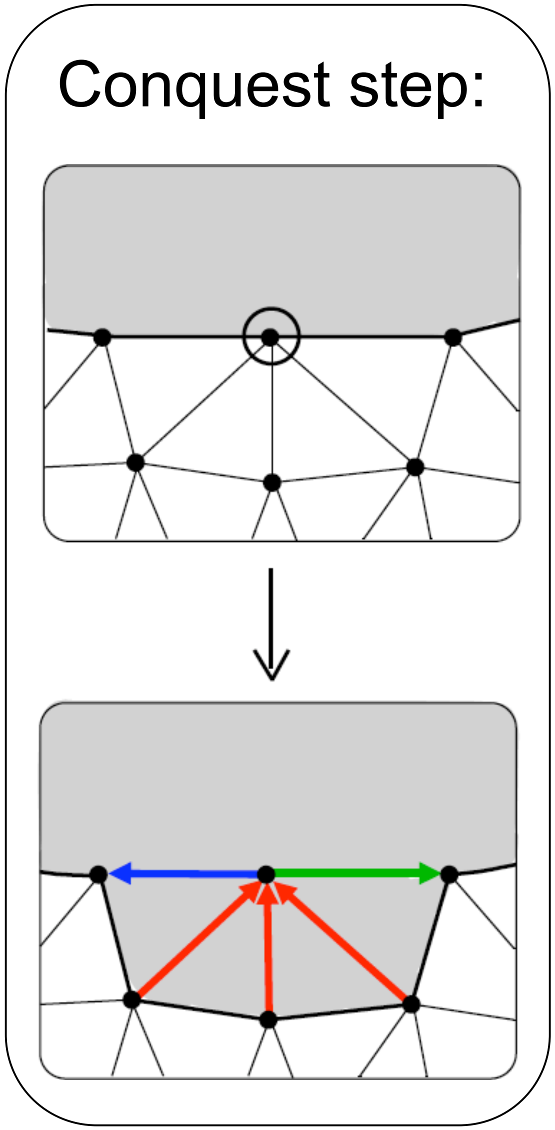
Conquest in higher genus



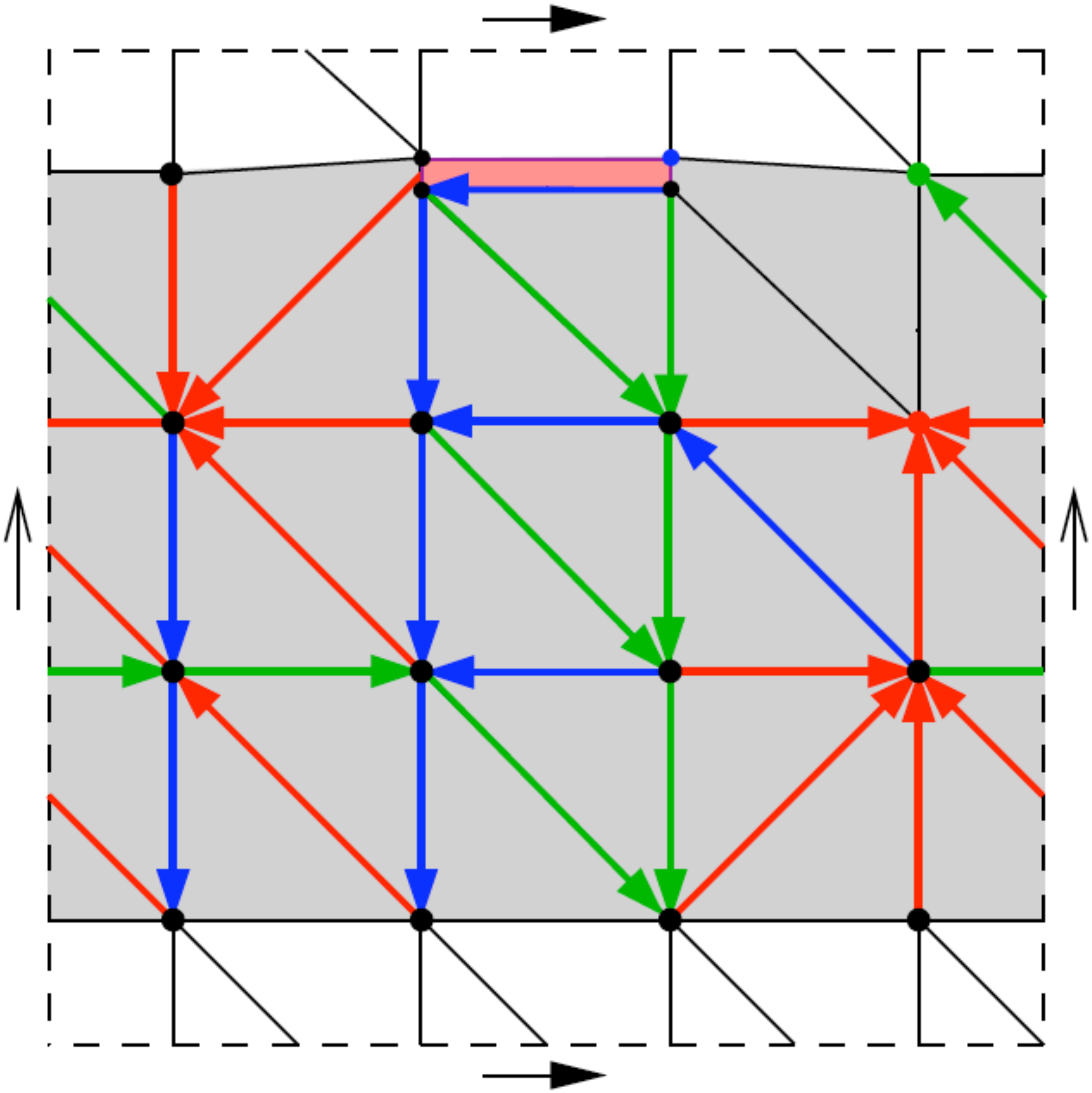


Conquest in higher genus



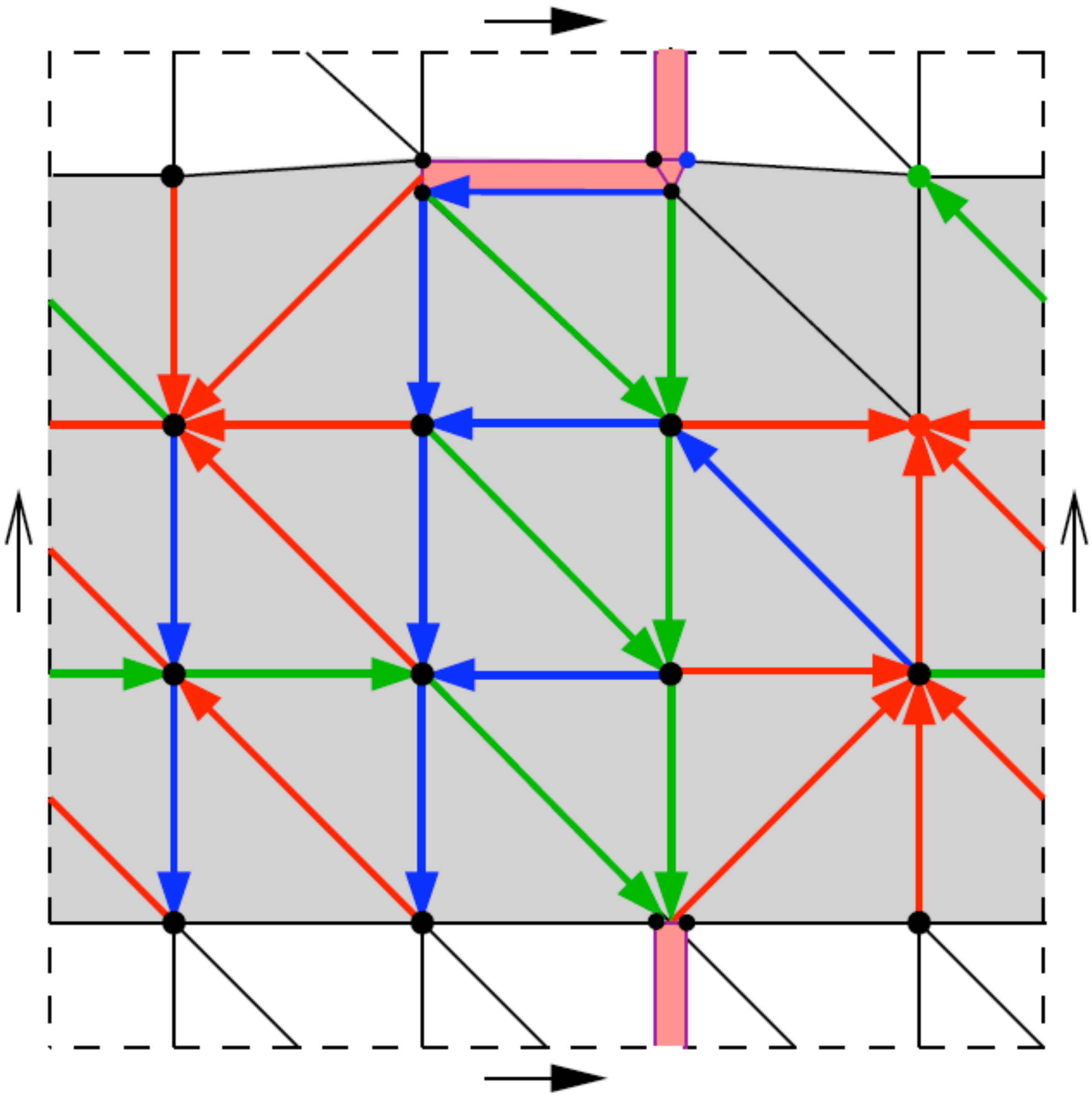


Conquest in higher genus



Can not extend conquered area C

Conquest in higher genus



Can not extend conquered area *C*

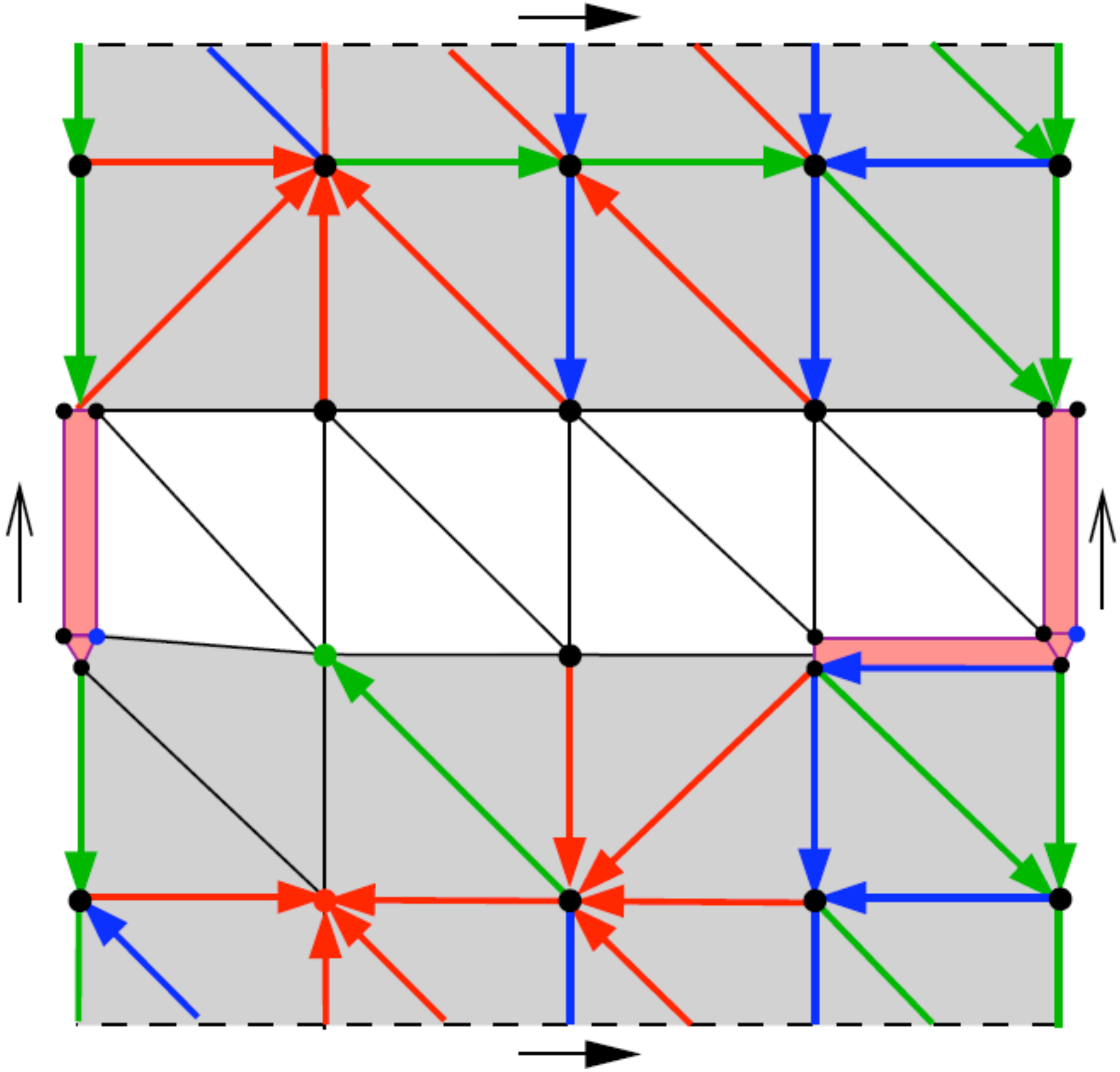
Special step:

- choose chord *e*
- make it fat
- add it to *C*

C : cylinder->torus

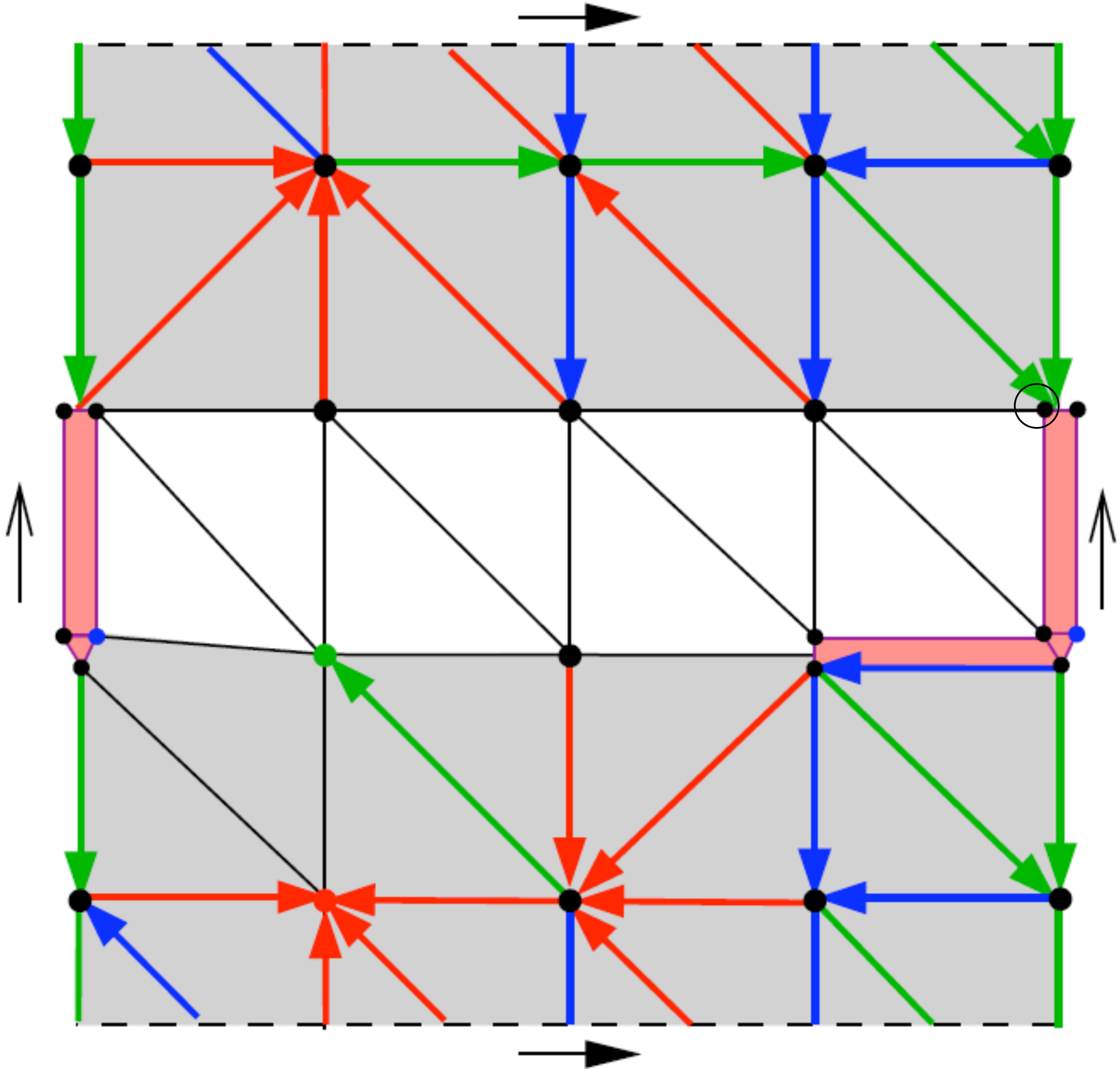
$S_g \backslash C$: cylinder->disk

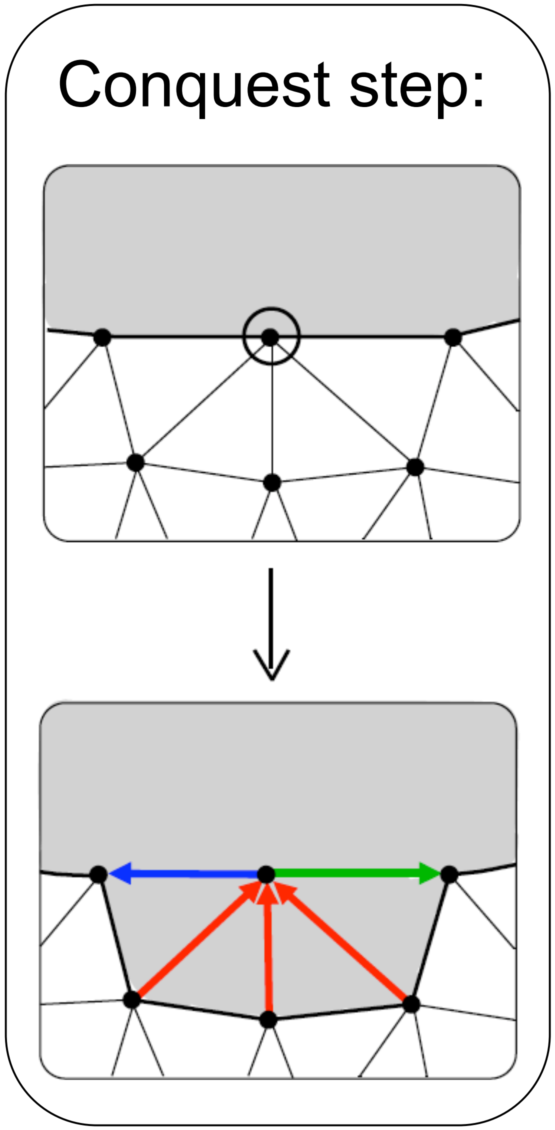
Conquest in higher genus



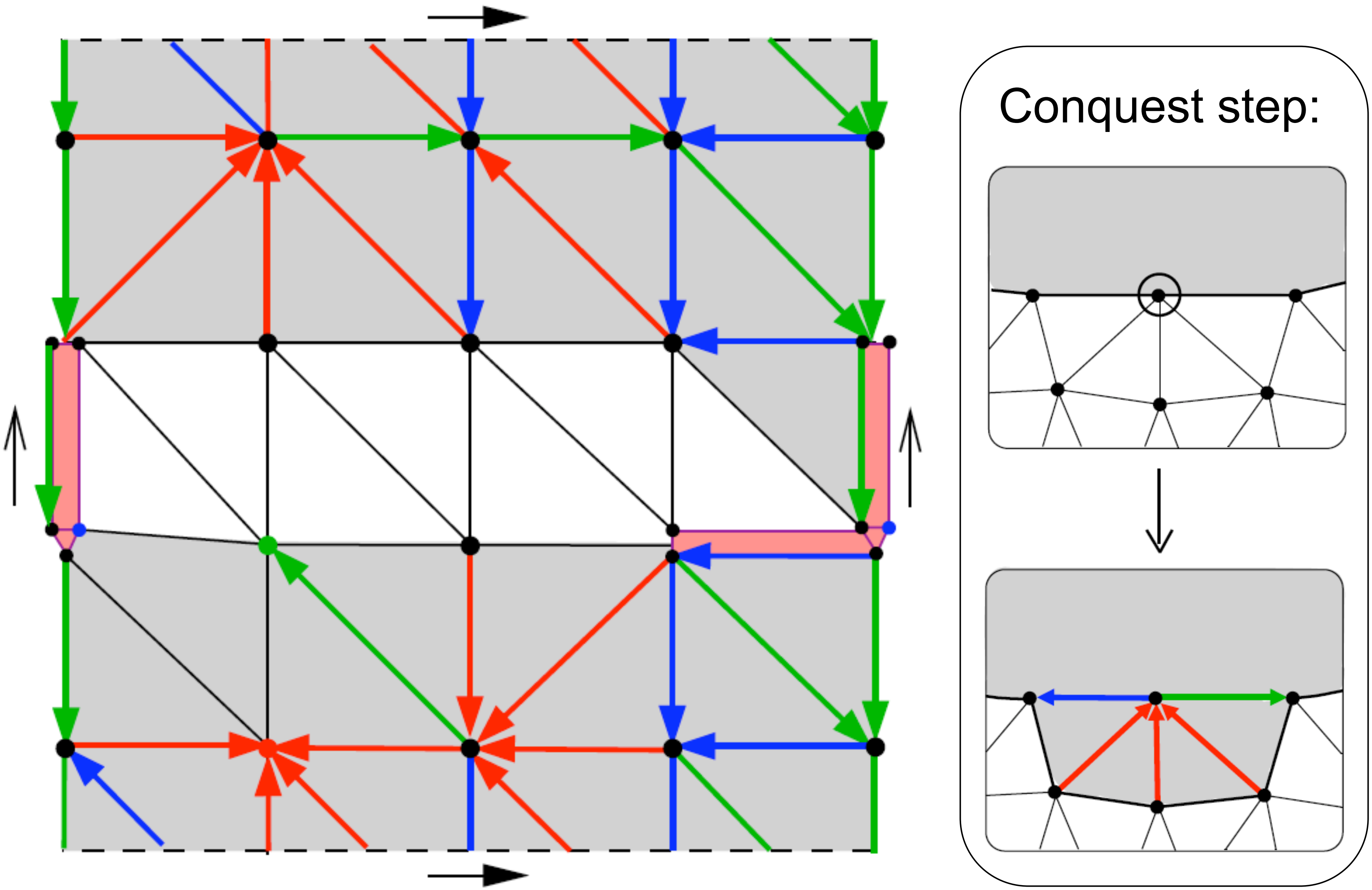
Continue and finish !

