Duality relations for constrained walks

Éric Fusy (CNRS/LIX)

joint work with Mireille Bousquet-Mélou, Julien Courtiel, Mathias Lepoutre, Marni Mishna, and Kilian Raschel

TU Berlin, Jan 13, 2020

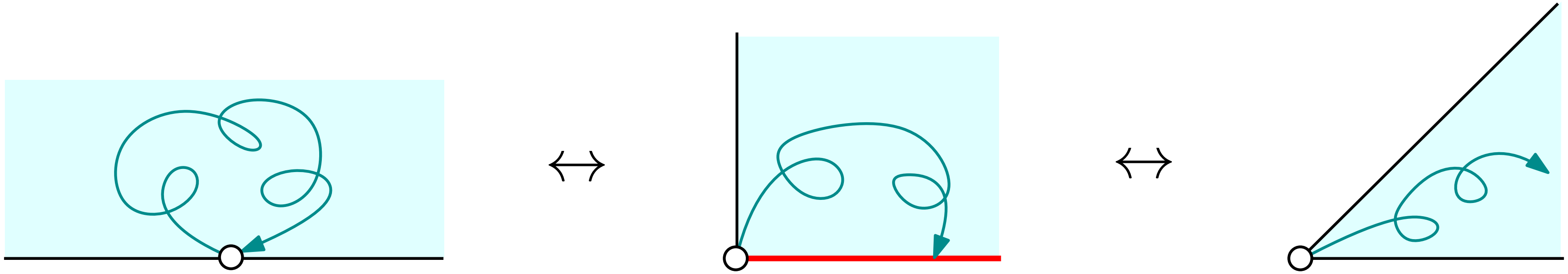
Duality phenomenon for paths

We say two path families $\mathcal{A}$ and $\mathcal{B}$ are **dual** if

- both families use the same steps, such that
  $\mathcal{A}$ has stronger endpoint constraint, $\mathcal{B}$ has stronger domain constraint

- there is a length-preserving **bijection** between $\mathcal{A}$ and $\mathcal{B}$

Example in 2D: (step-set $\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$)



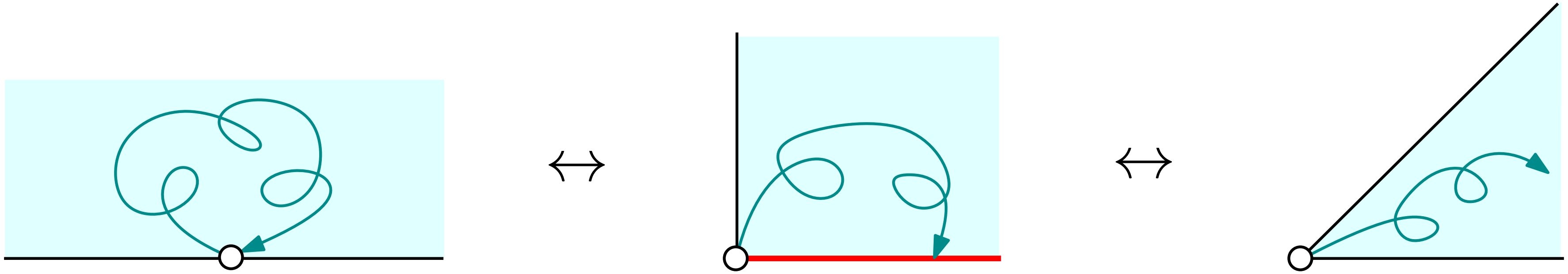
Duality phenomenon for paths

We say two path families $\mathcal{A}$ and $\mathcal{B}$ are **dual** if

- both families use the same steps, such that
  $\mathcal{A}$ has stronger endpoint constraint, $\mathcal{B}$ has stronger domain constraint

- there is a length-preserving **bijection** between $\mathcal{A}$ and $\mathcal{B}$

Example in 2D: (step-set $\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$)



Motivations:

- mapping $\mathcal{A} \to \mathcal{B}$ for **counting** ($\mathcal{A}$ easier)
- mapping $\mathcal{B} \to \mathcal{A}$ for **random generation** (early-abort rejection)

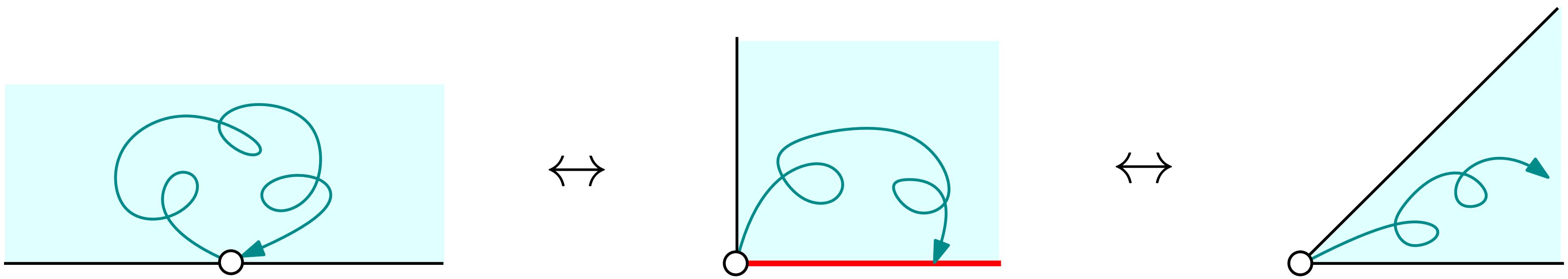
Duality phenomenon for paths

We say two path families $\mathcal{A}$ and $\mathcal{B}$ are **dual** if

- both families use the same steps, such that
  $\mathcal{A}$ has stronger endpoint constraint, $\mathcal{B}$ has stronger domain constraint

- there is a length-preserving **bijection** between $\mathcal{A}$ and $\mathcal{B}$

Example in 2D: (step-set $\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$)



Motivations:

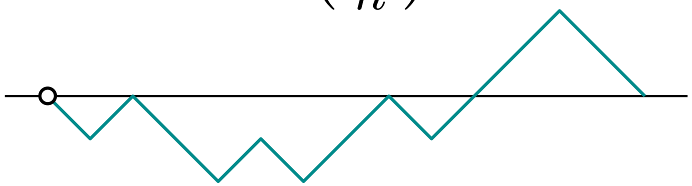
- mapping $\mathcal{A} \rightarrow \mathcal{B}$ for **counting** ($\mathcal{A}$ easier)
- mapping $\mathcal{B} \rightarrow \mathcal{A}$ for **random generation** (early-abort rejection)

gen$\mathcal{B}$: while not fails
  generate random walk step by step
  reject as soon as walk leaves domain for $\mathcal{B}$ (if not, success!)

Classical 1D example