Conquest in higher genus

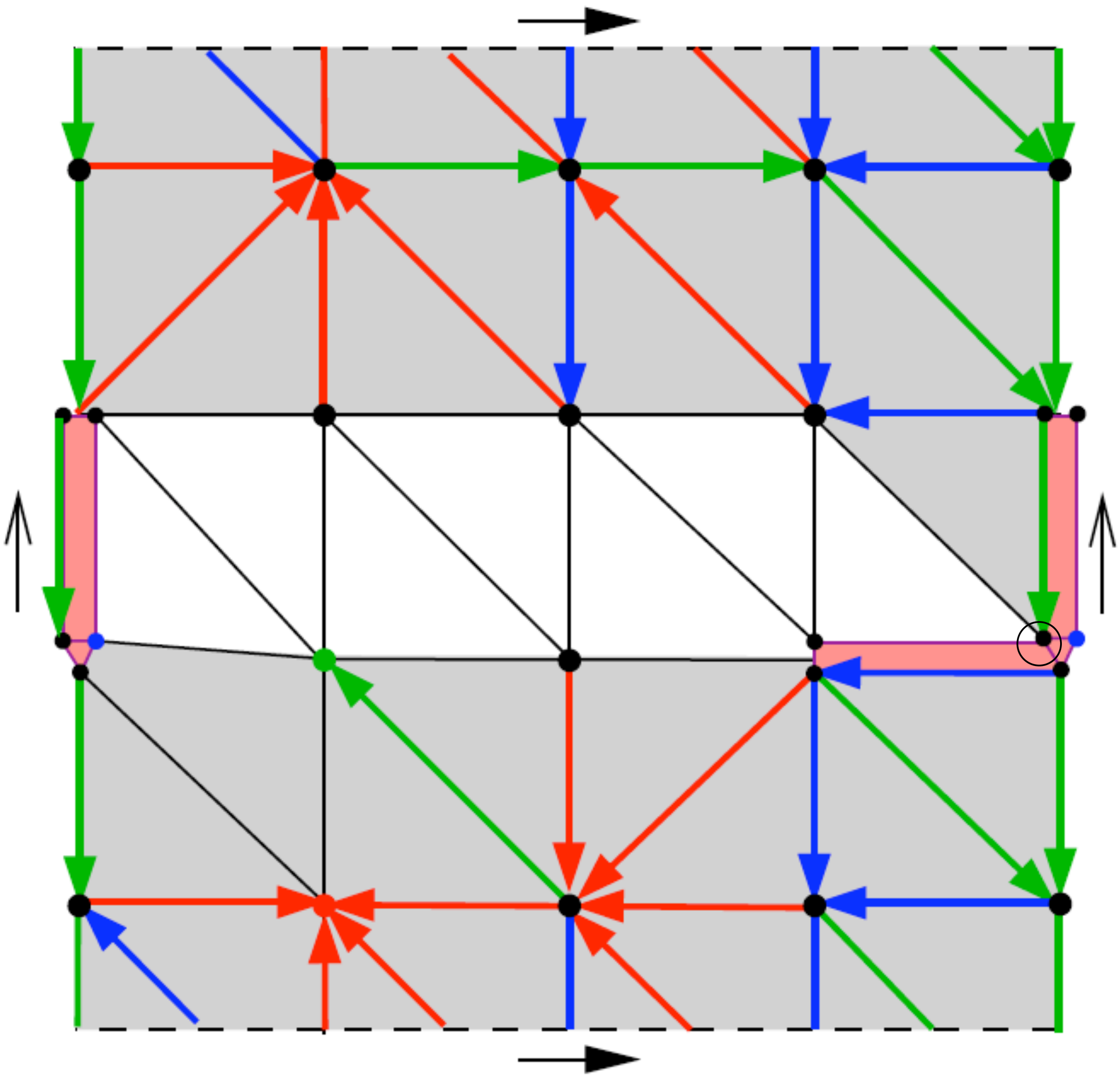


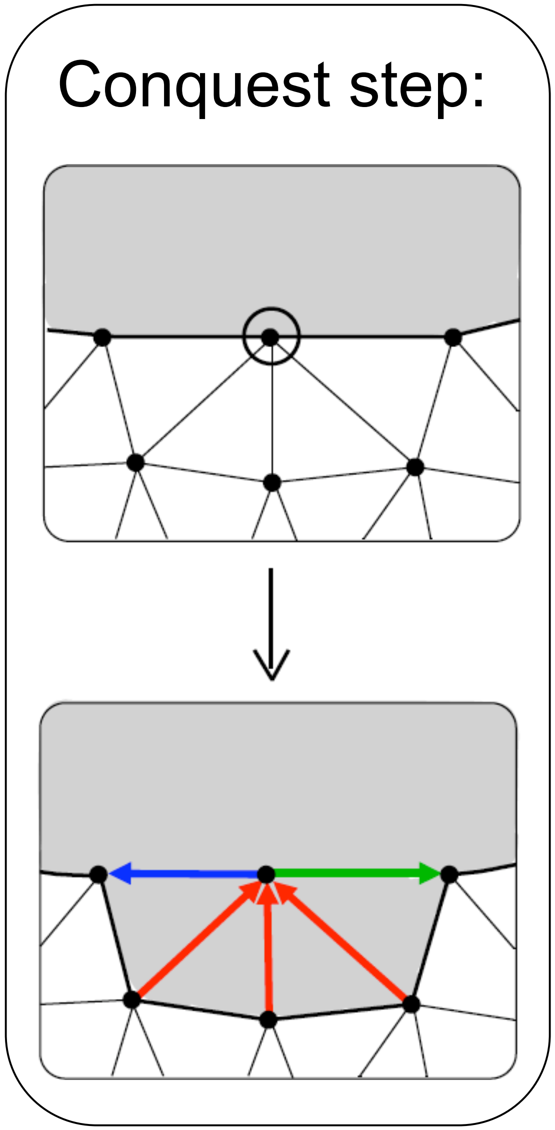


Conquest in higher genus

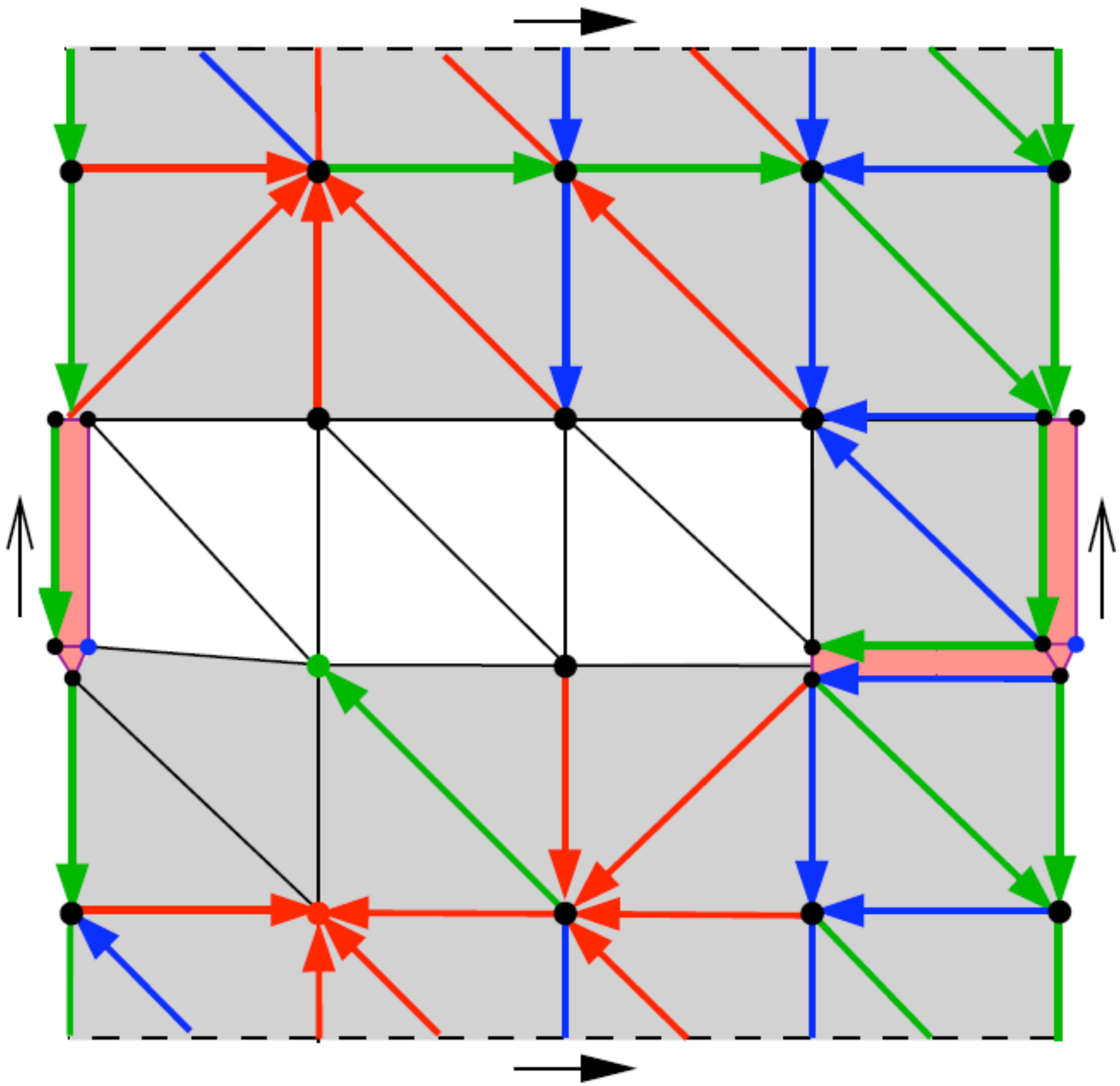


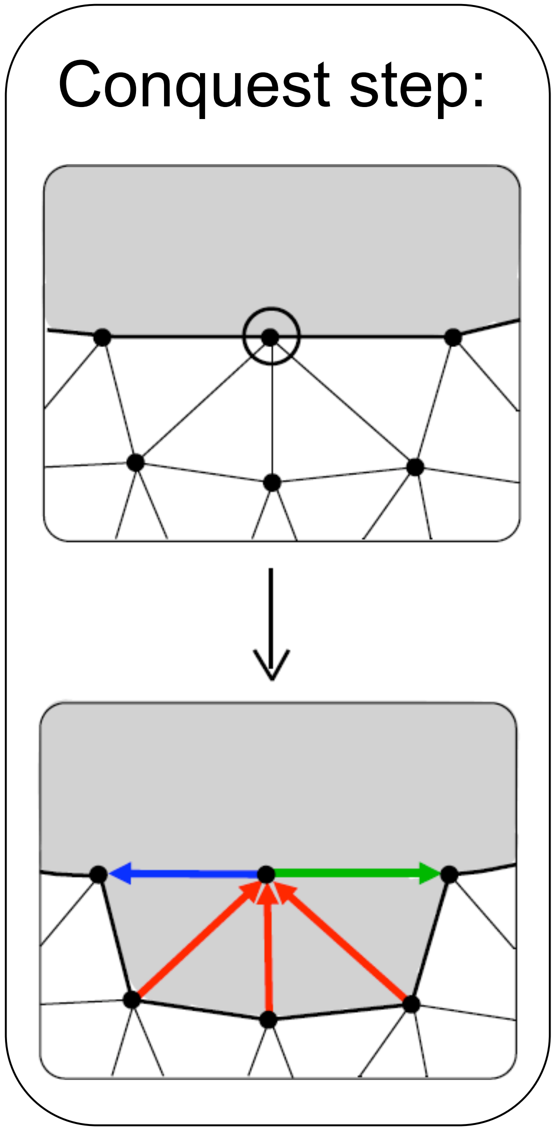
Conquest in higher genus



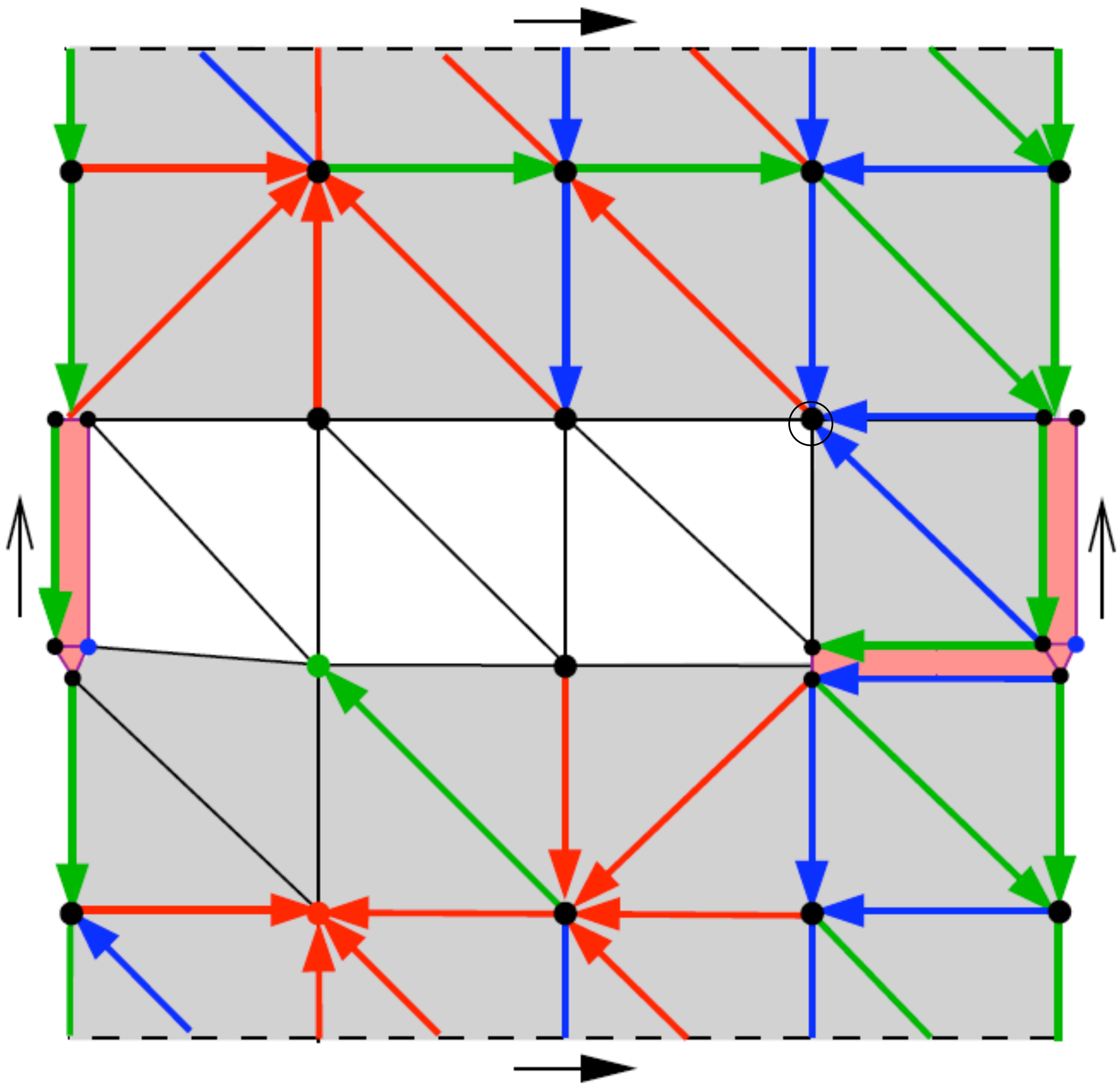


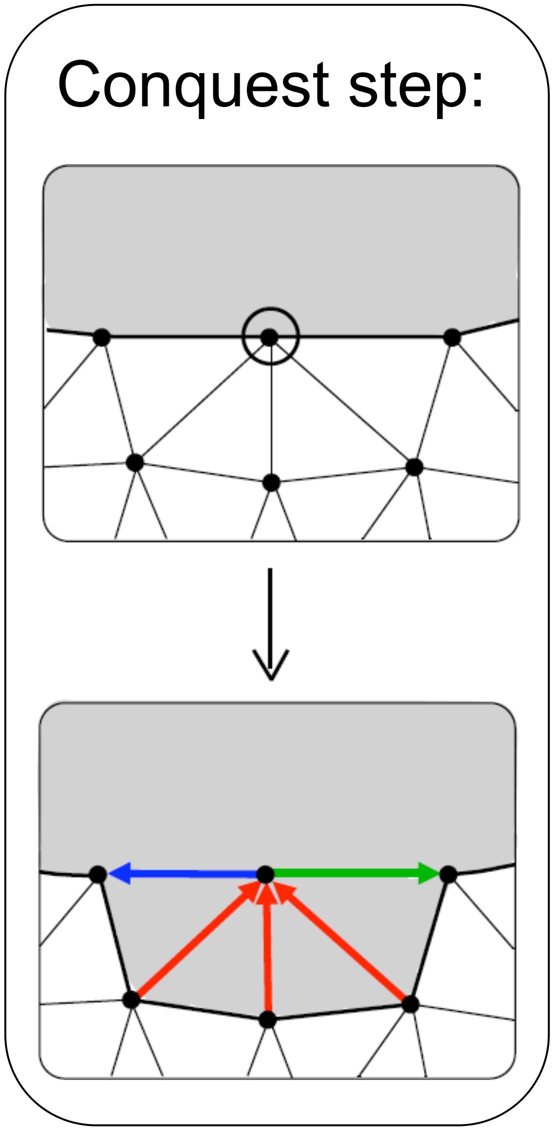
Conquest in higher genus



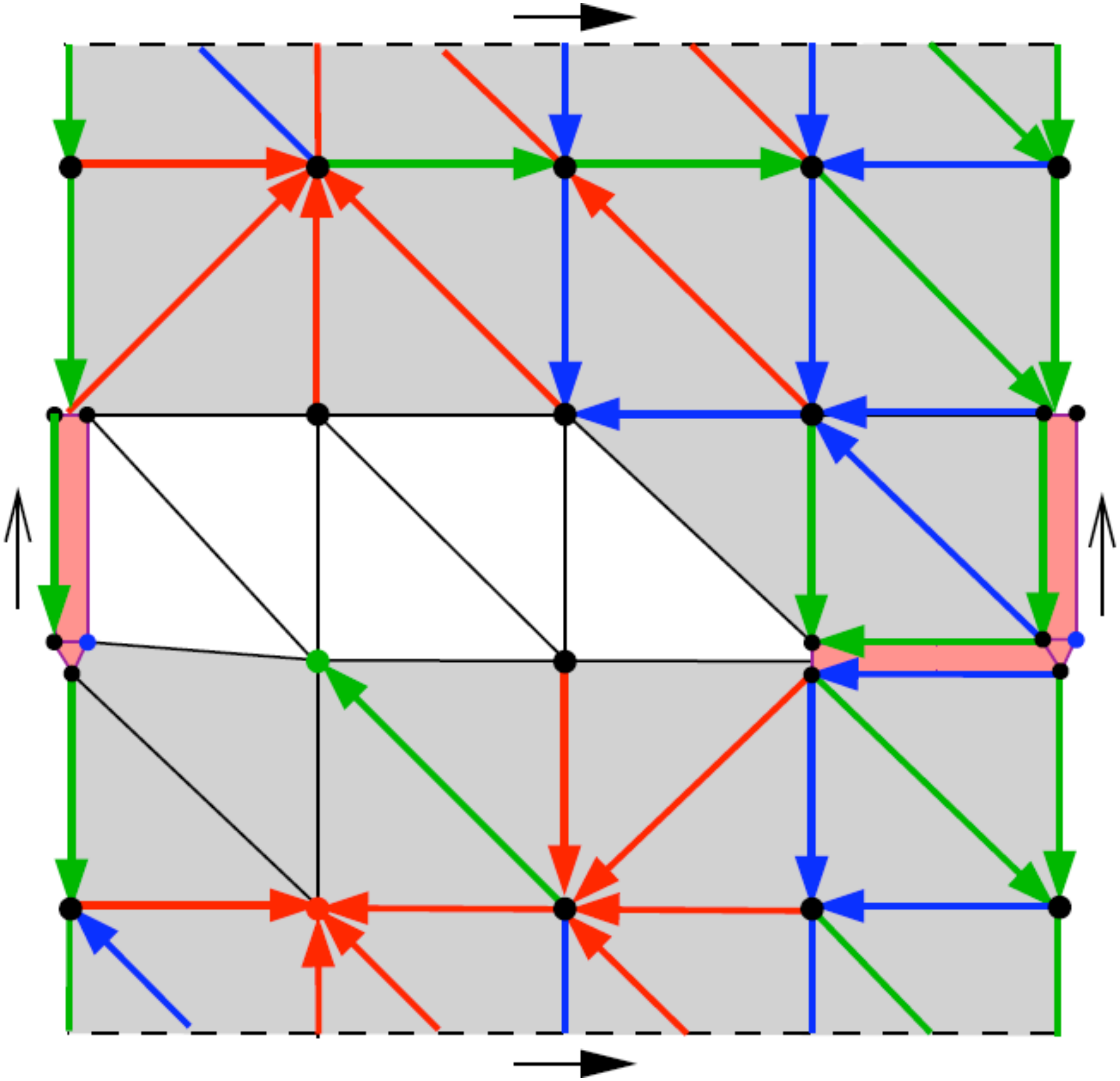


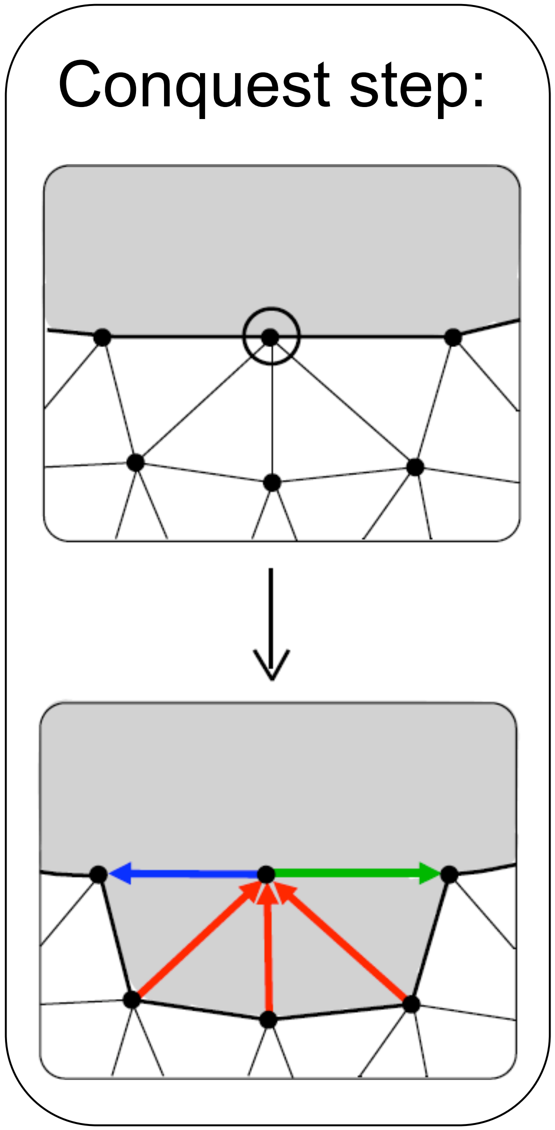
Conquest in higher genus



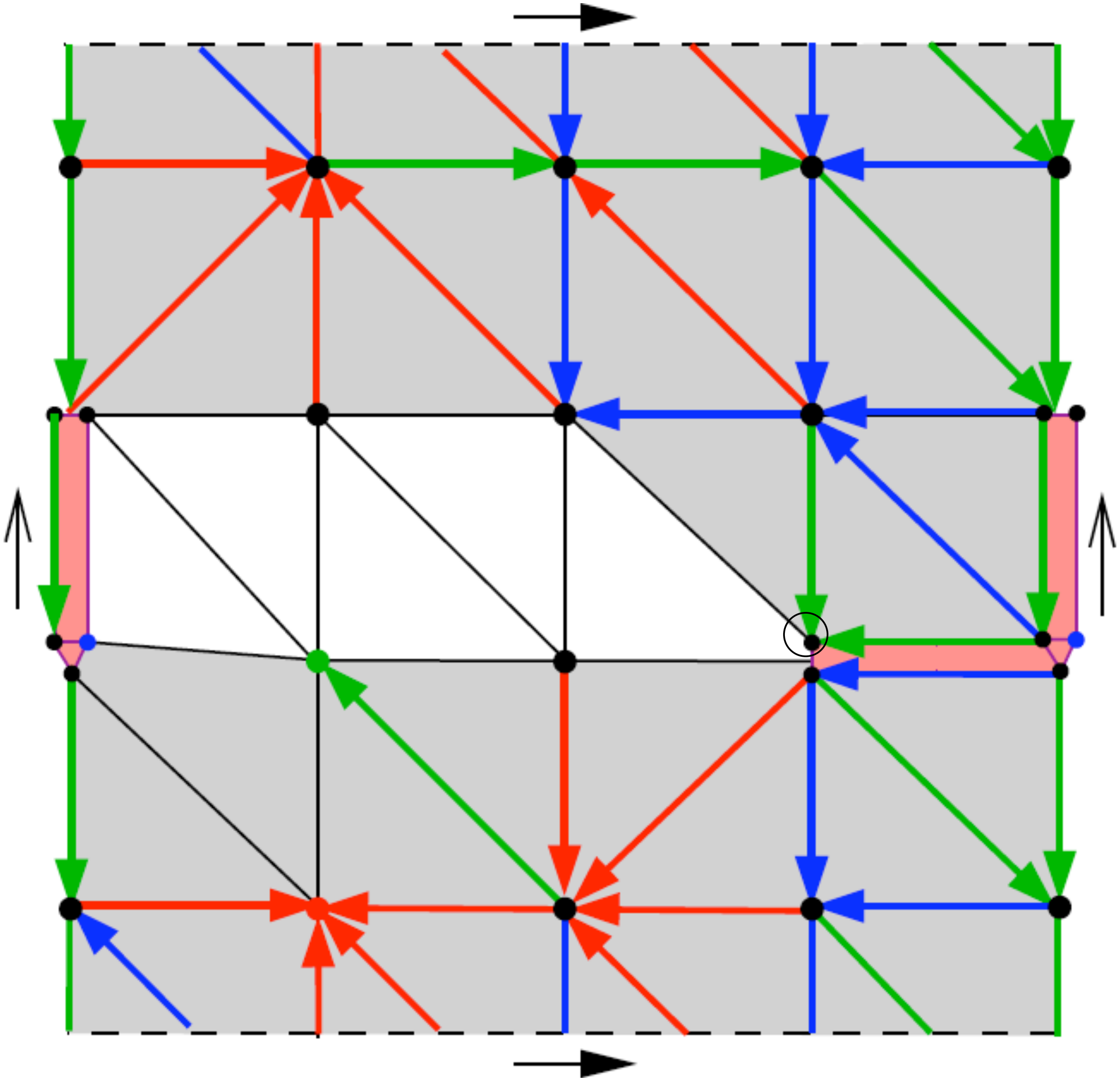


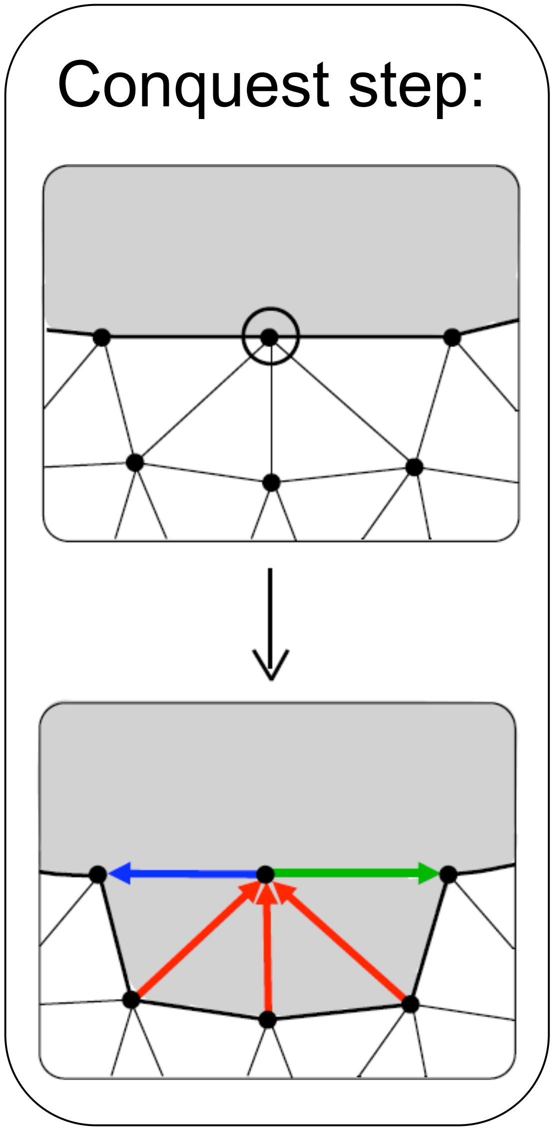
Conquest in higher genus



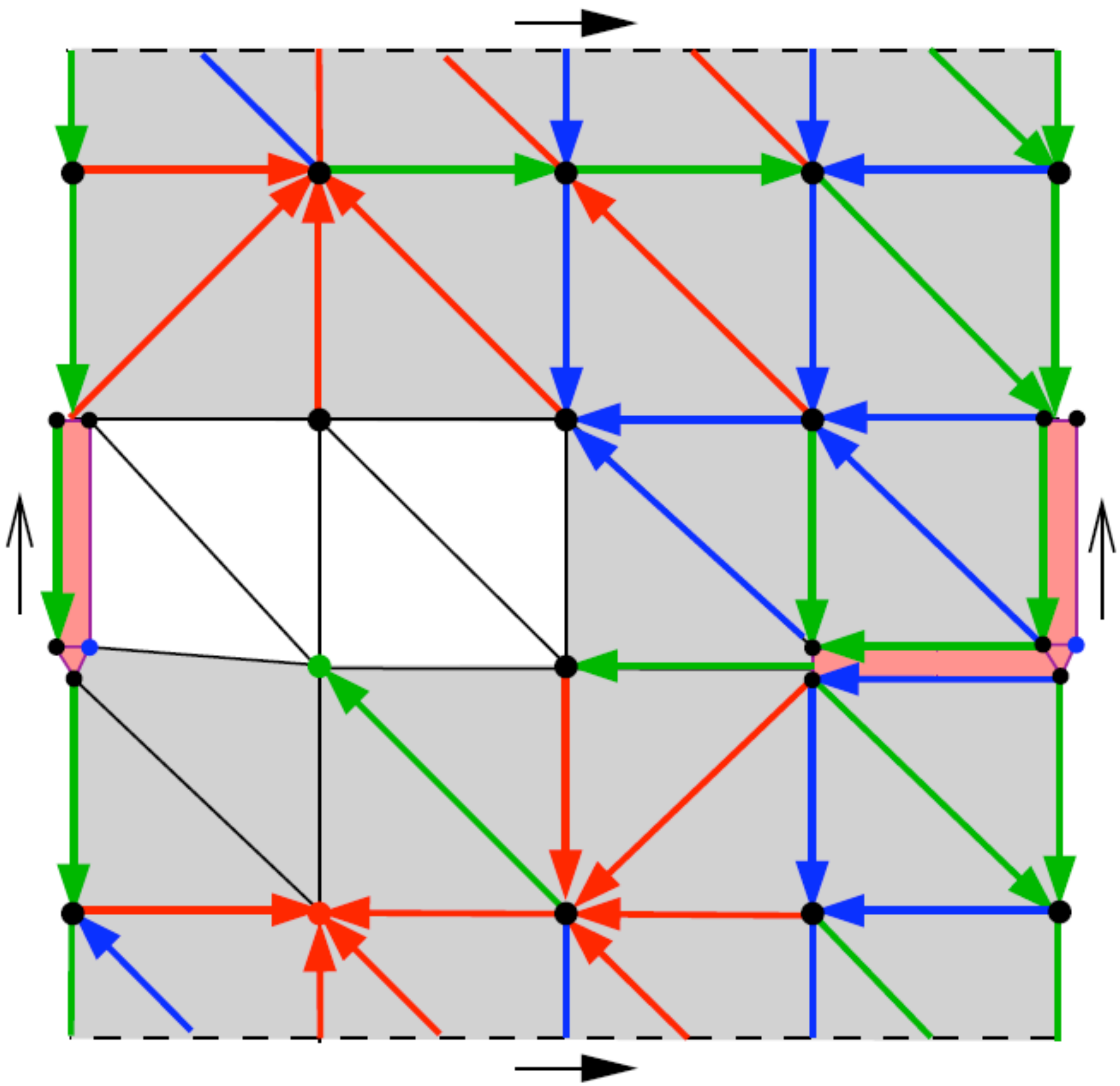


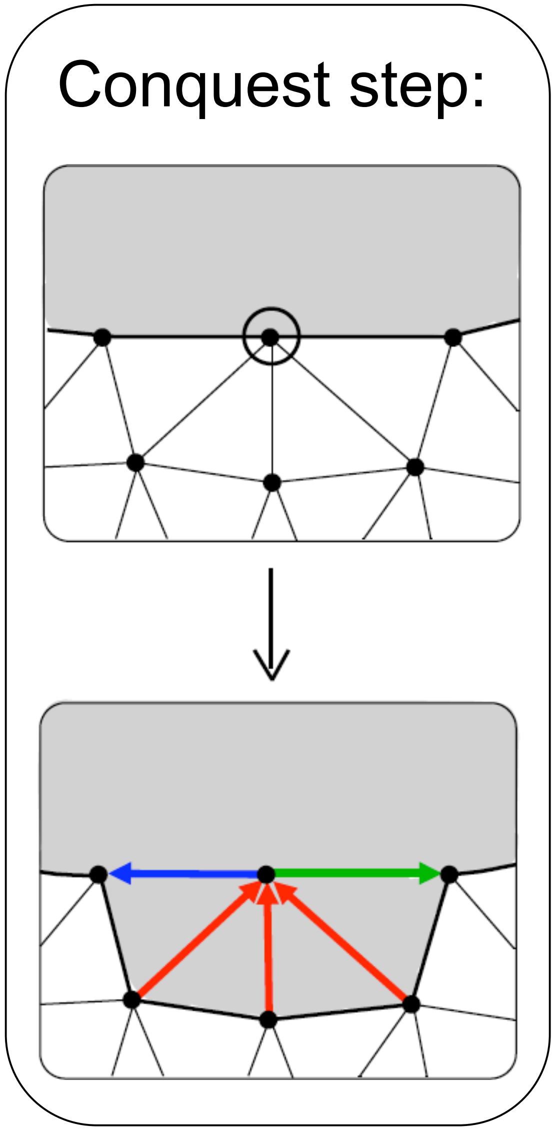
Conquest in higher genus



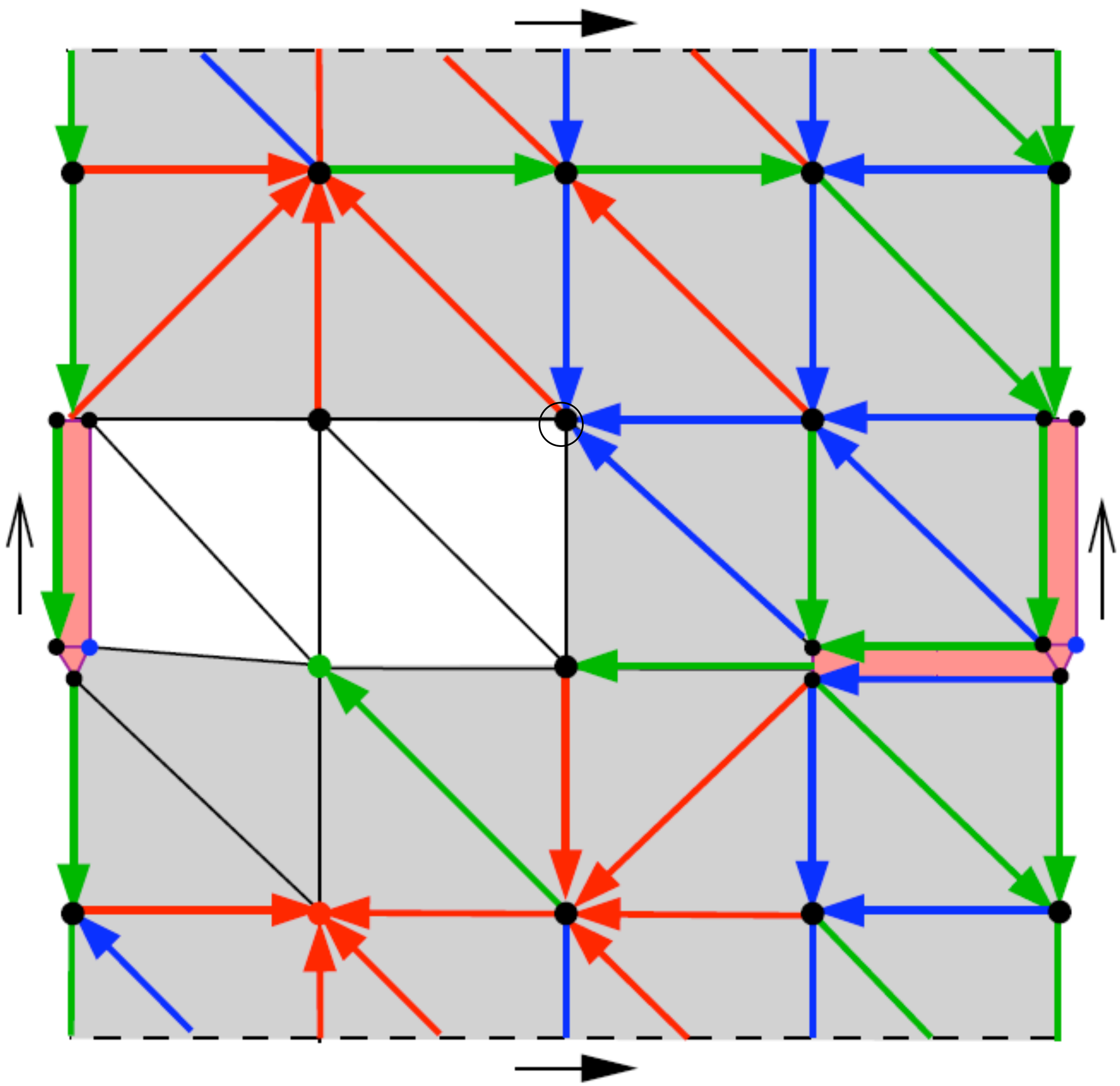


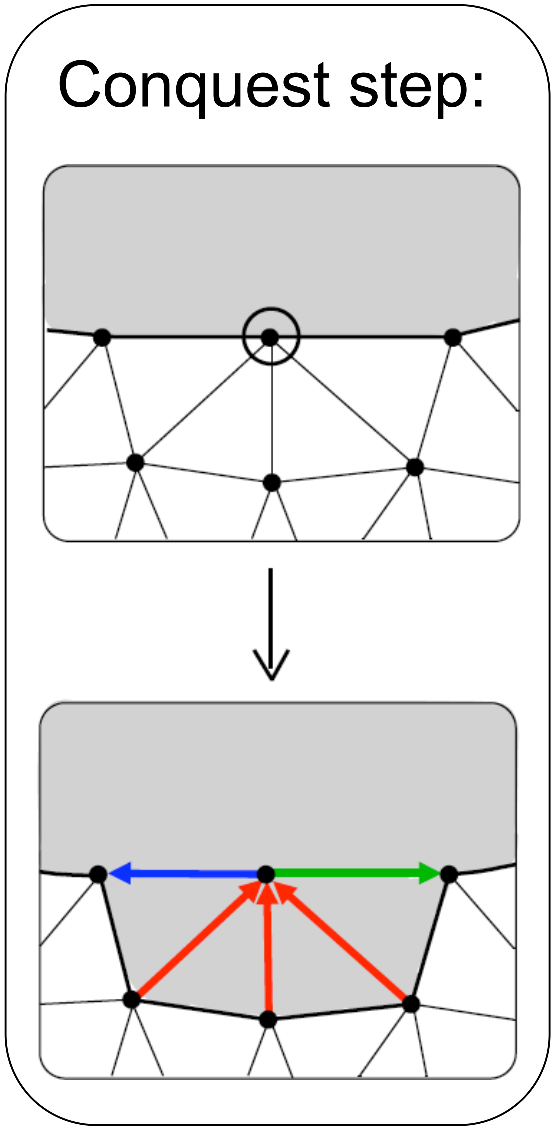
Conquest in higher genus



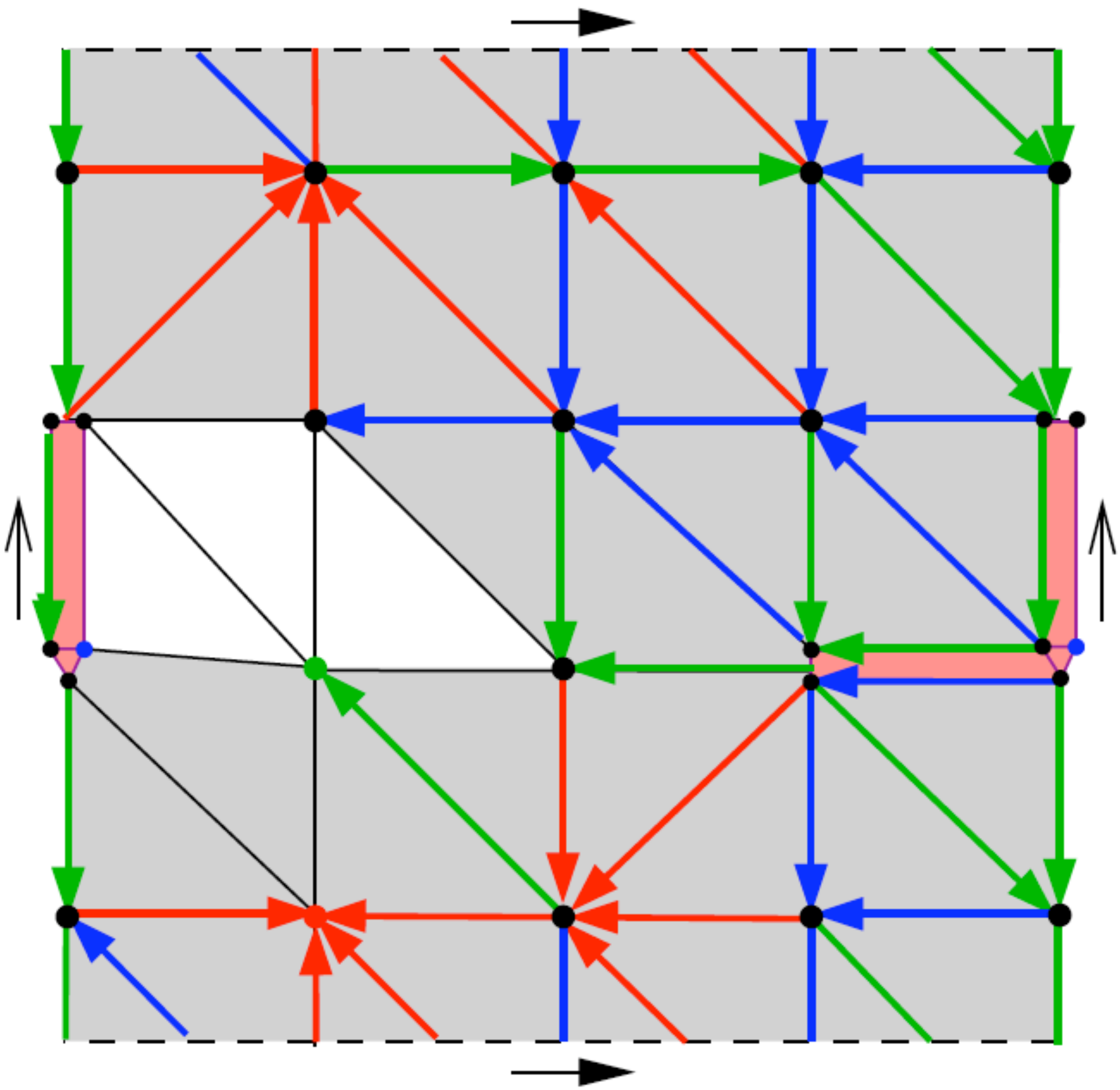


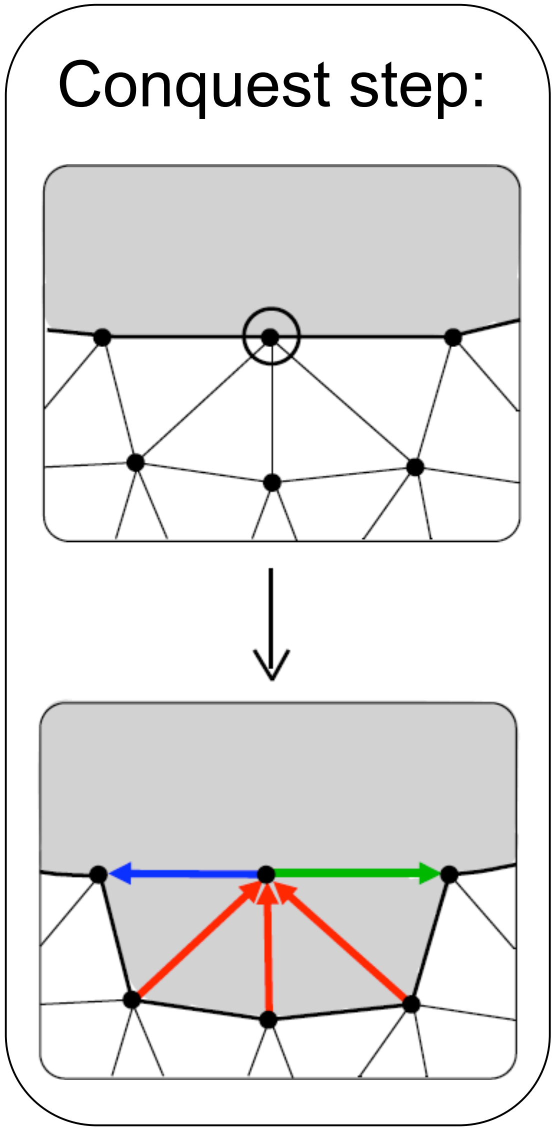
Conquest in higher genus



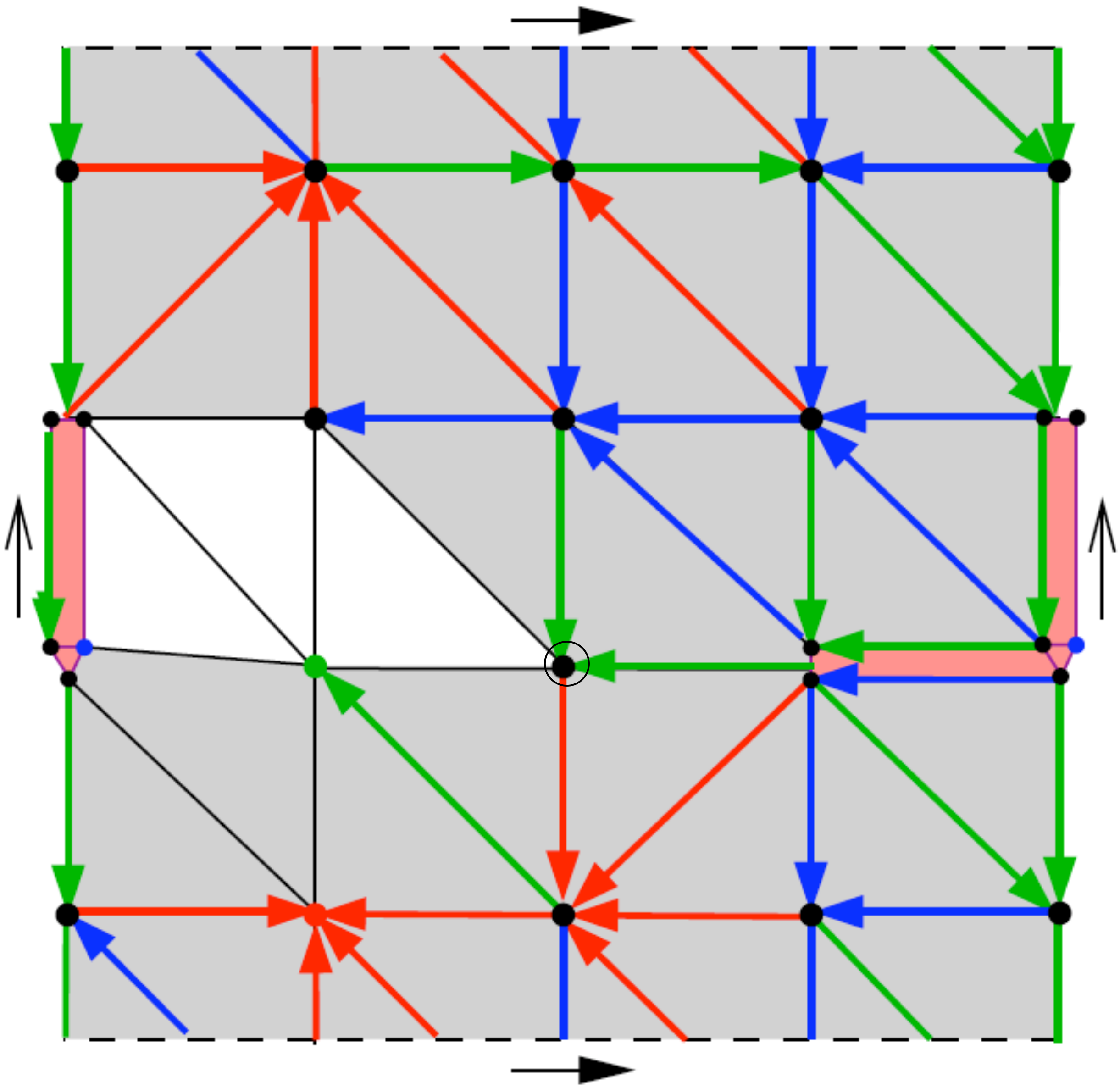


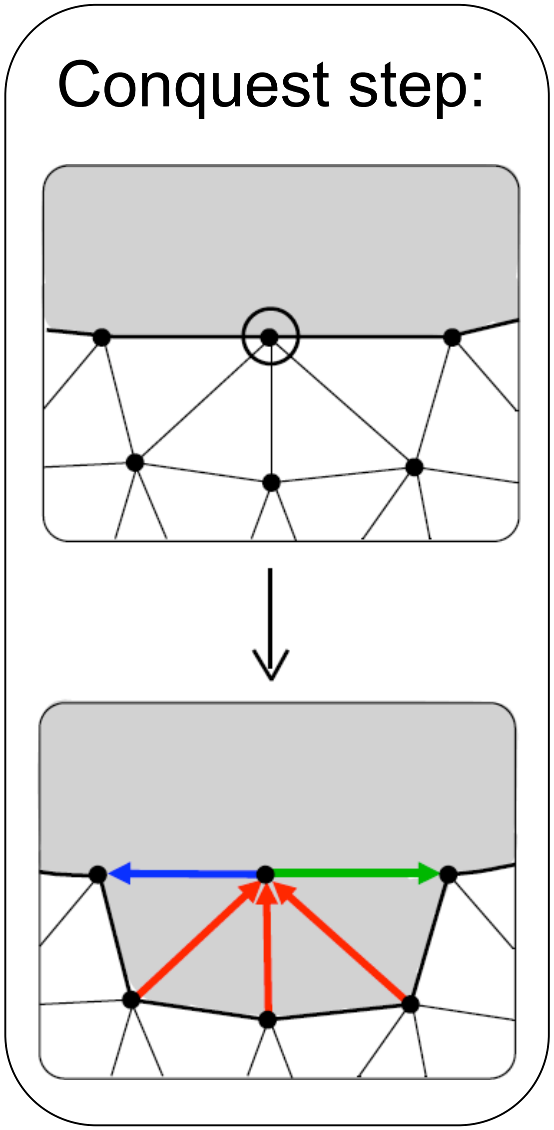
Conquest in higher genus



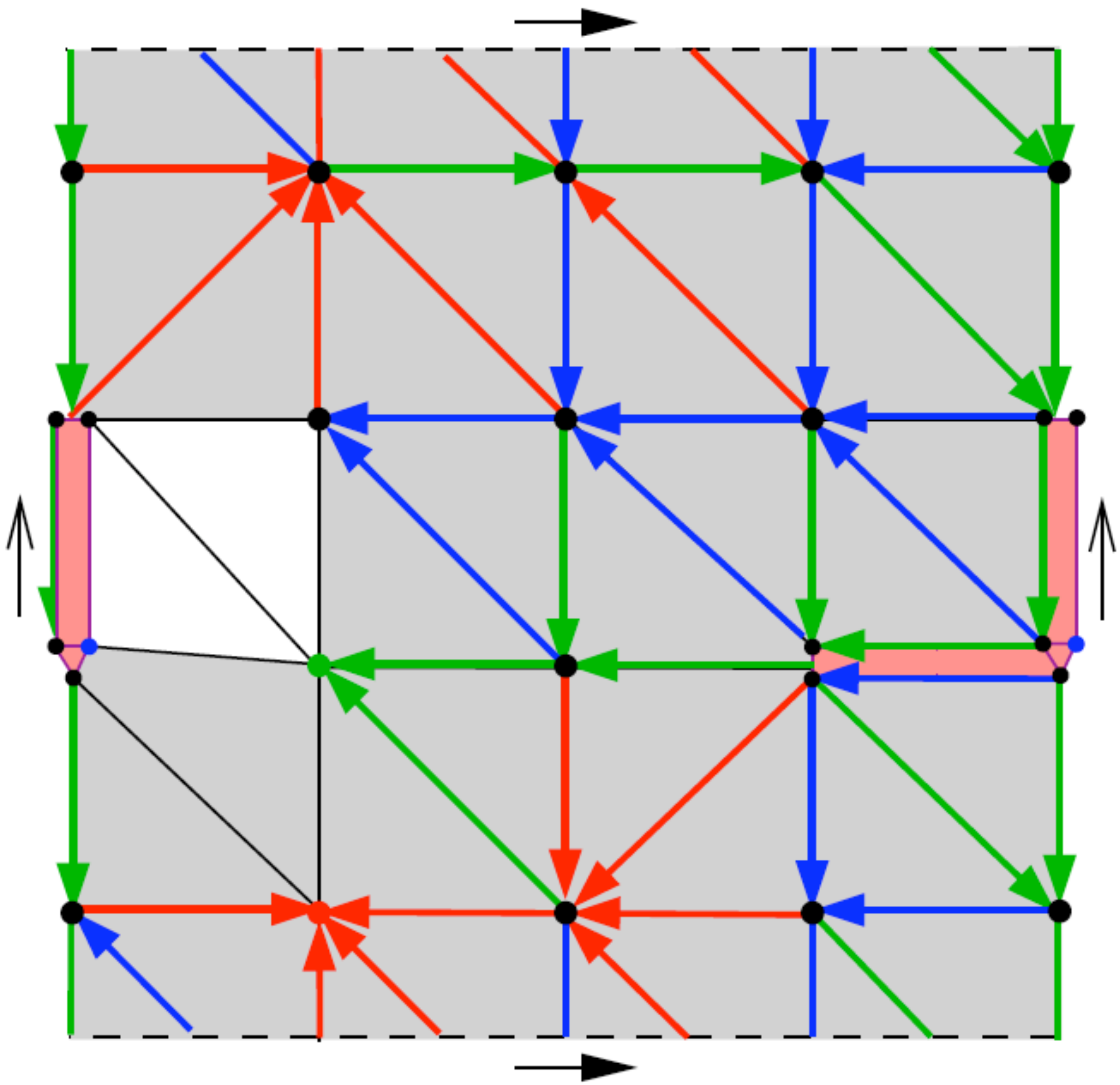


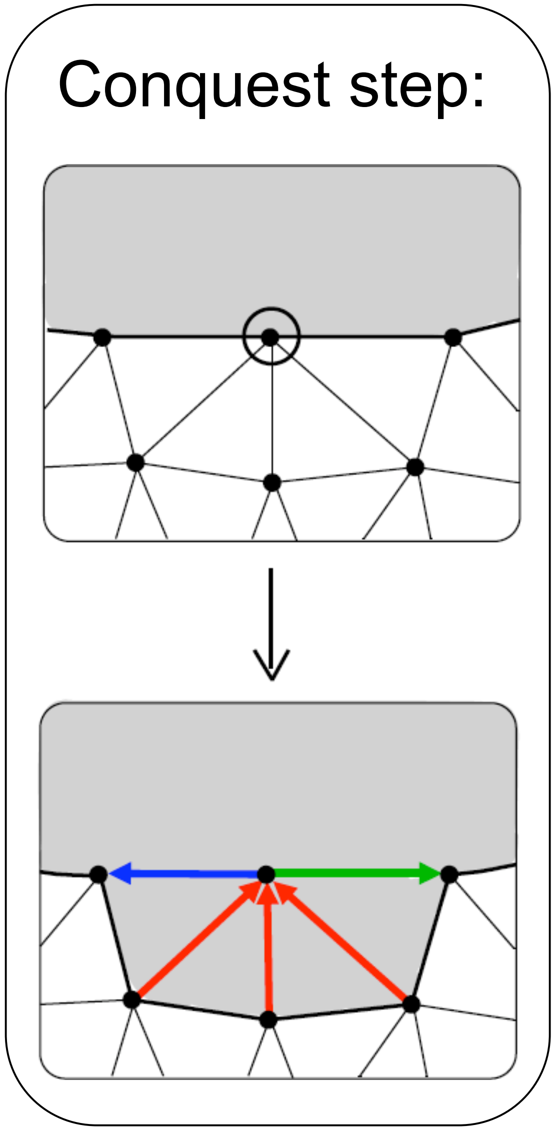
Conquest in higher genus



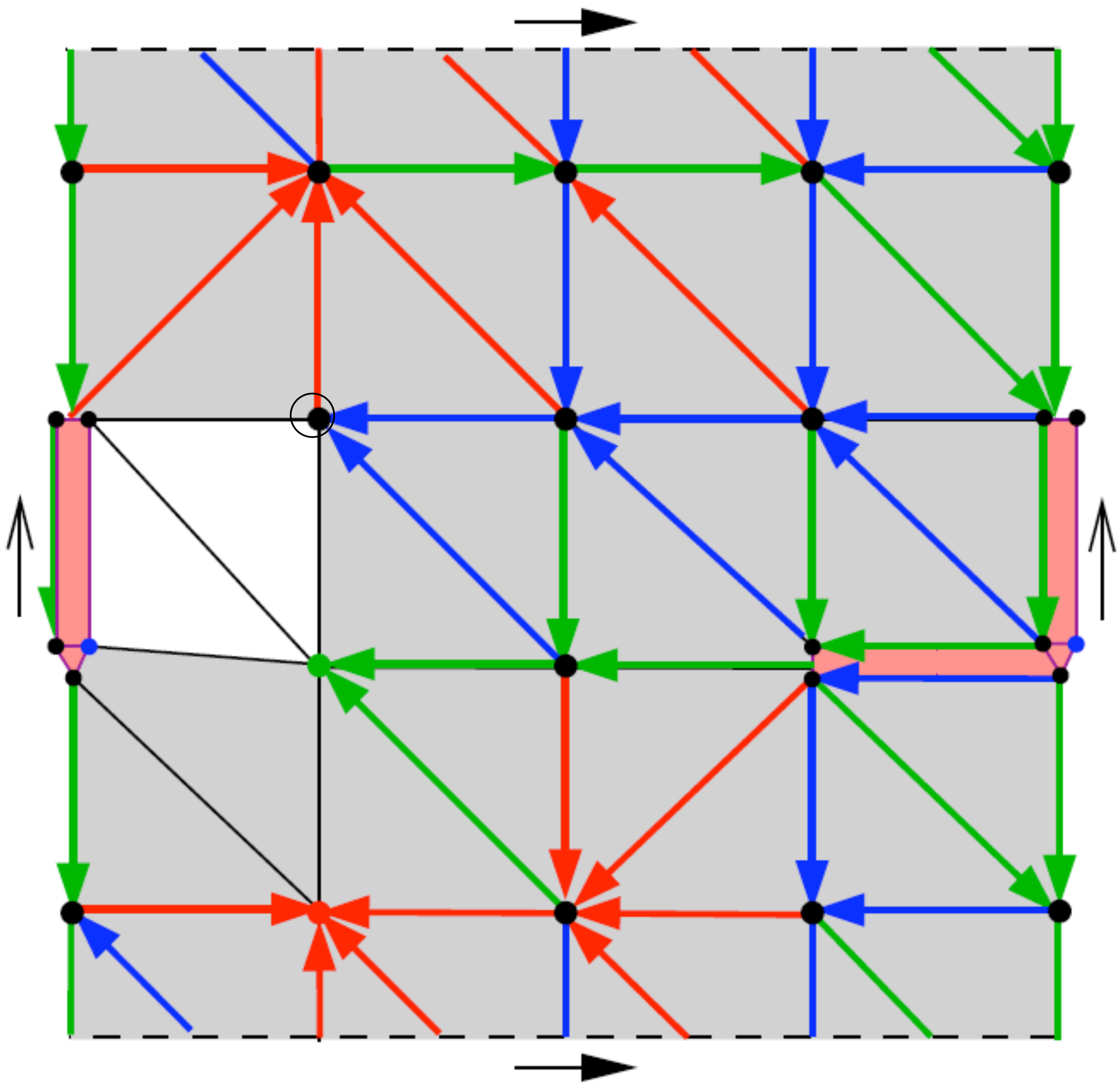


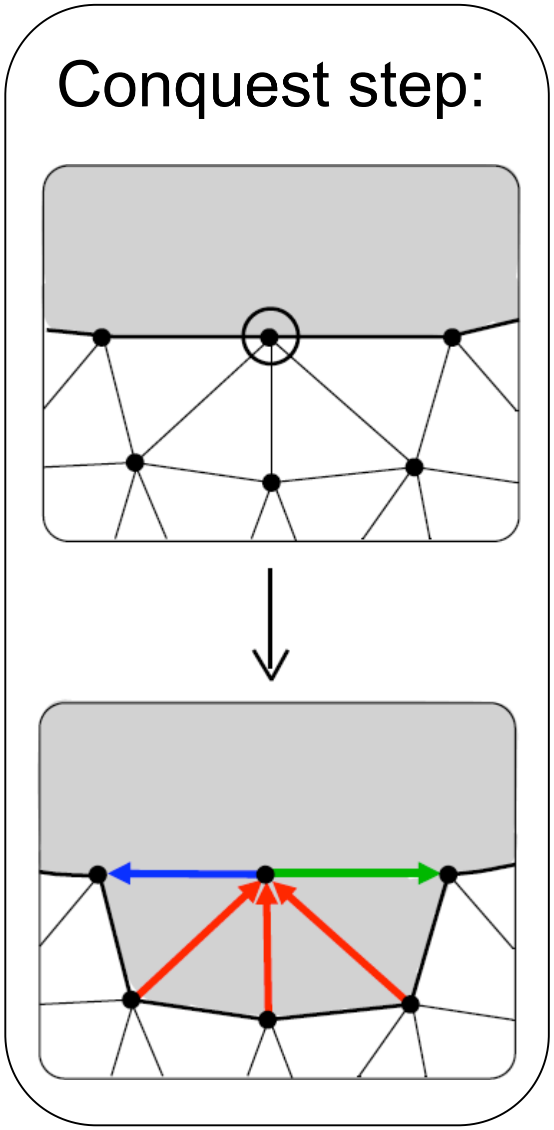
Conquest in higher genus



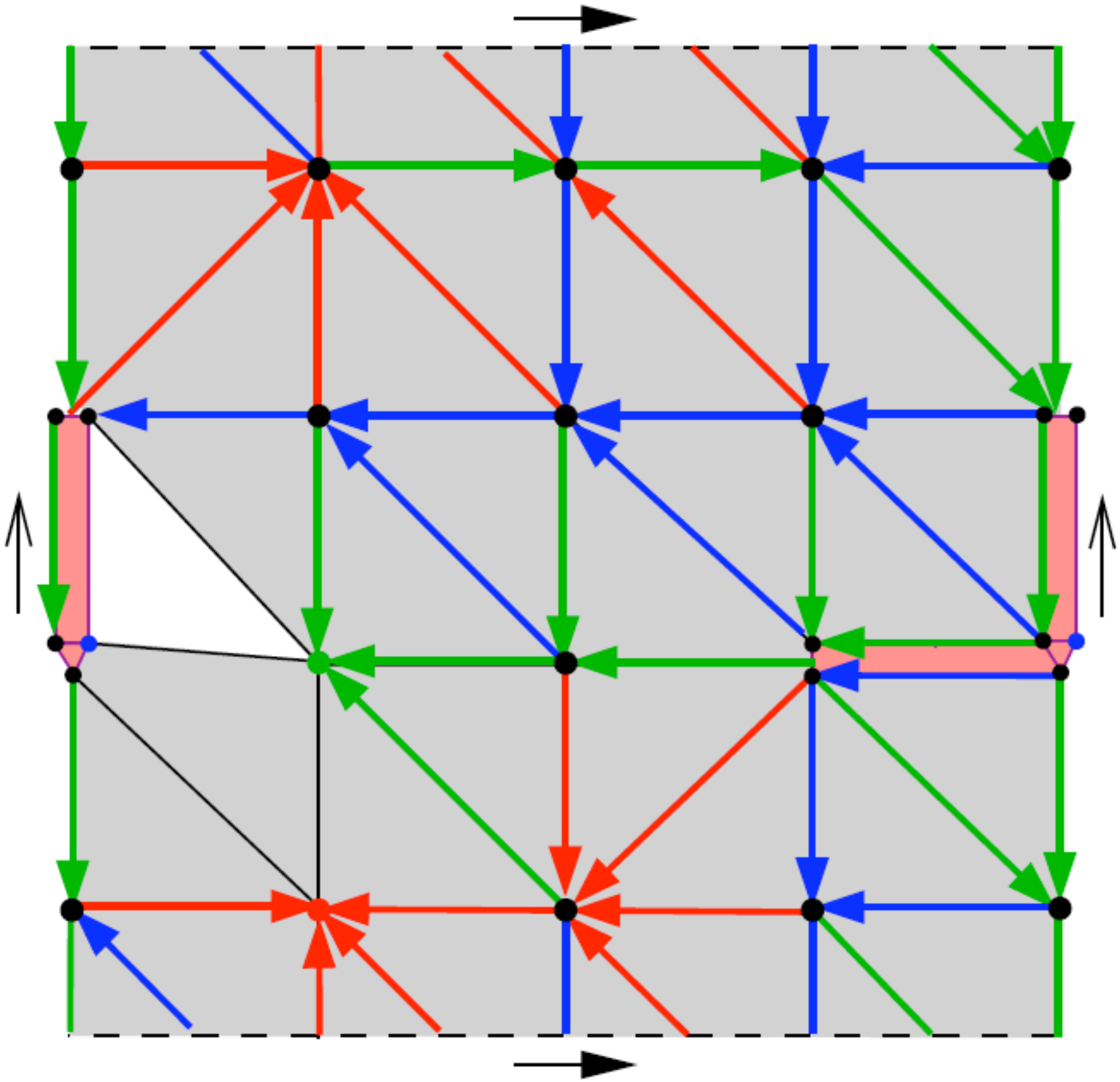


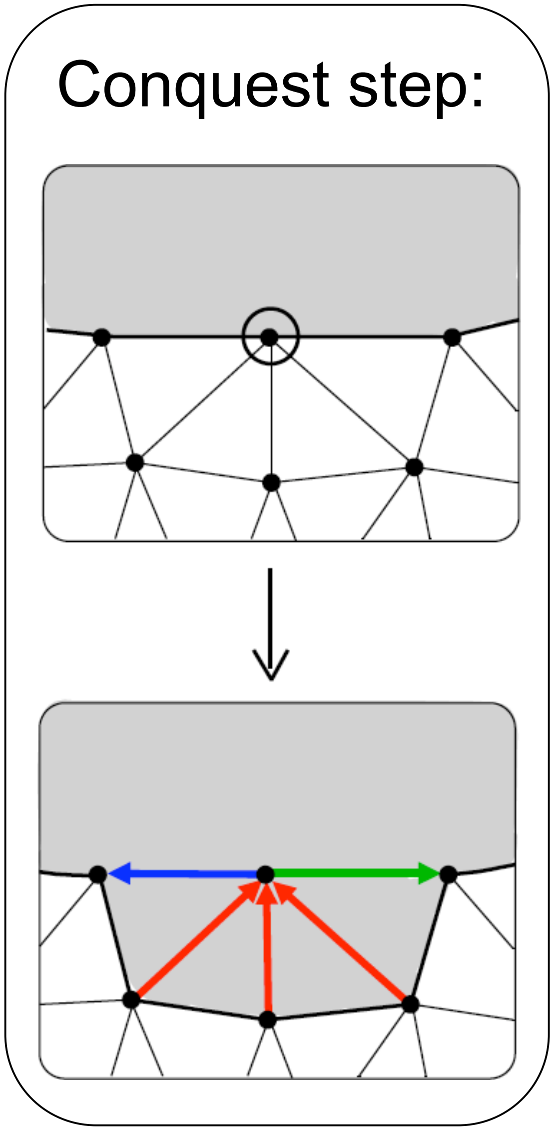
Conquest in higher genus



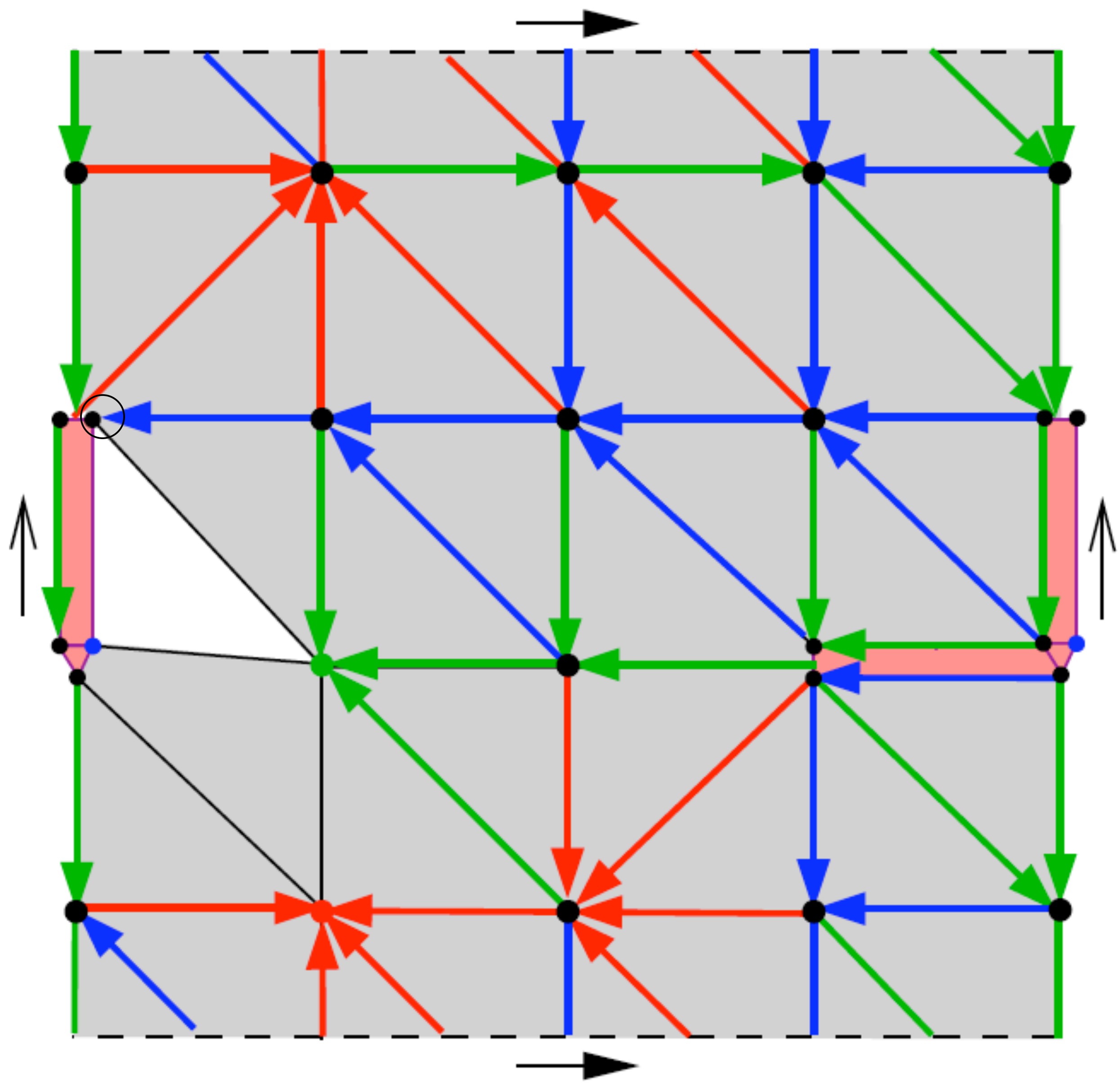


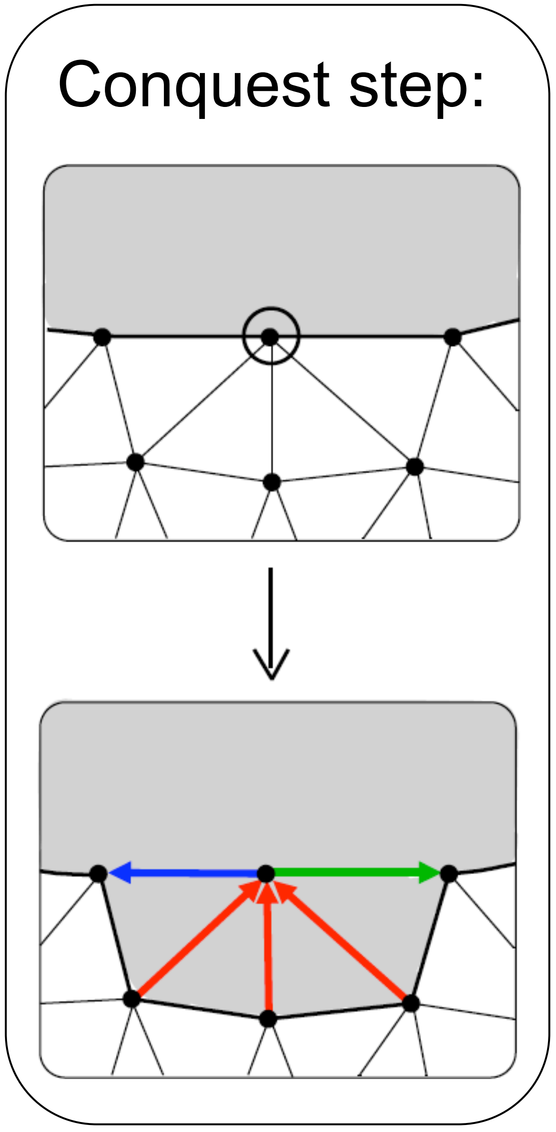
Conquest in higher genus



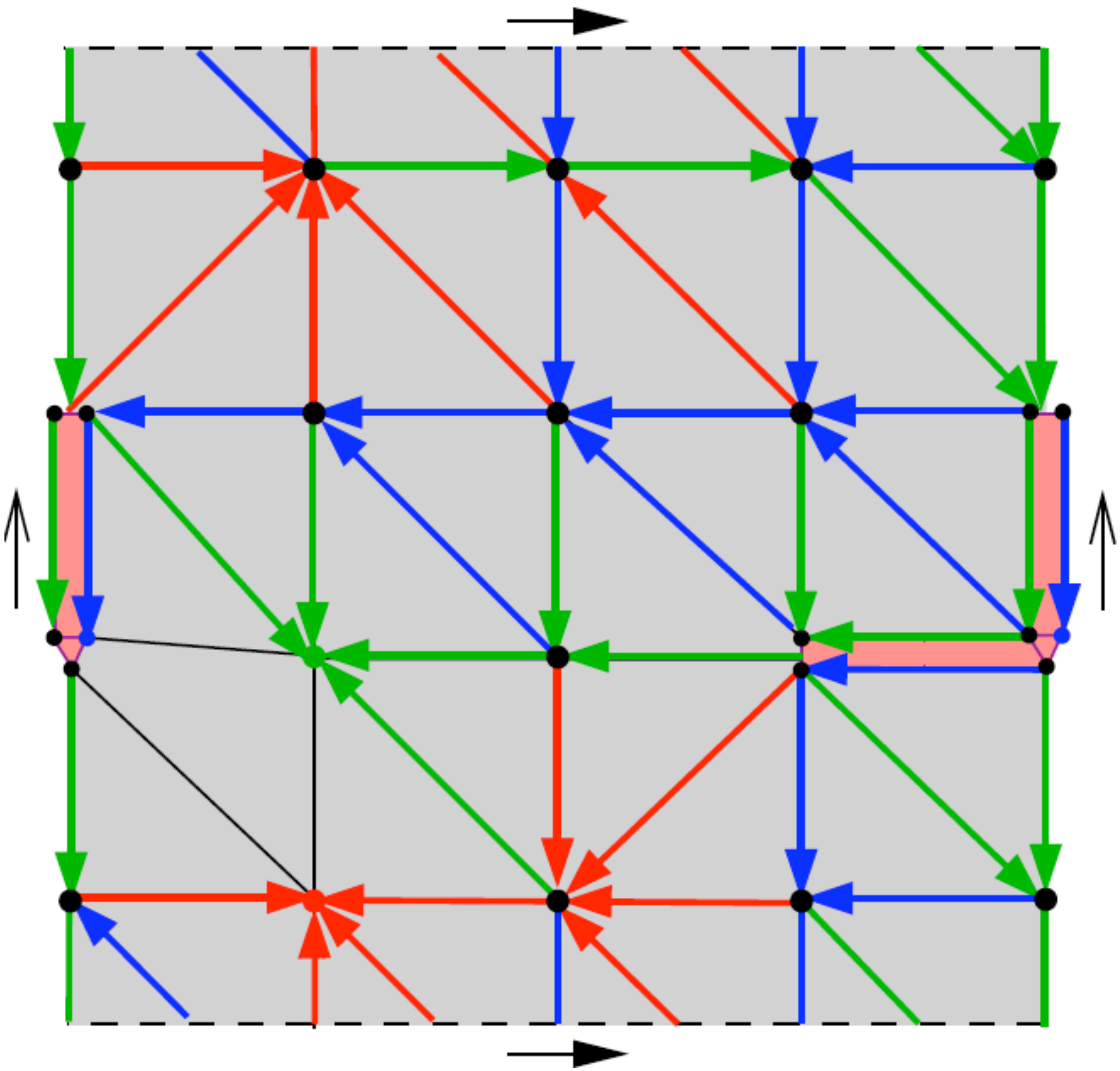


Conquest in higher genus





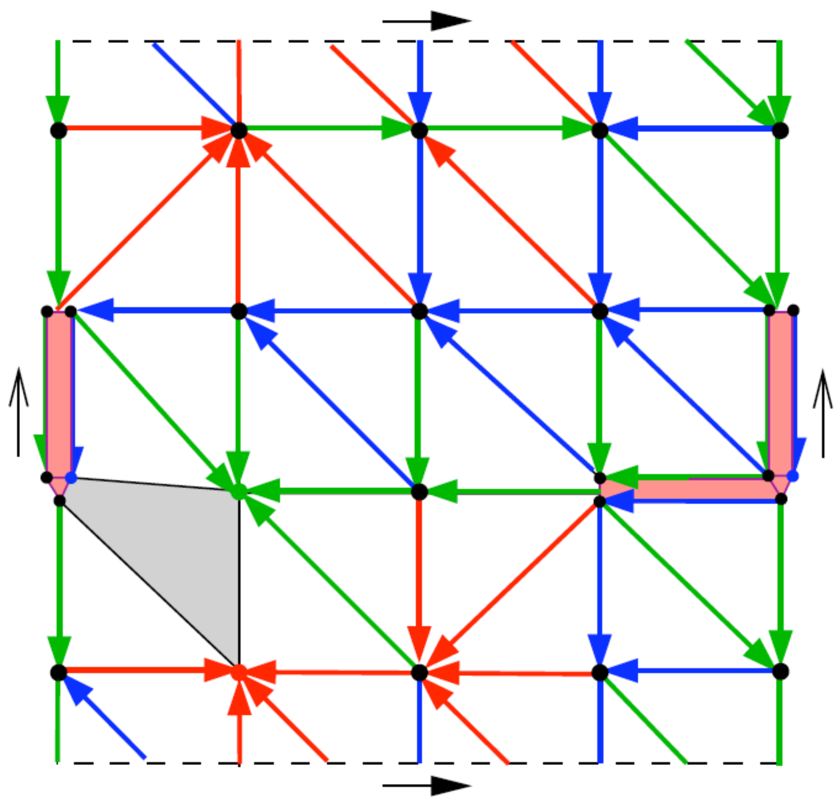
Conquest in higher genus



Finished !

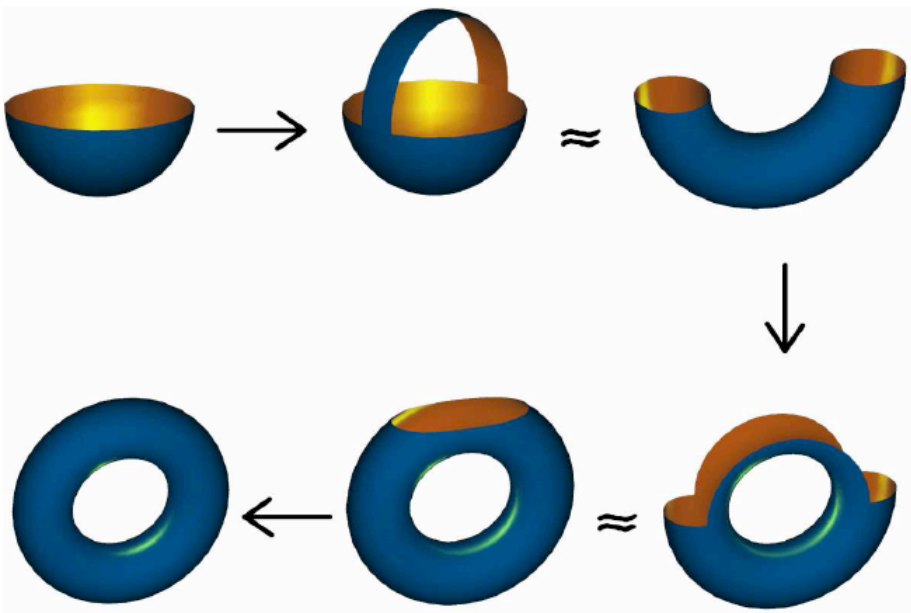
Main result

- **Theorem** [Castelli, F, Lewiner'08]: The conquest (with 2*g* special steps) terminates. Running time is O((*n*+*g*)*g*).
- The structure computed is called a *g*-Schnyder wood



Main result

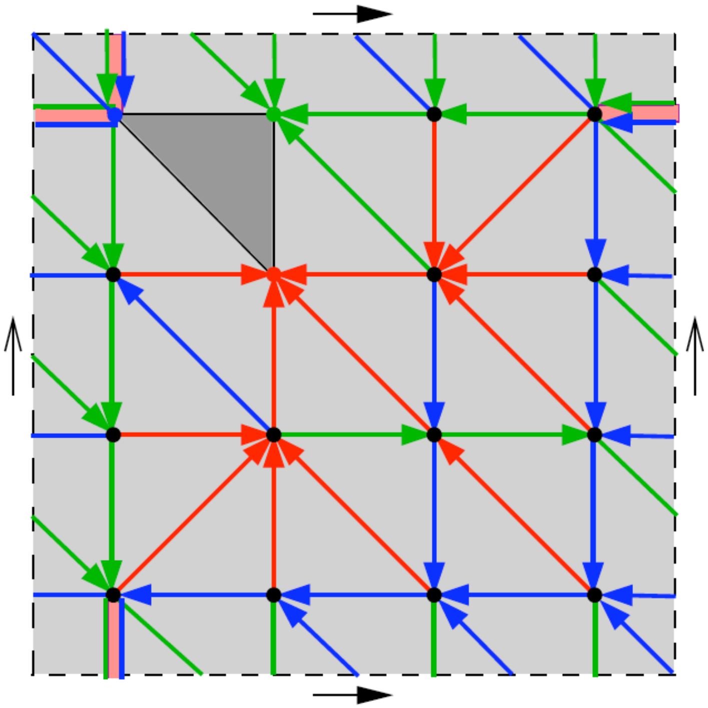
- **Theorem** [Castelli, F, Lewiner'08]: The conquest (with $2g$ special steps) terminates. Running time is $O((n+g)g)$.
- The structure computed is called a g-Schnyder wood
- Our traversal procedure is inspired by handlebody theory:



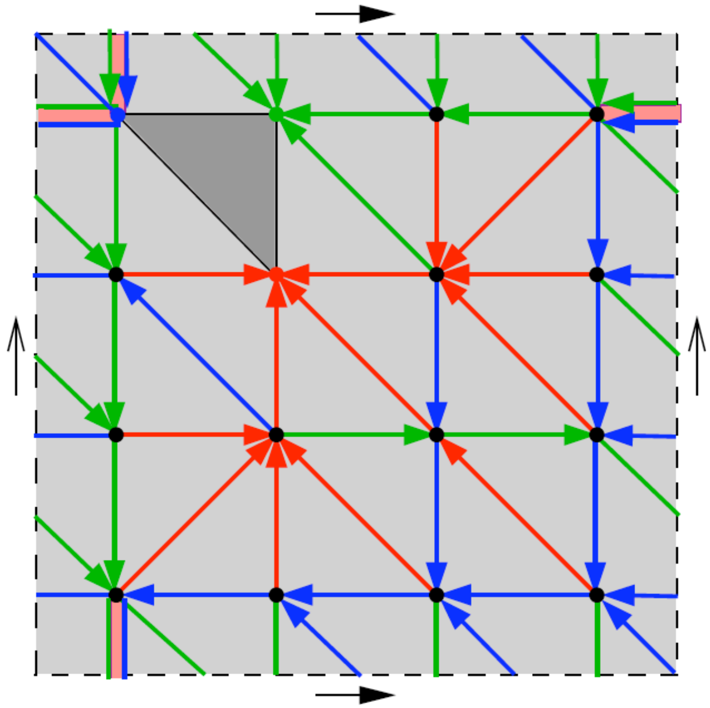
Handlebody decomposition of a torus

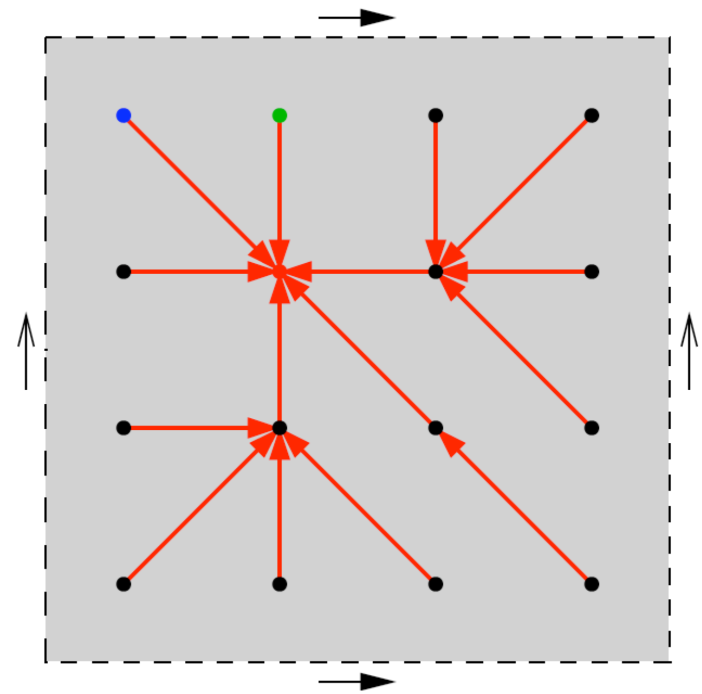
From [Rossignac et al'03]: ``EdgeBreaker'' procedure

Properties in higher genus



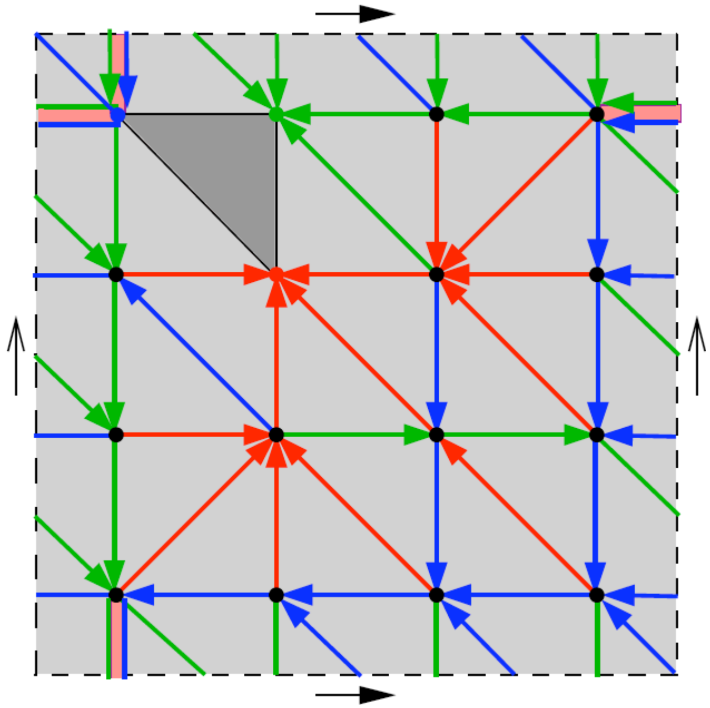
Properties in higher genus

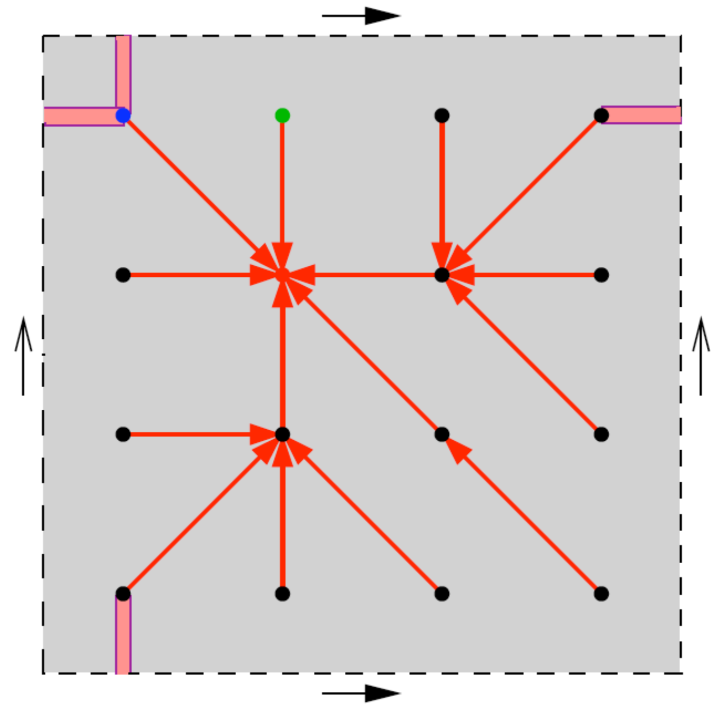




- T_R={red edges}+{R-B}+{R-G} is a spanning tree

Properties in higher genus





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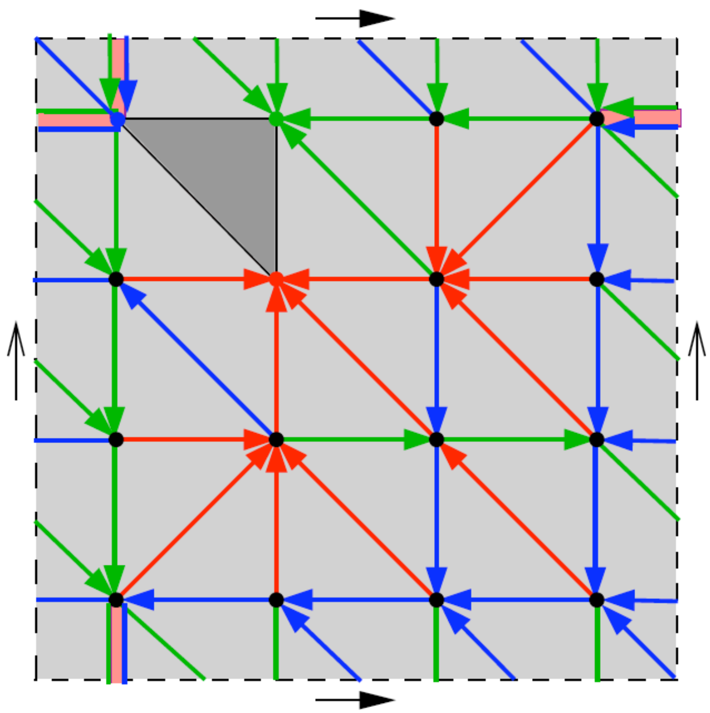
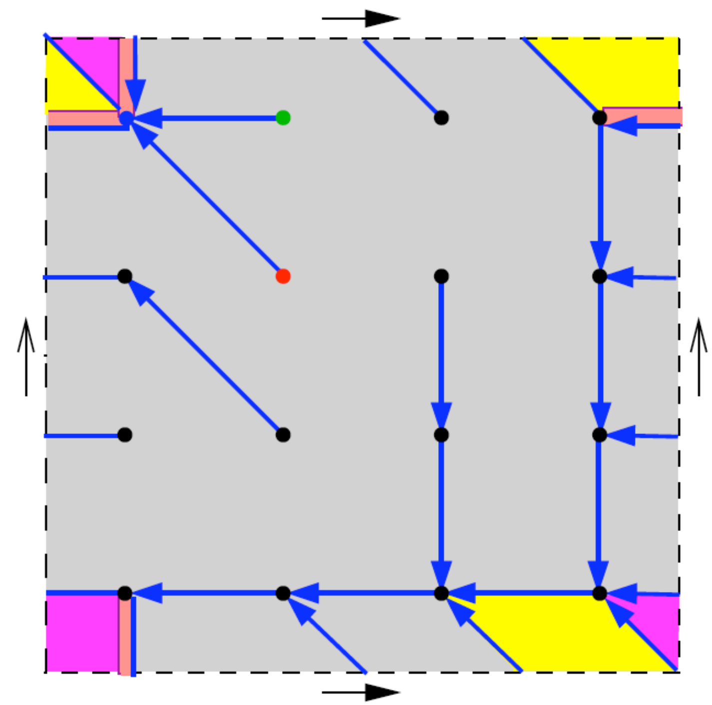
Properties in higher genus





- T_R={red edges}+{R-B}+{R-G} is a spanning tree
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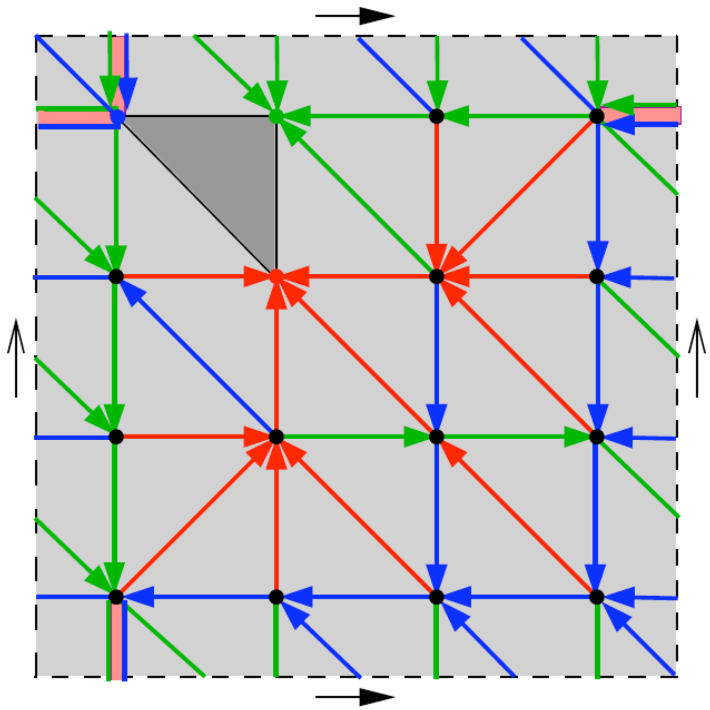
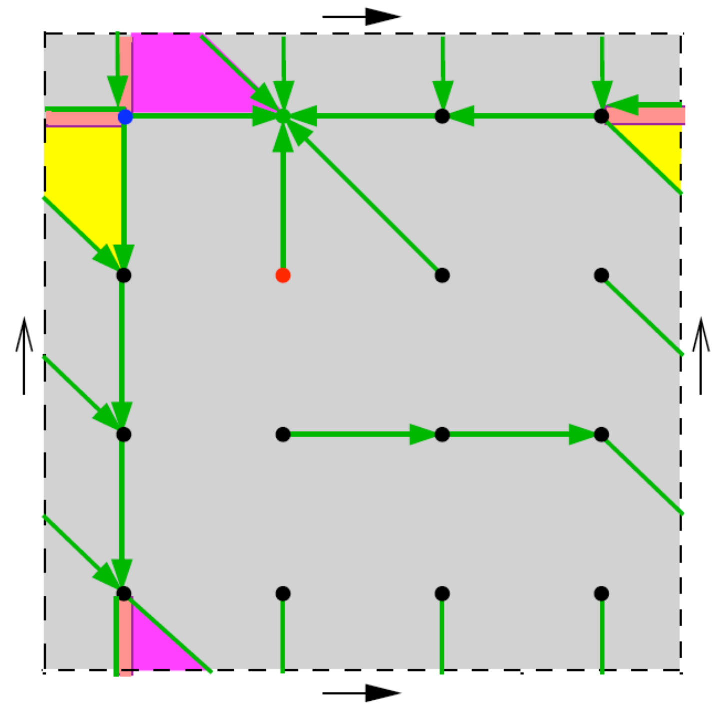
Properties in higher genus





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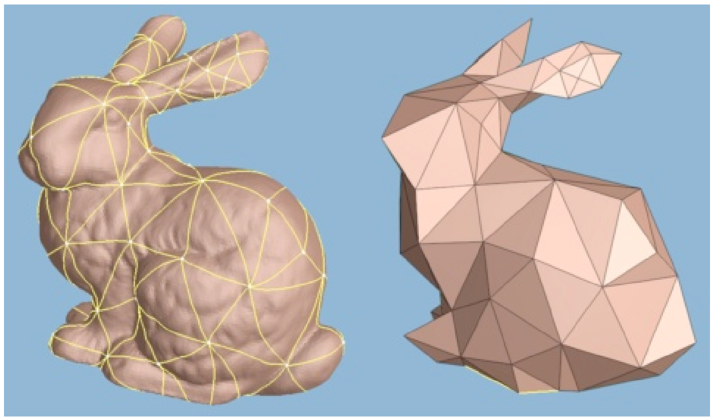
- G_G={green edges}+{G-R}+{G-B} is a spanning submap with 1+2g faces

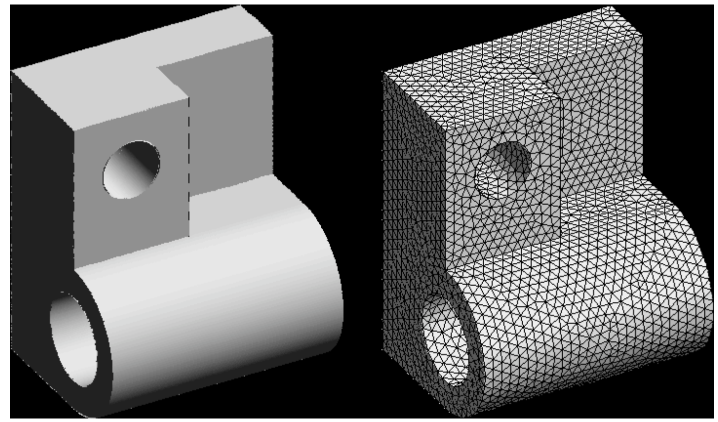
Application to coding

Motivation: mesh compression

- Triangulations are the combinatorial part of triangular meshes



mesh of genus 0



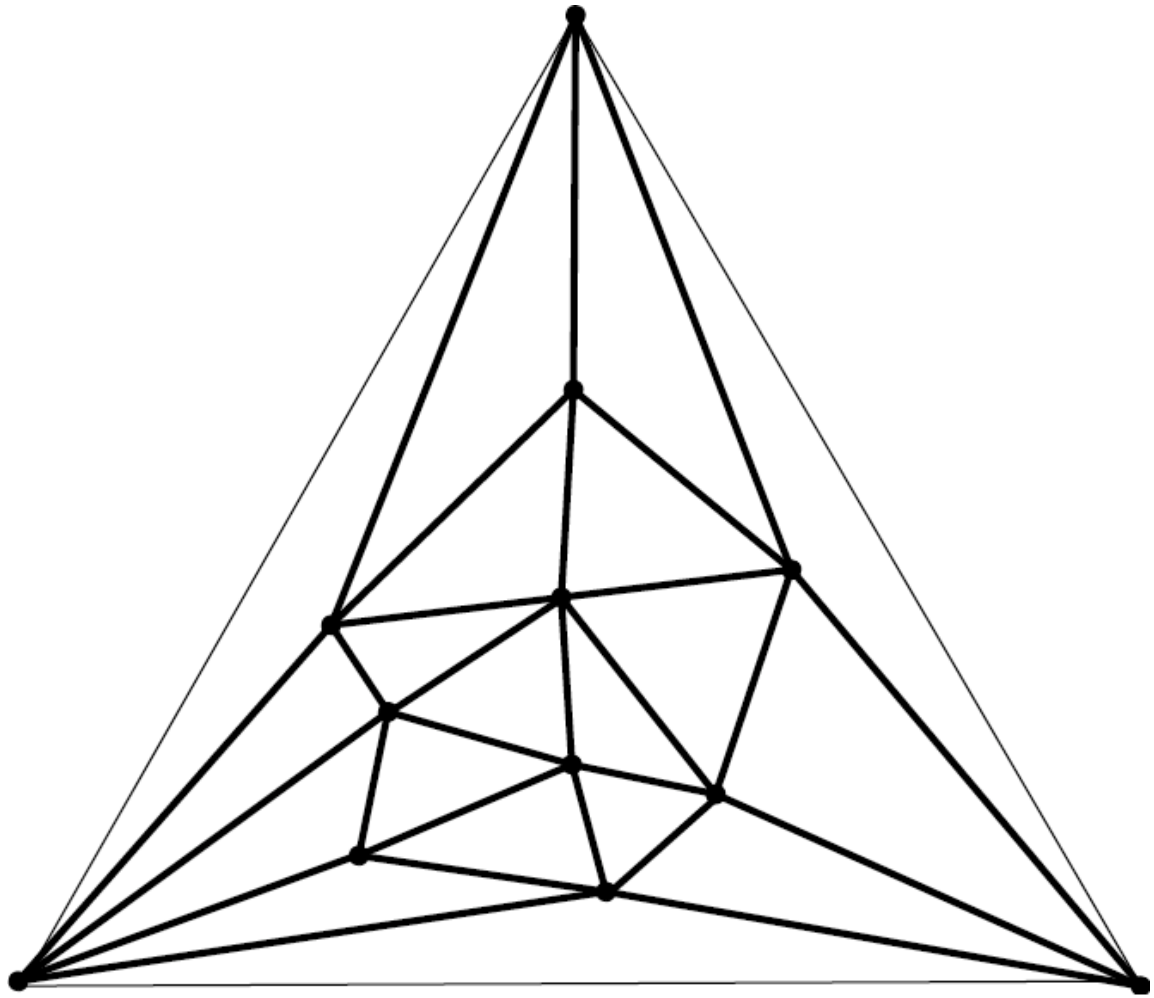
mesh of genus 2

- Naïve encoding: vertices are labelled $\{1,2,\ldots,n\}$
  store the faces (vertex-triples), takes memory of order $n \log(n)$

- This talk: Schnyder woods → encoding in $4n+O(g \log(n))$ bits
  (extends encoding procedure of [He-Kao-Lu'99, Bernardi-Bonichon'07] to any genus)

Encoding a planar triangulation

- Reduces to encoding a Schnyder wood



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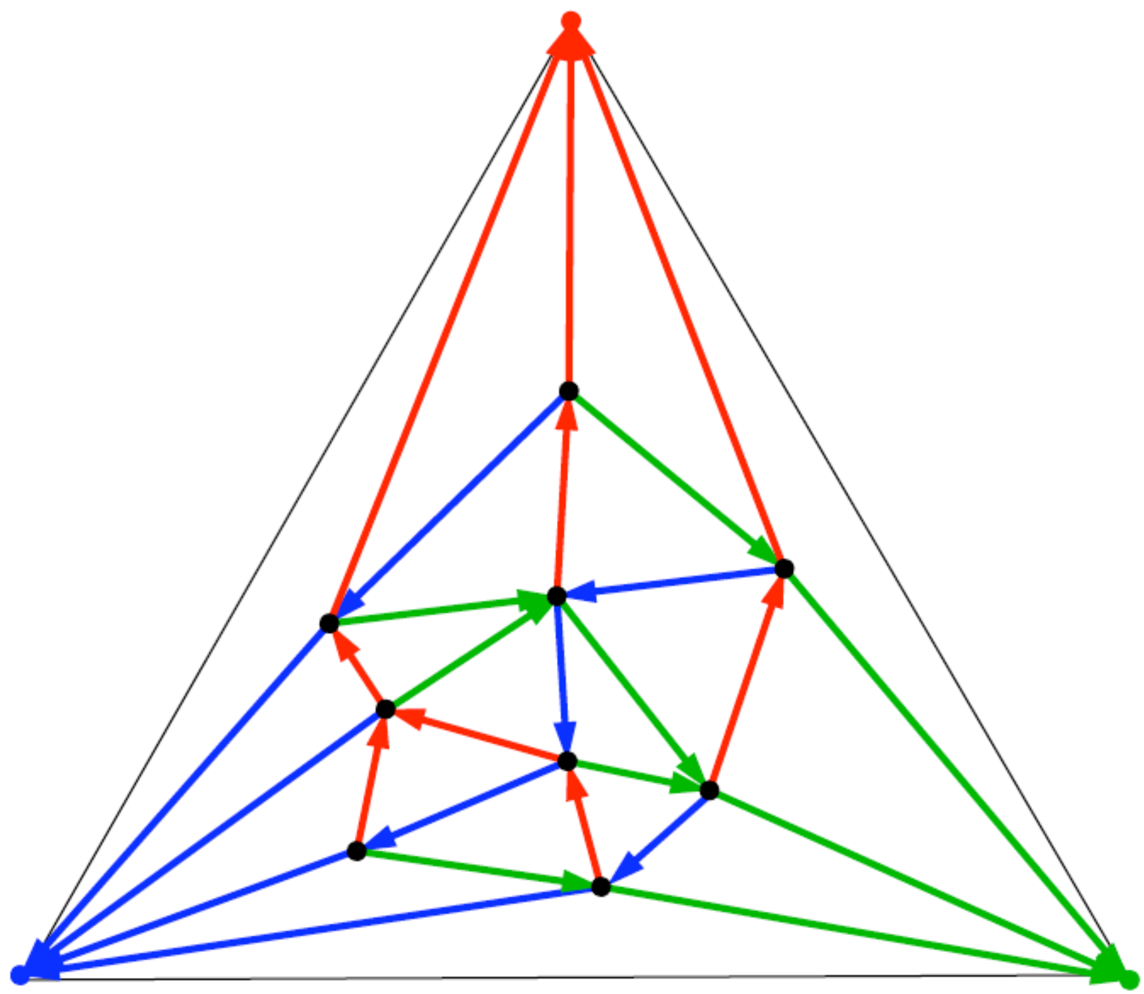
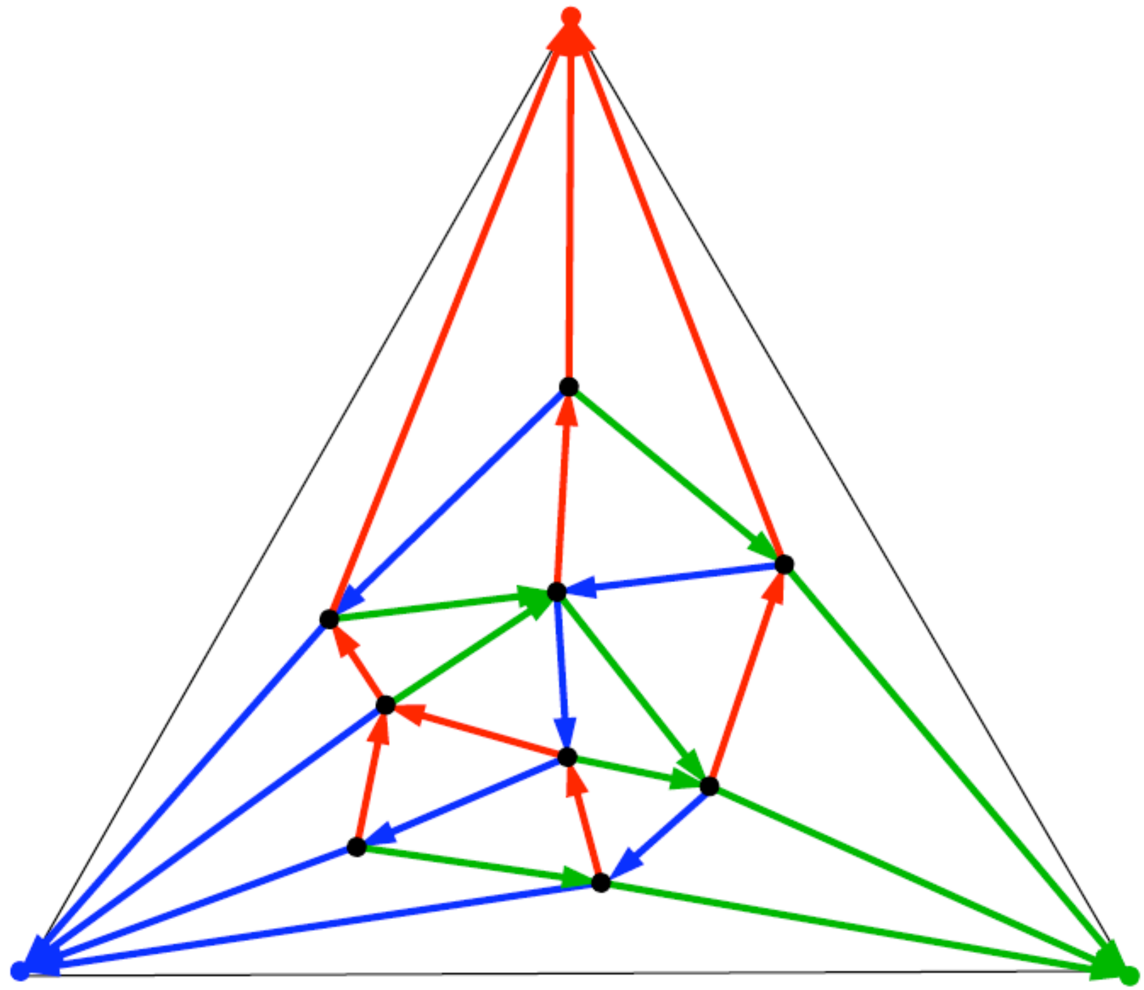


Encoding a planar triangulation

- Some information is **redundant**



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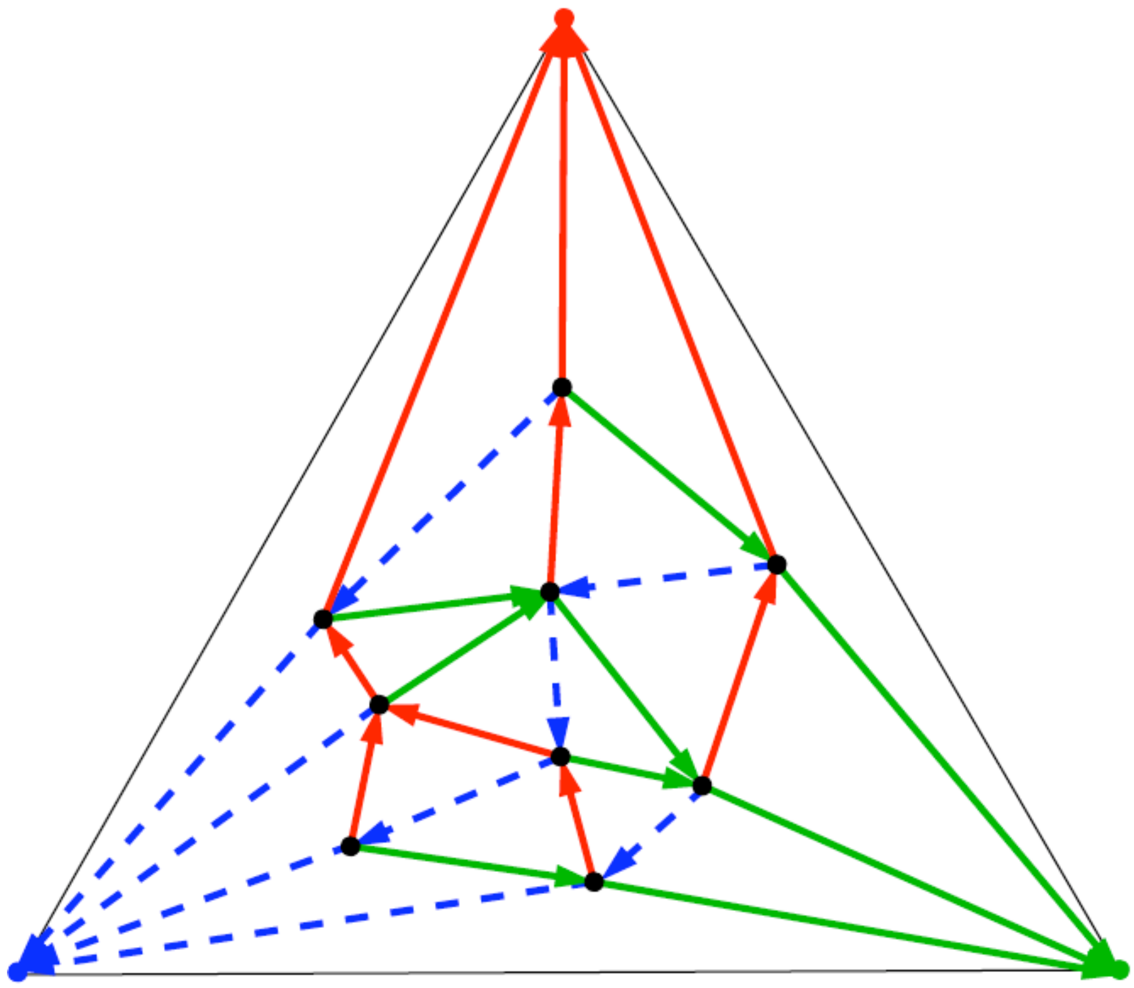

can erase blue edges

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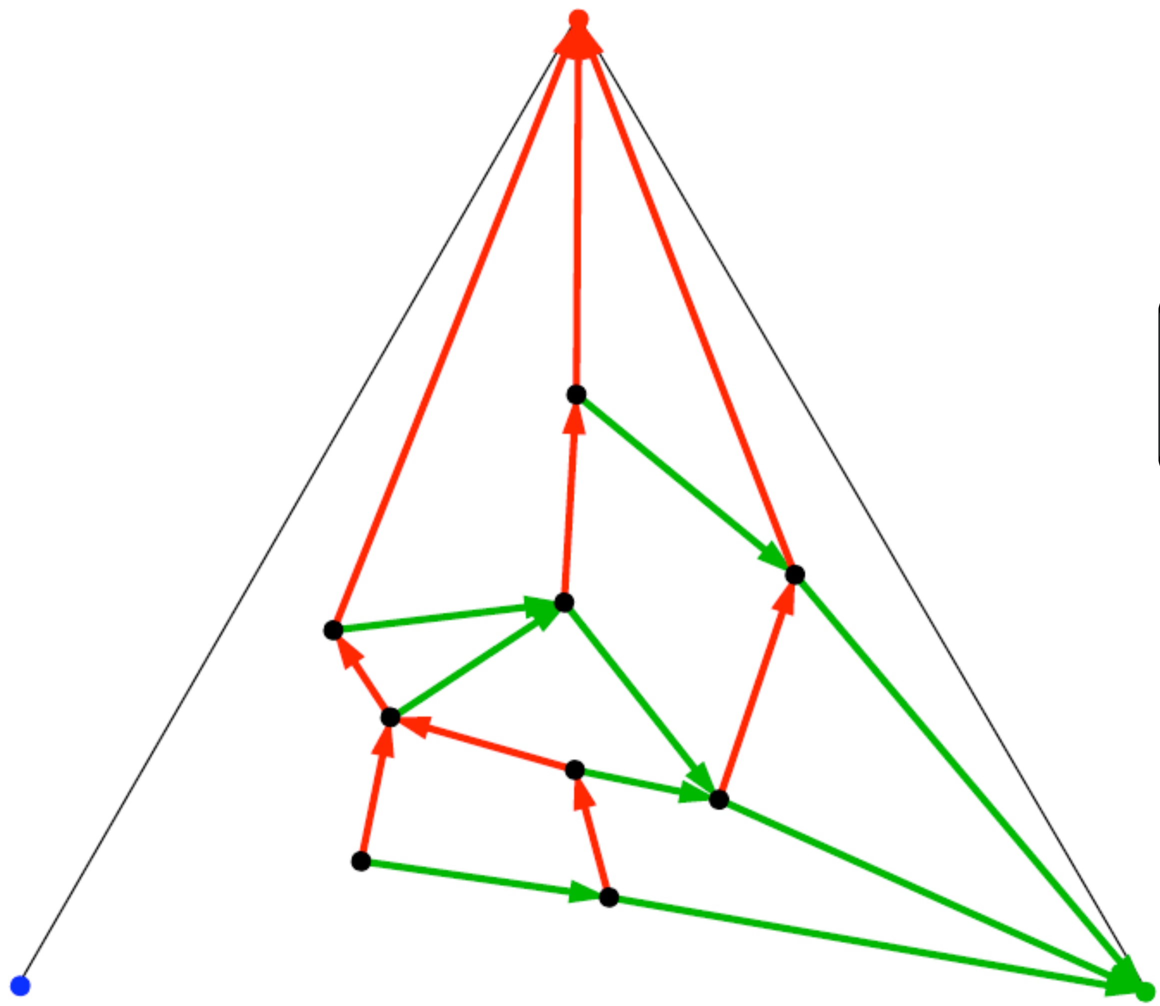

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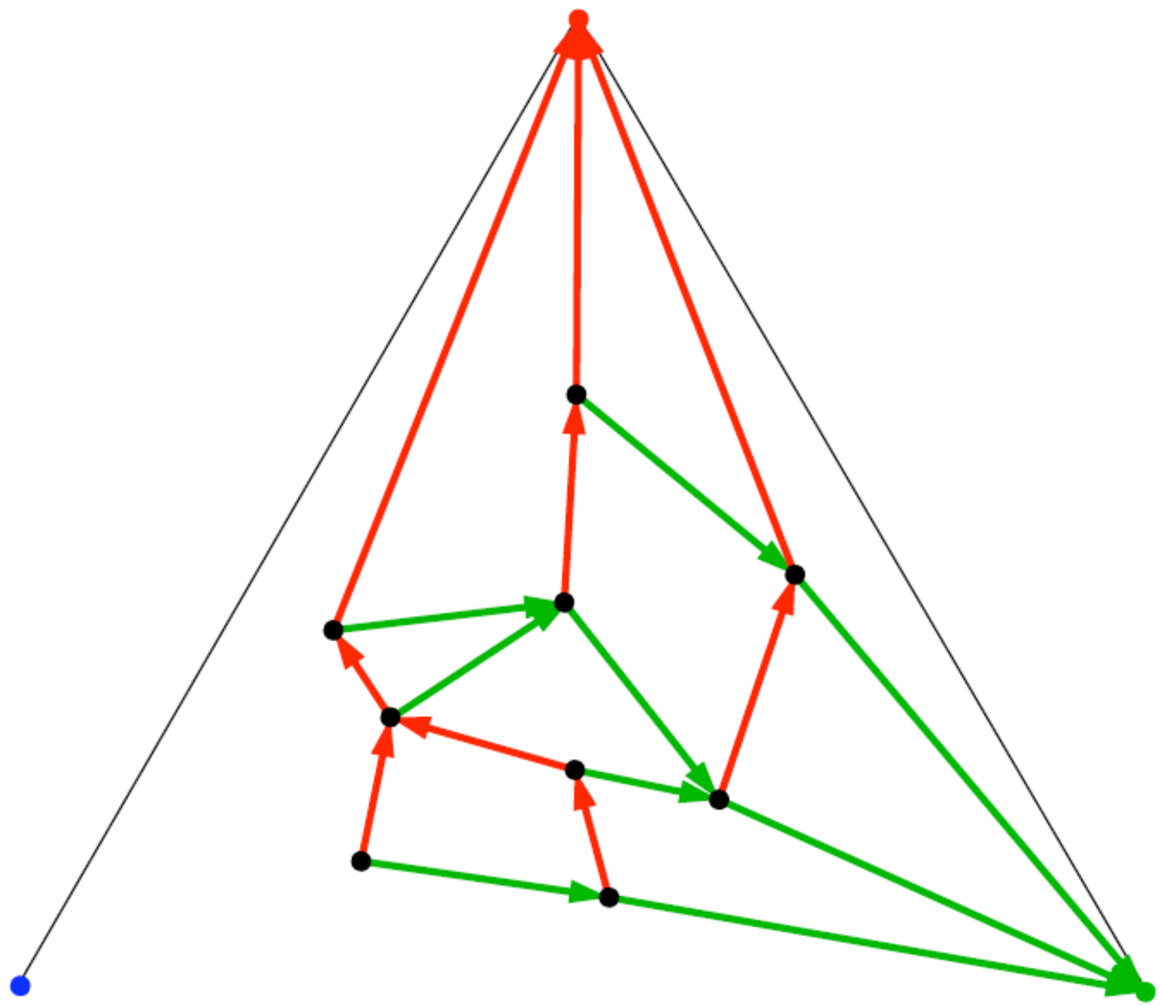

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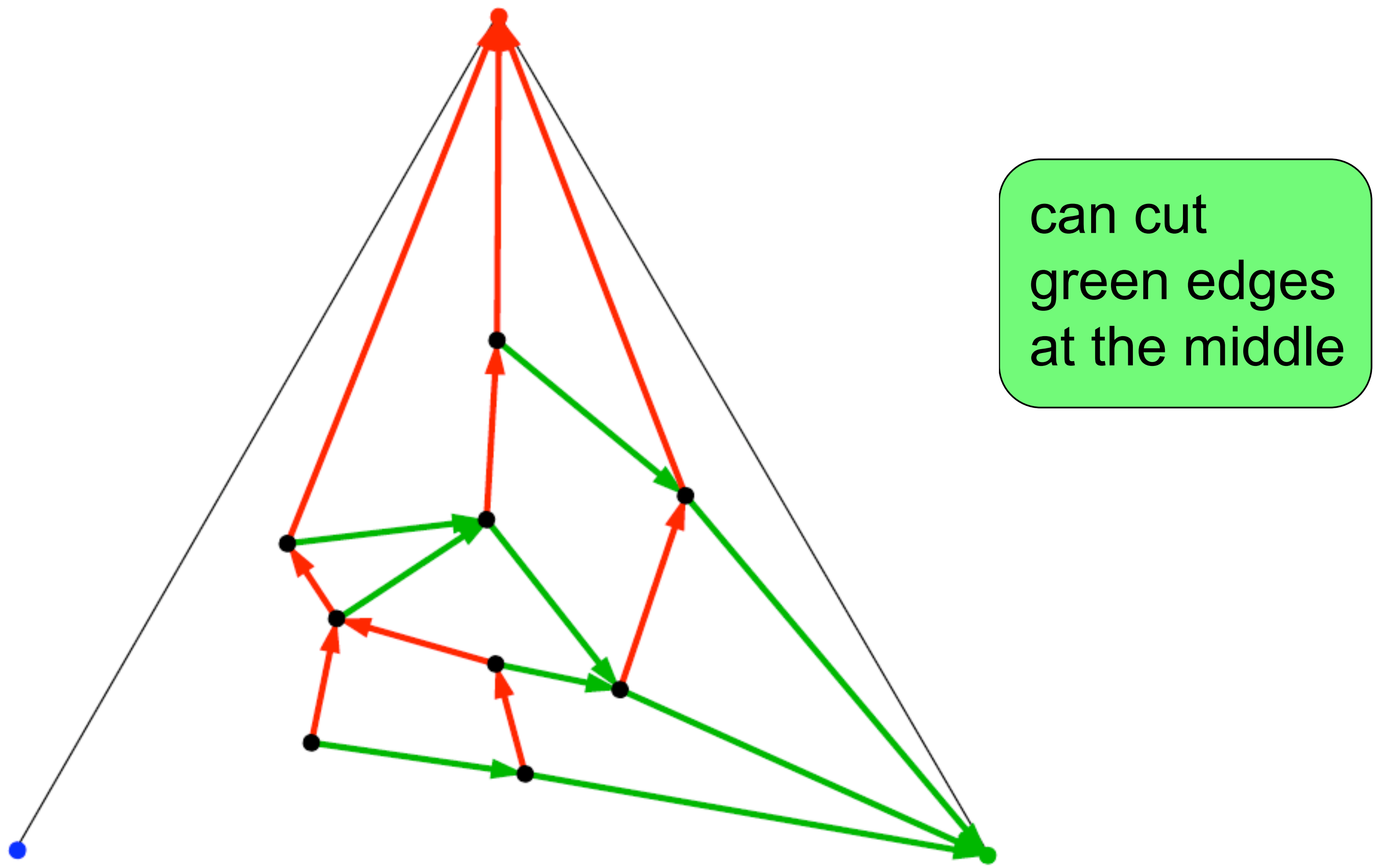

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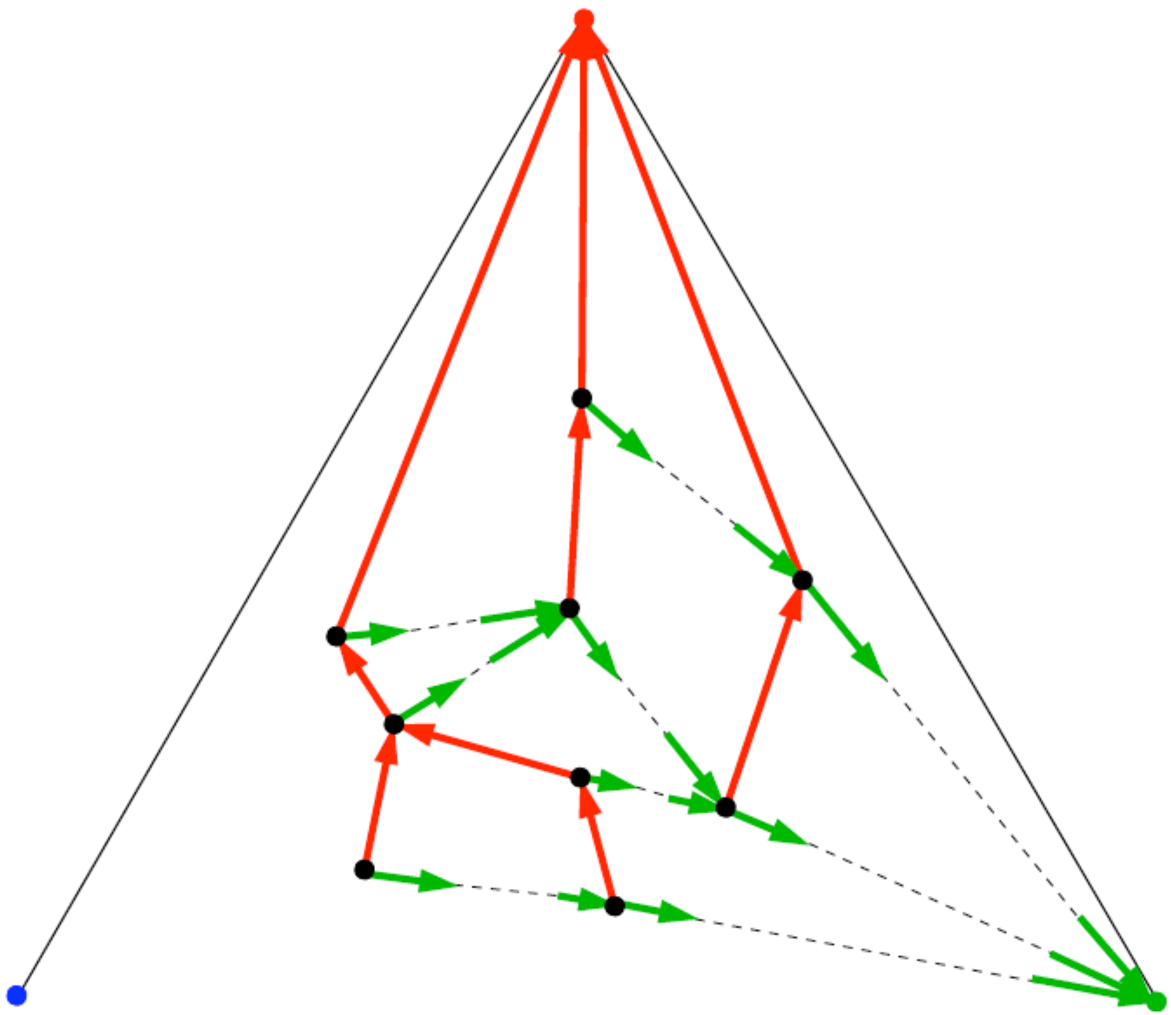

can cut green edges at the middle

Encoding a planar triangulation

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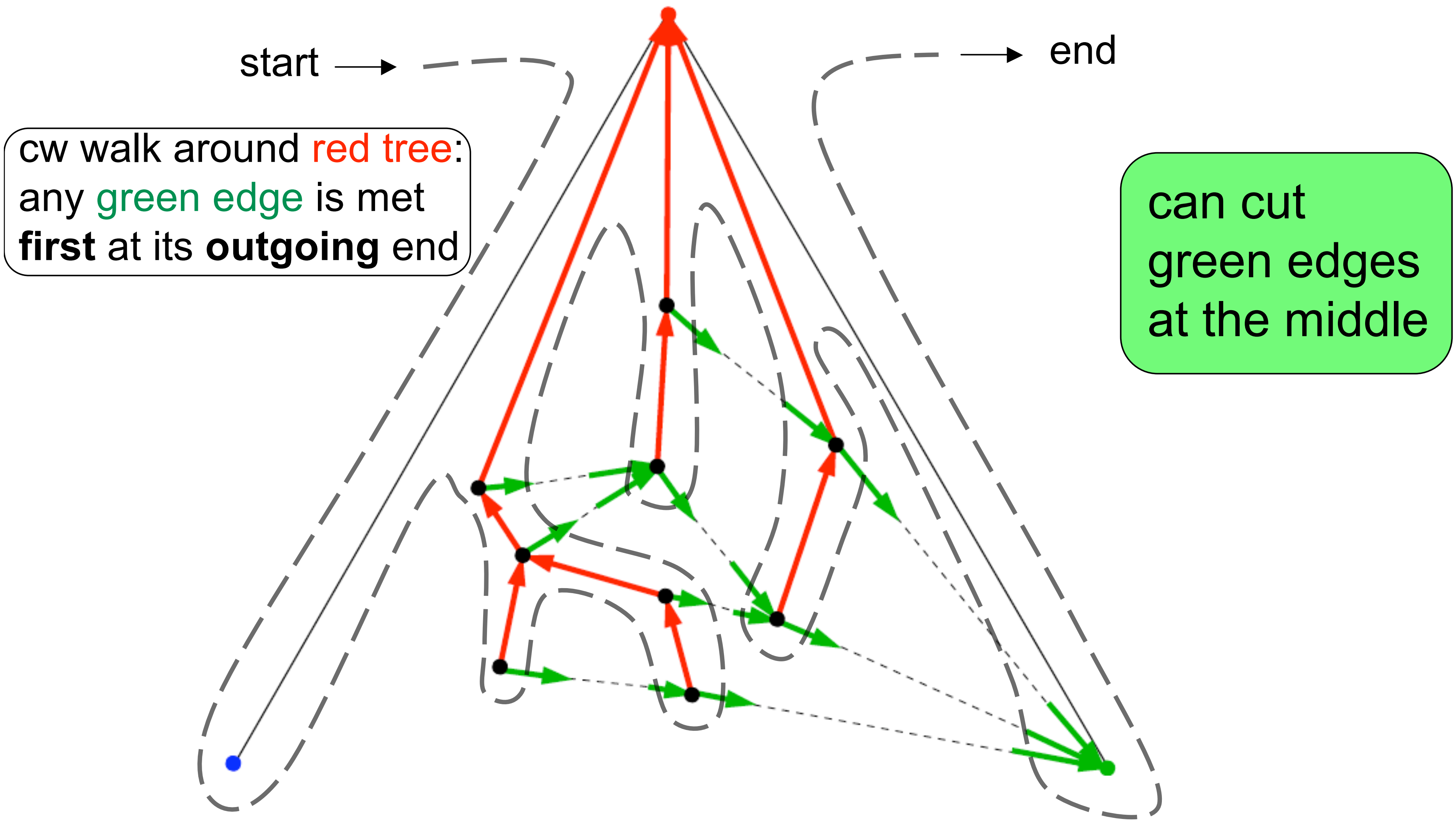
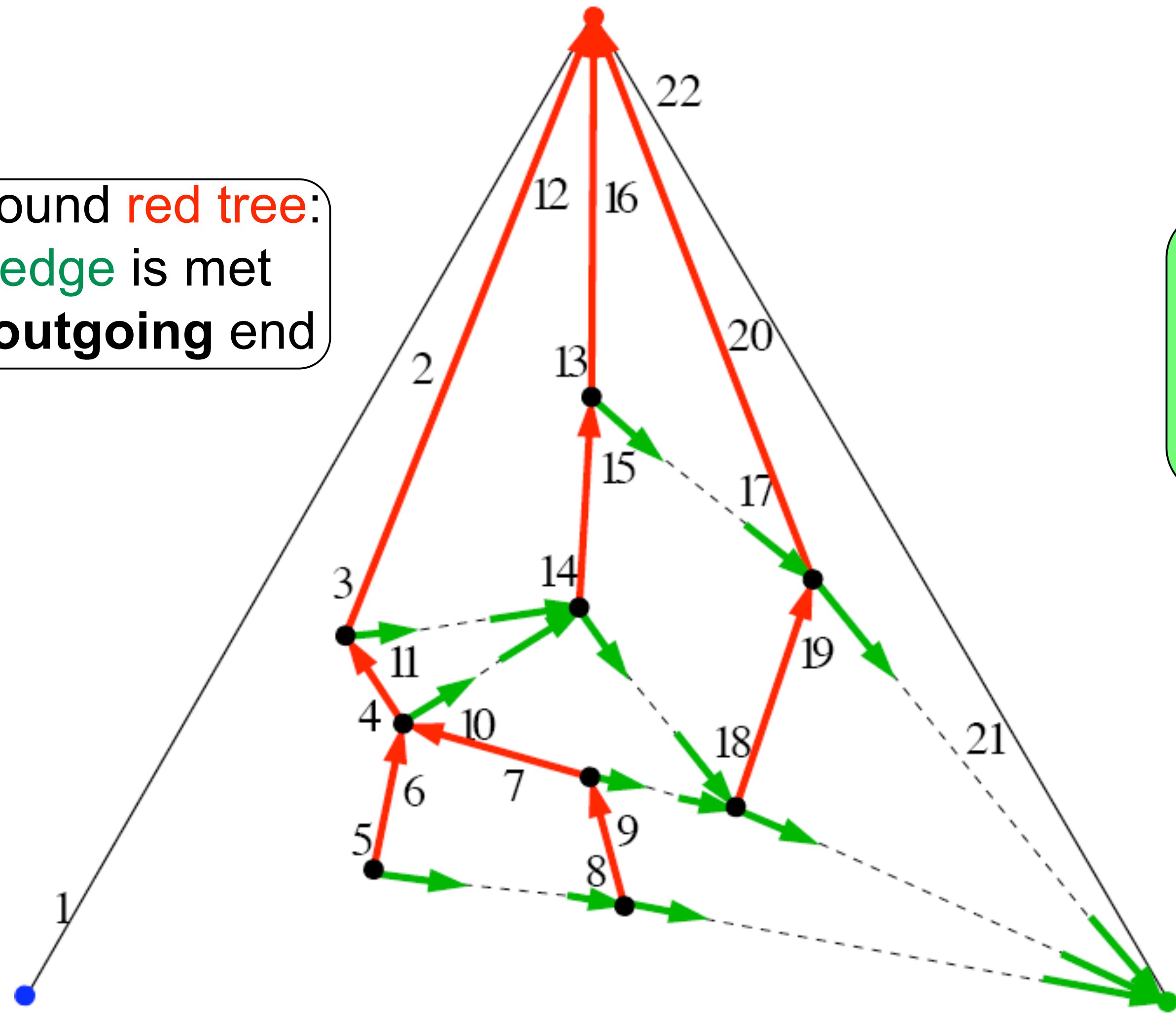


Encoding a planar triangulation

- Some information is **redundant**

cw walk around red tree: any green edge is met **first** at its **outgoing** end



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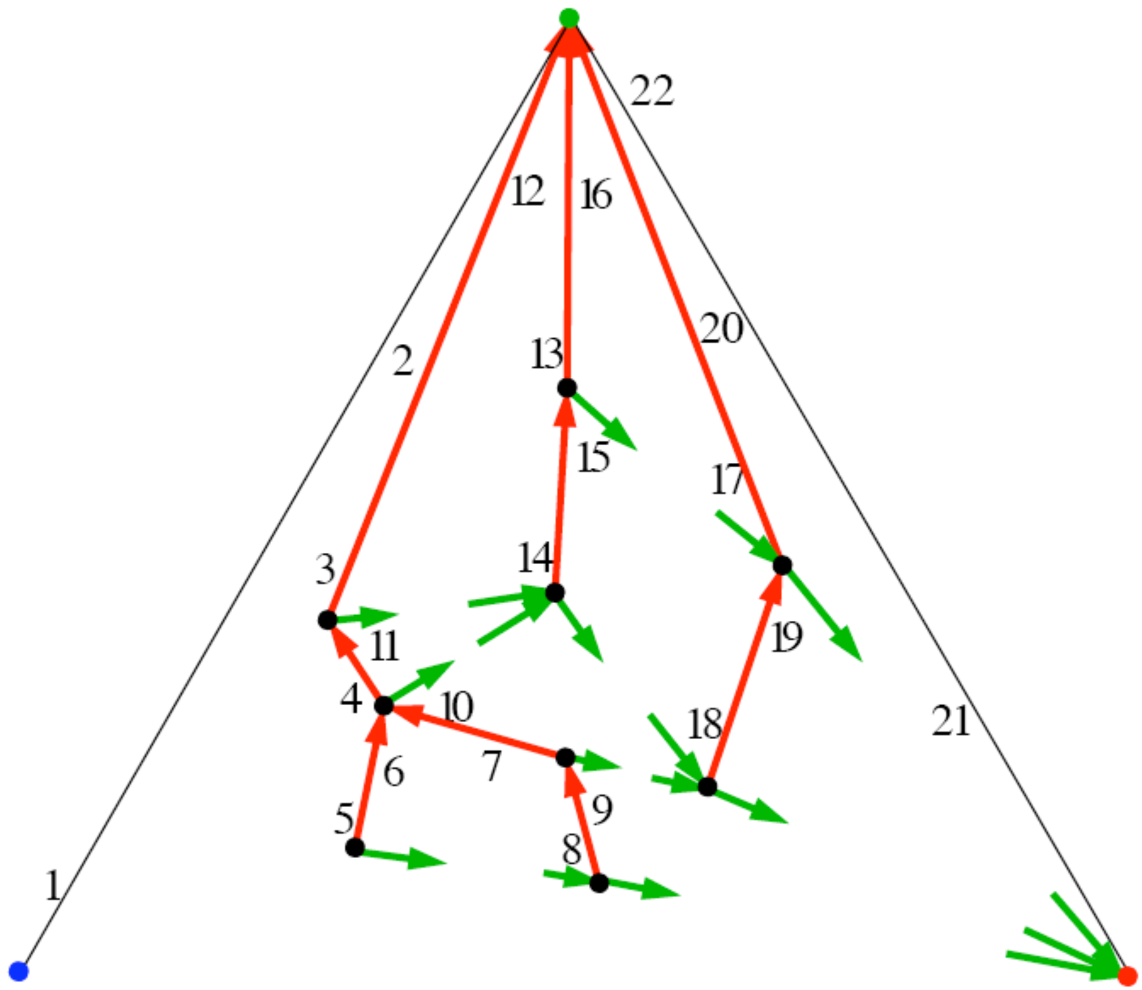

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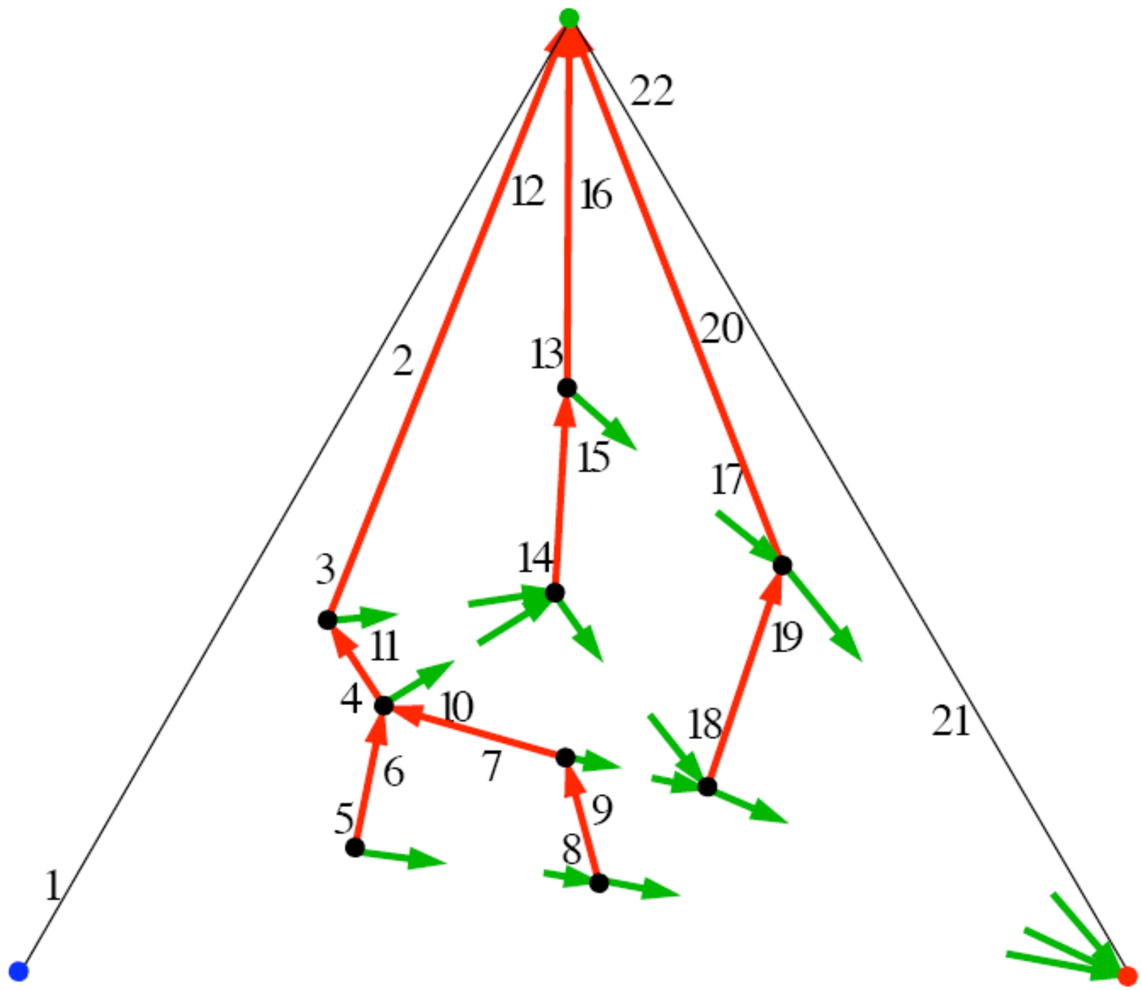

can erase green outer half-edges

Encoding a planar triangulation

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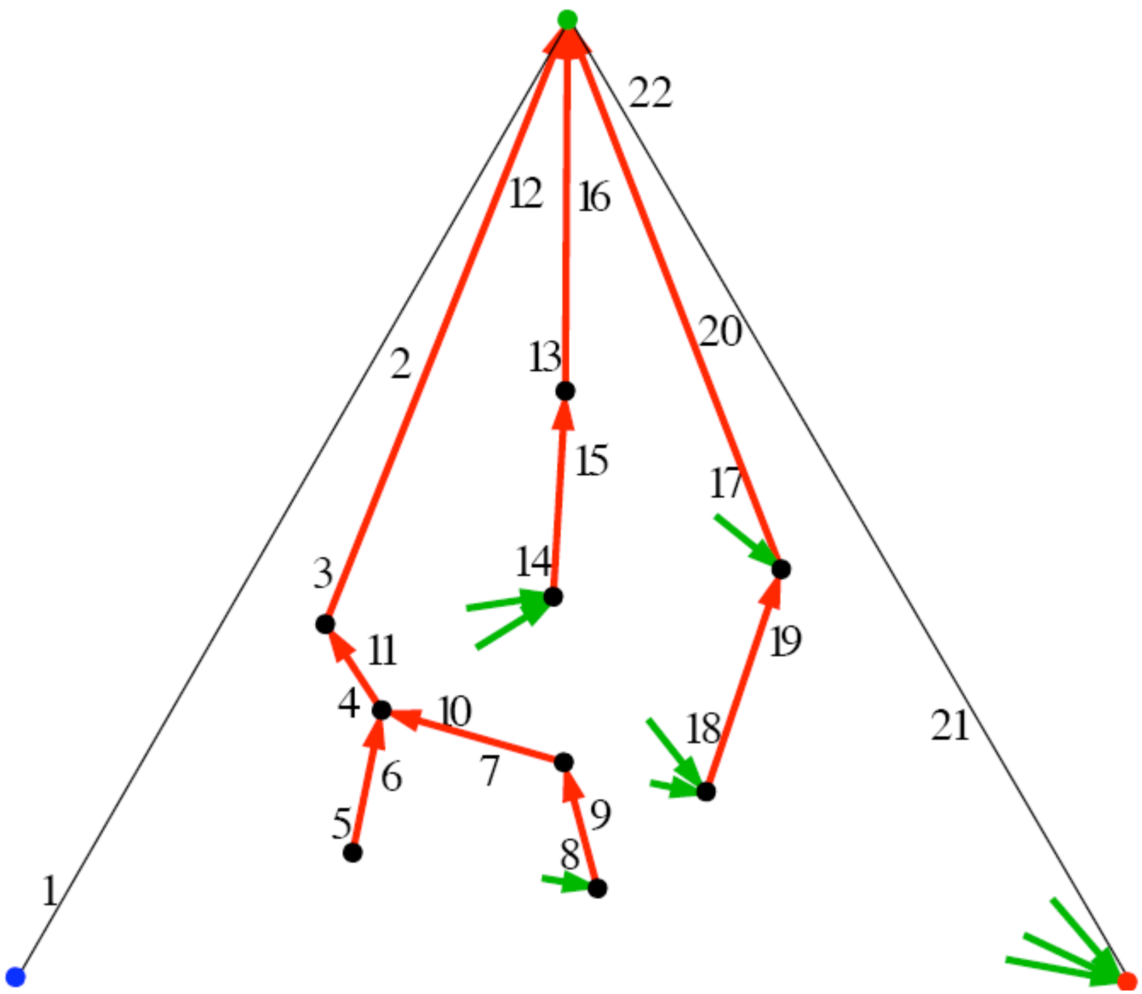

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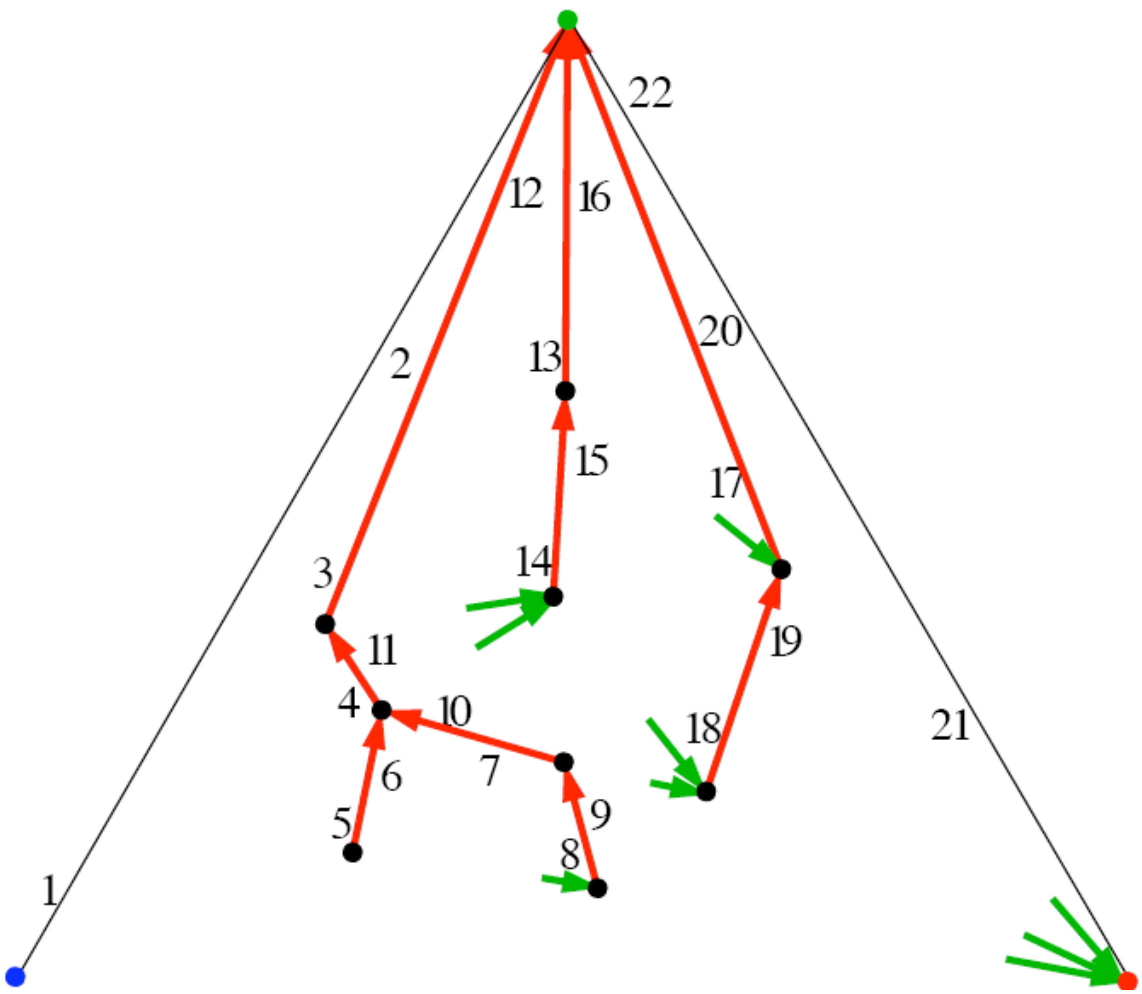

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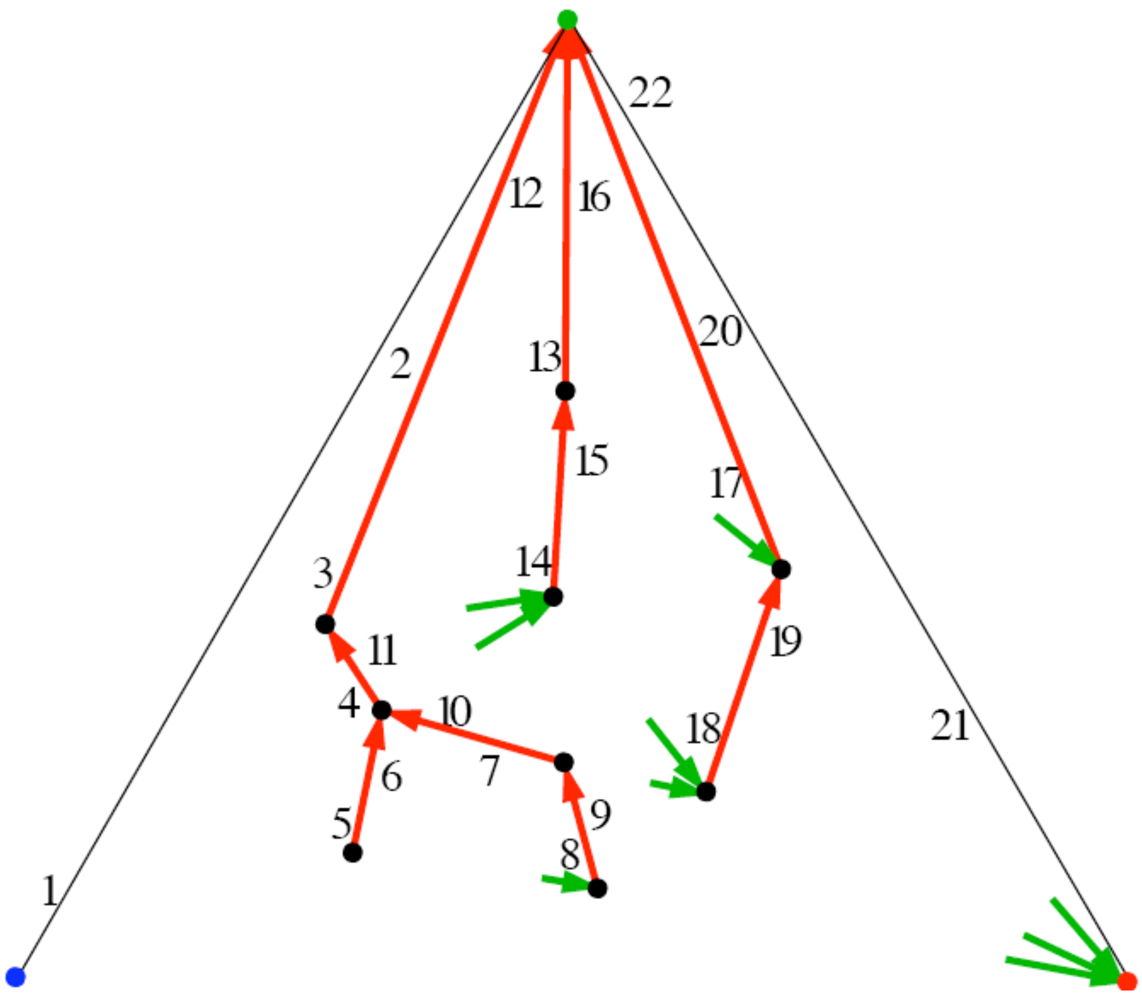

locate corners that can have ingoing green half-edges

Encoding a planar triangulation

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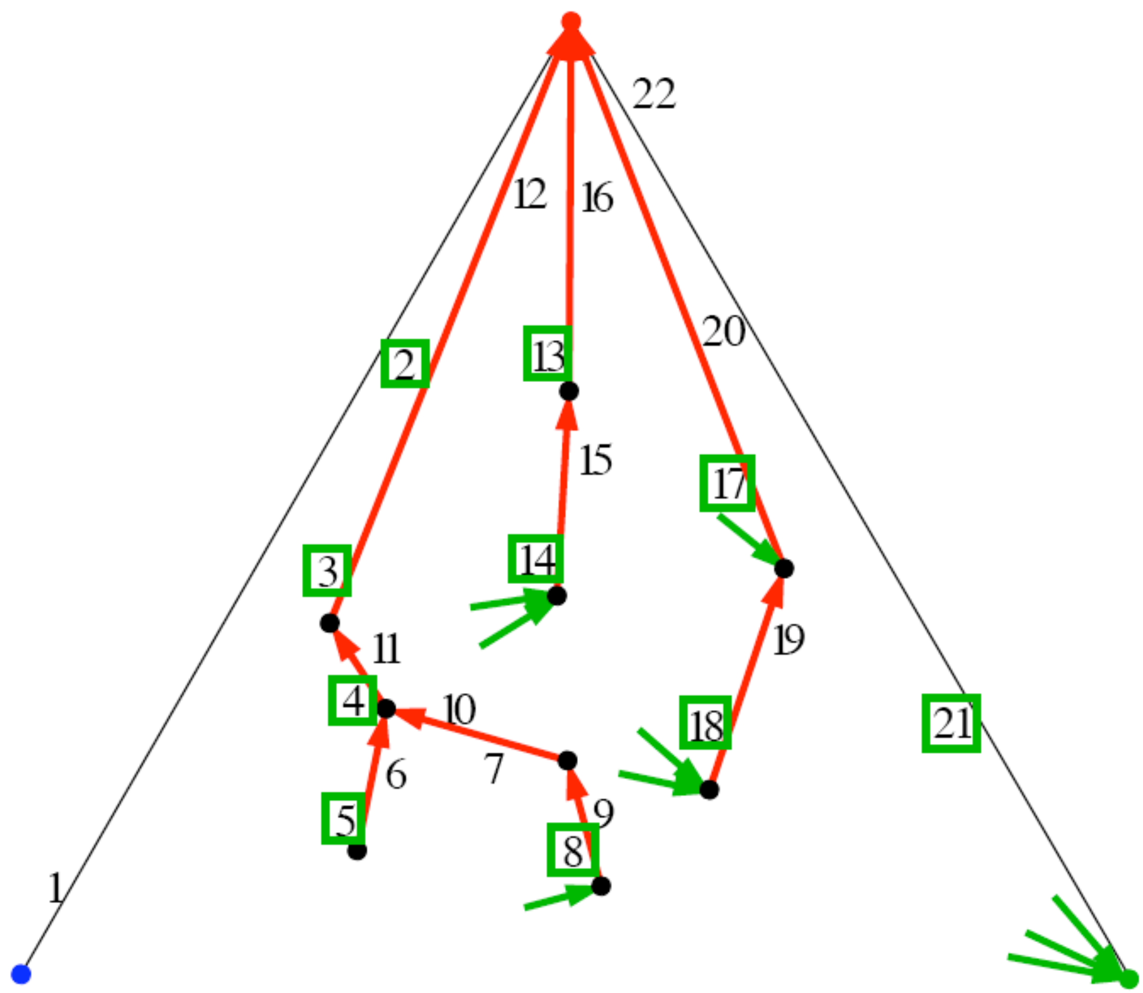
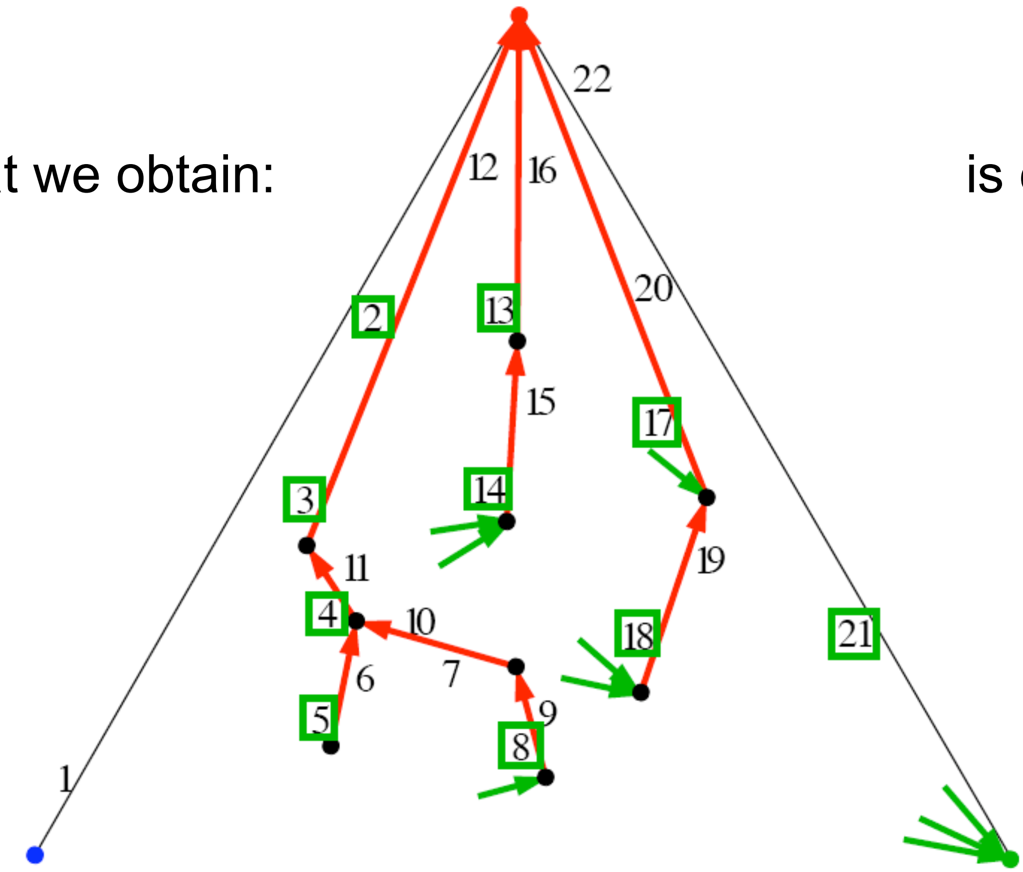


locate corners that can have ingoing green half-edges

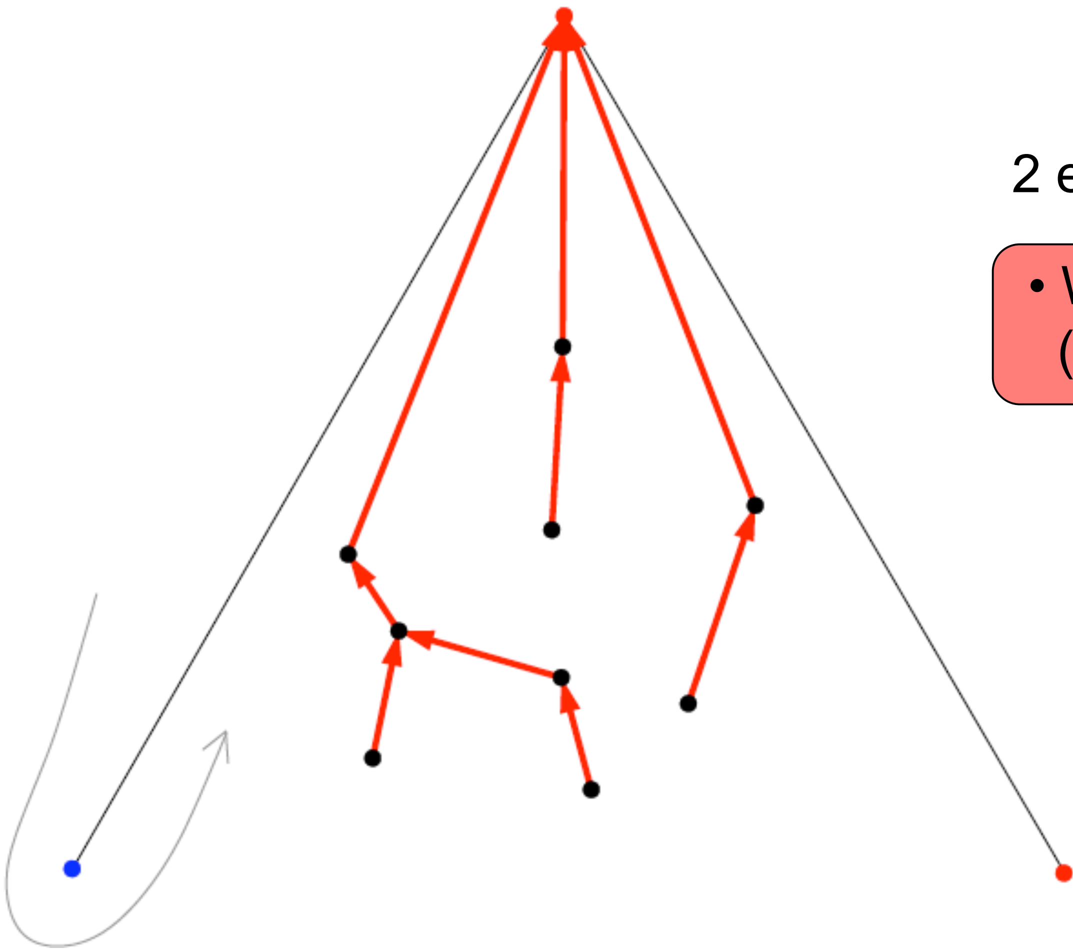
Encoding a planar triangulation

What we obtain:



is coded by **2 words**:

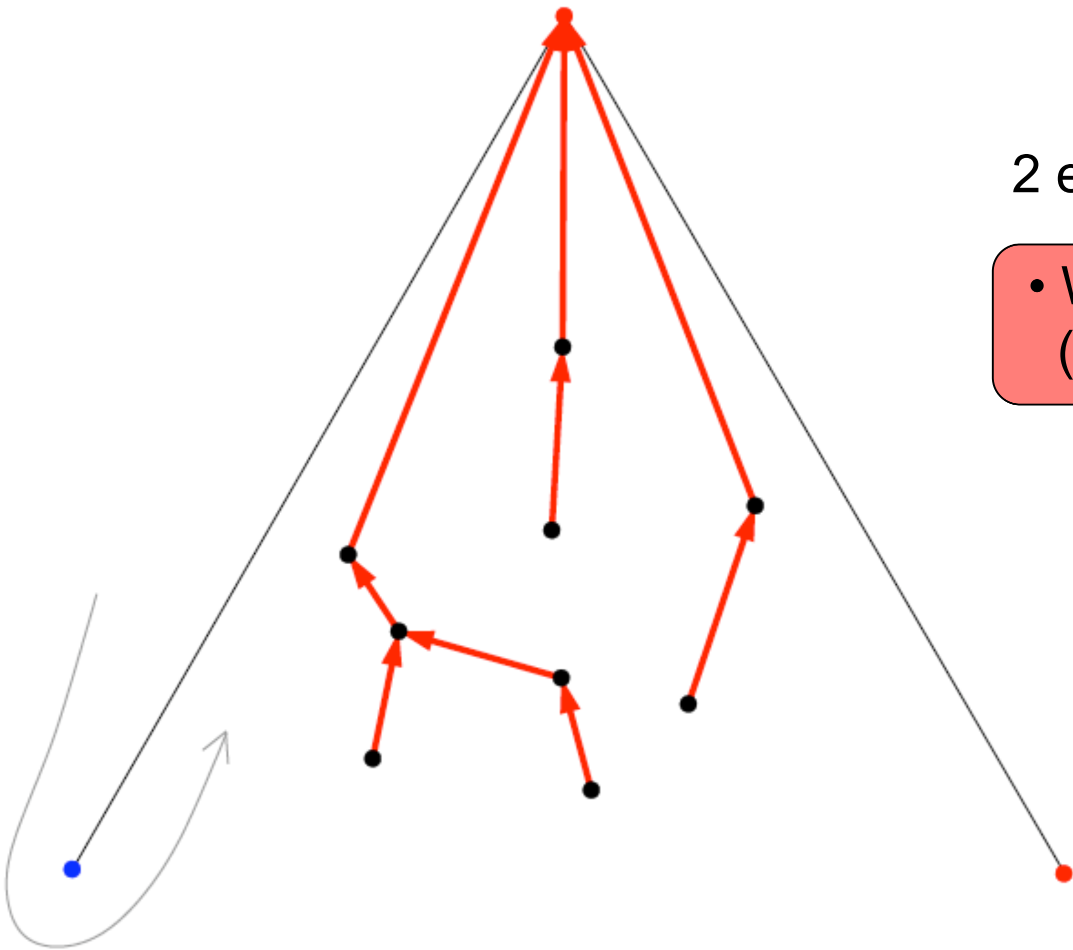
Encoding a planar triangulation



2 encoding words:

- W_R codes the red tree (Dyck word)

Encoding a planar triangulation



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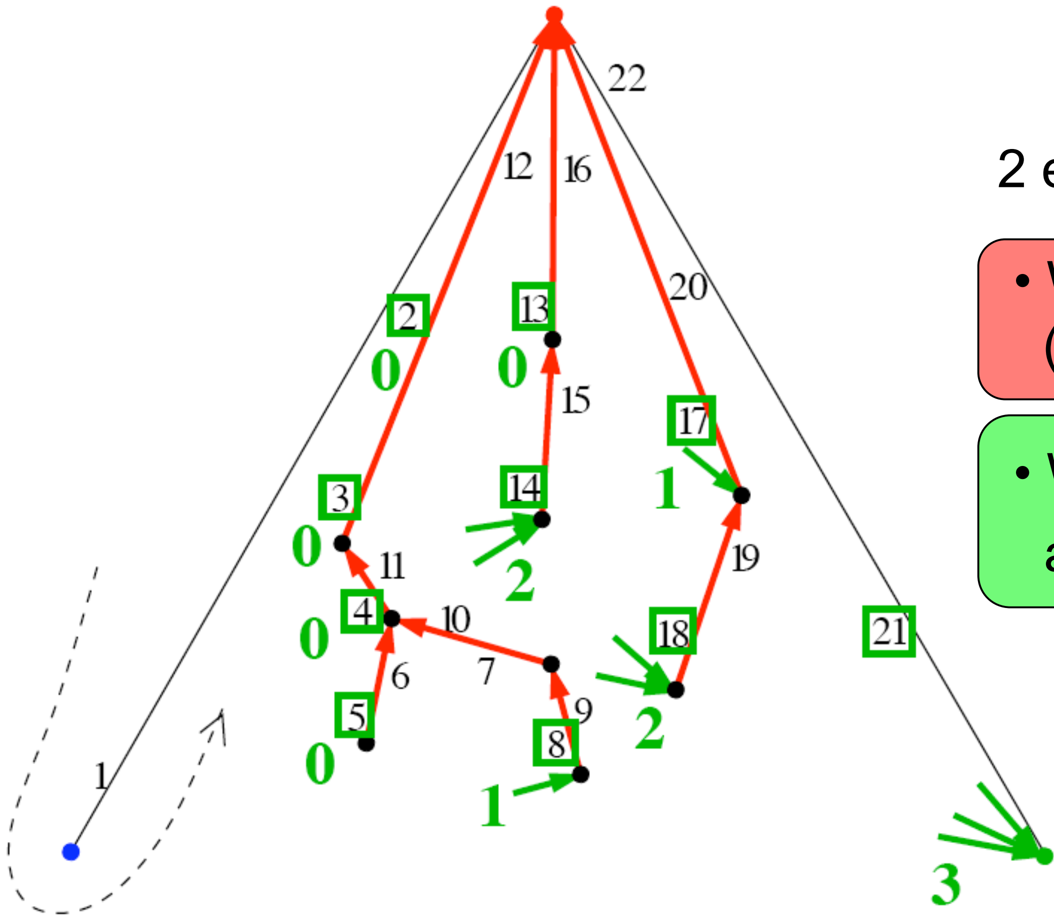
1) W_R=abaaabaabbbbaabbaabbab W_R has length 2n-2,

Encoding a planar triangulation



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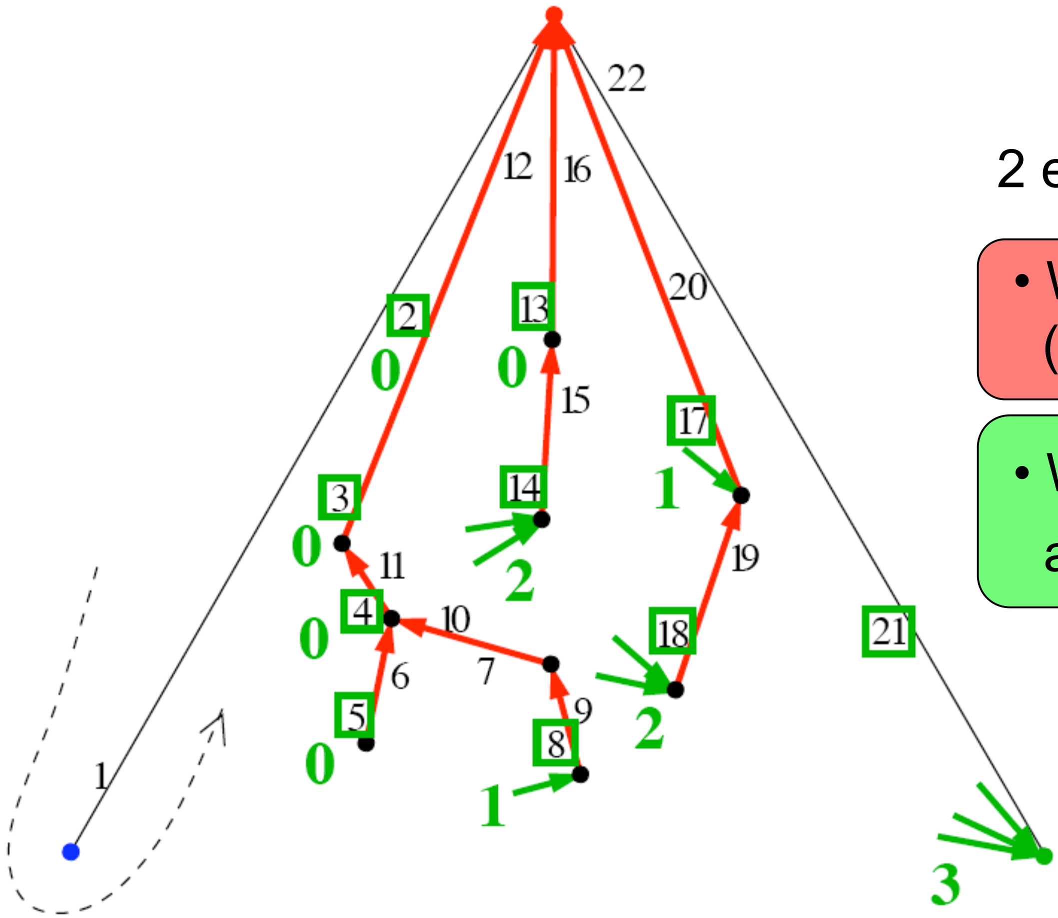
Encoding a planar triangulation



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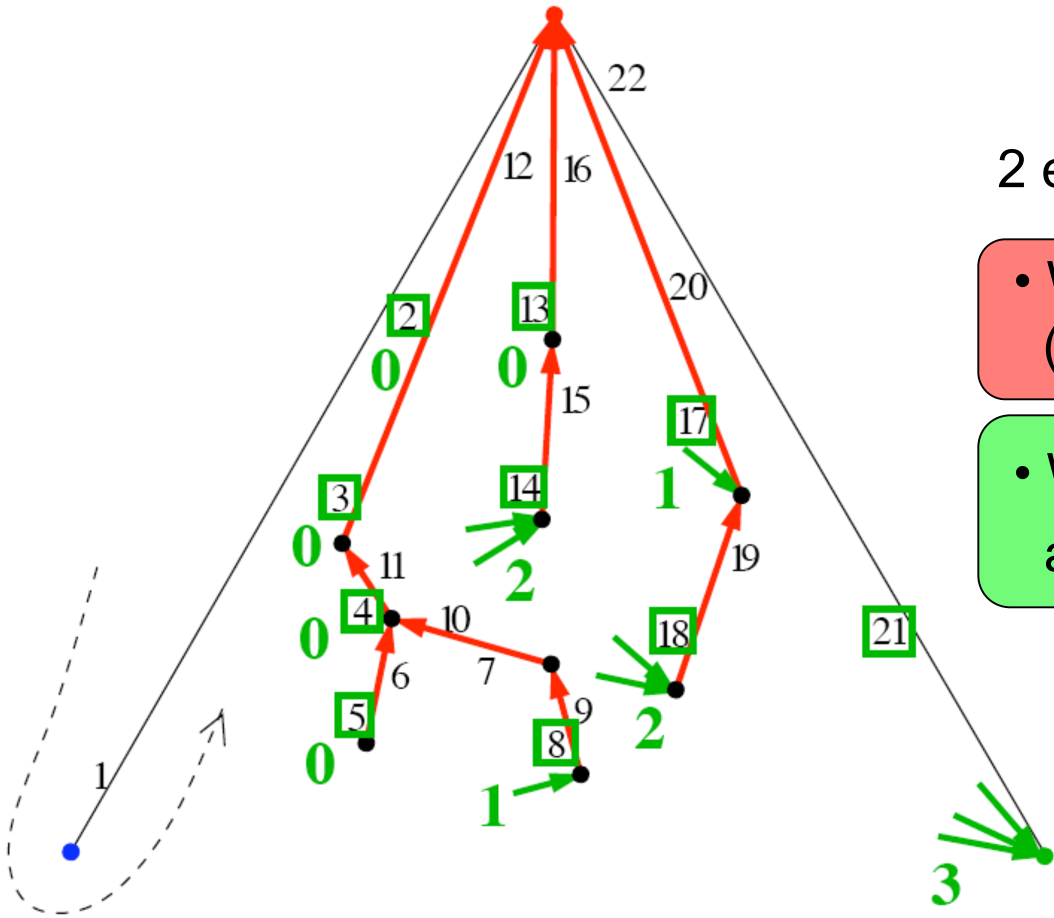
2) W_G=0,0,0,0,1,0,2,1,2,3

W_R has length 2n-2,

$W_G \simeq$ binary word length 2n-6

Encoding a planar triangulation



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W_R has length 2n-2,

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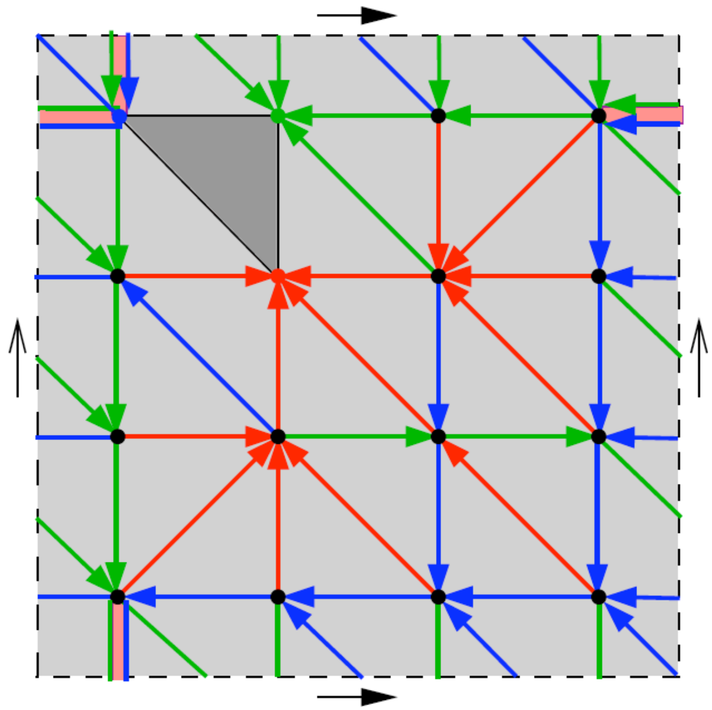
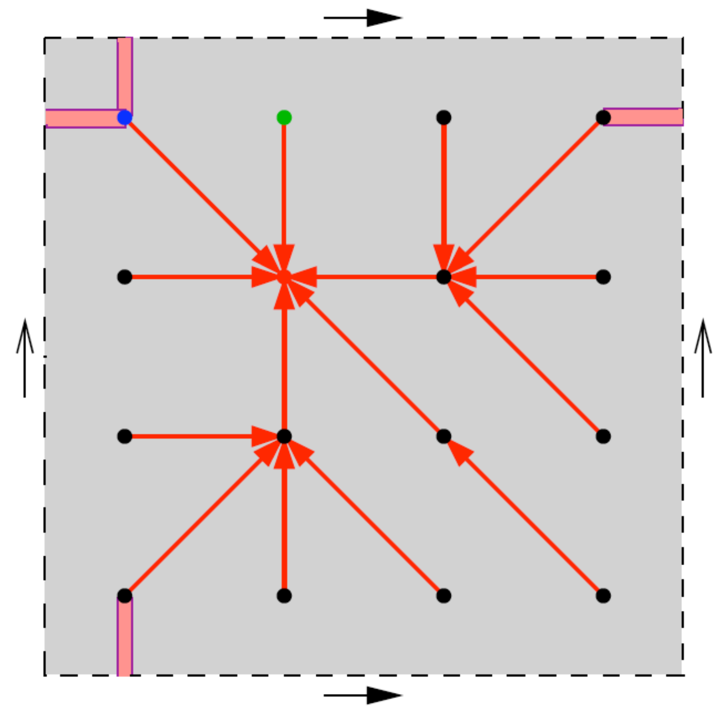
$\Rightarrow$ code length is 4n-8

Encoding in higher genus

- Everything works the same ! (walk along red cut-graph)





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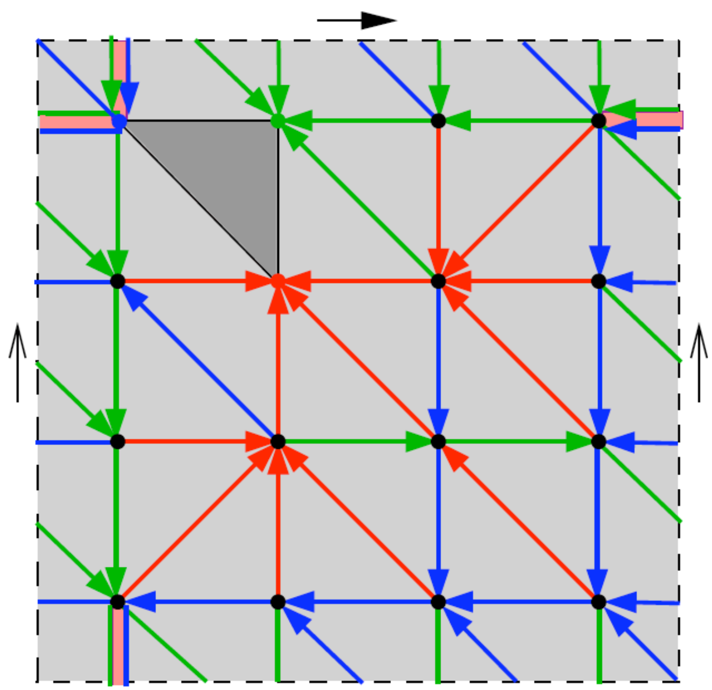

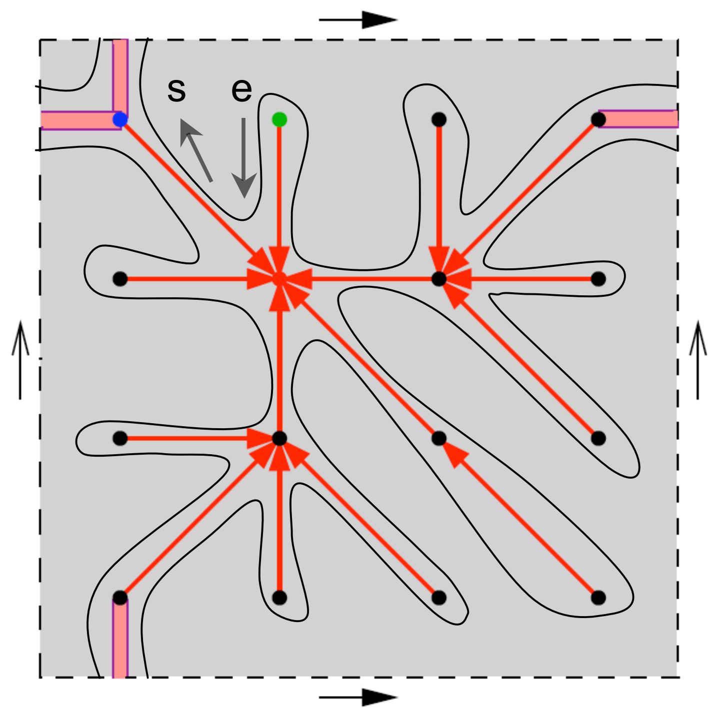

- Code-length is $4n+O(g \log(n))$

Results (coding)

- **Theorem** [Castelli, F, Lewiner'08]: A genus *g* triangulation can be encoded with $4n+O(g \log(n))$ bits. Coding and decoding take time $O((n+g)g)$

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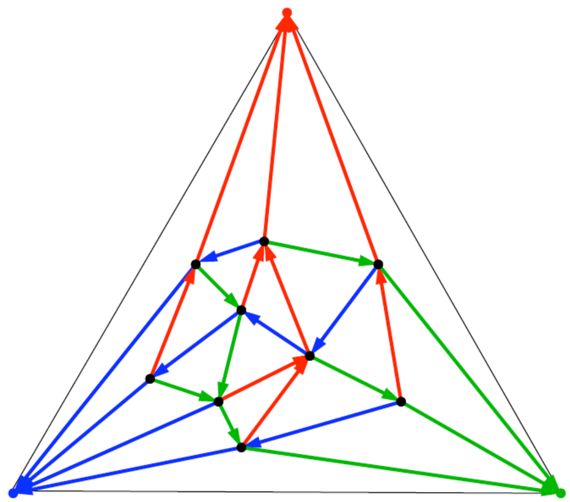
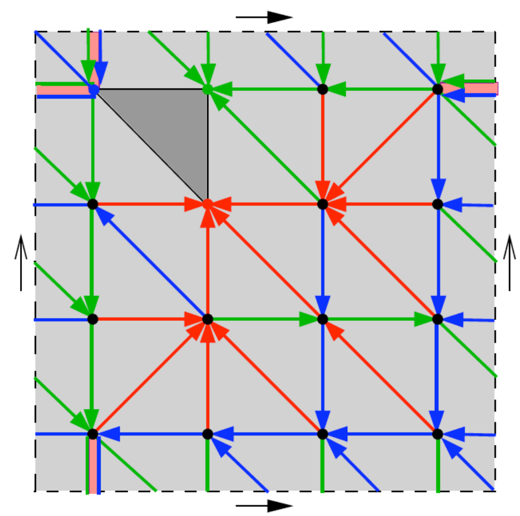
- In genus 0, bijective coding [Poulalhon-Schaeffer'03] optimal
- In higher genus, best known rate is 4n+O(g log(n)) bits:
  - our encoding based on Schnyder woods
  - Edgebreaker of [Rossignac et al]

Conclusion

- We extend definition/computation of Schnyder woods to higher genus



planar



toroidal

- In genus g>0, there are 2g `special' edges

- Schnyder wood -> code triangulation of genus g>0 in 4n+O(g log(n)) bits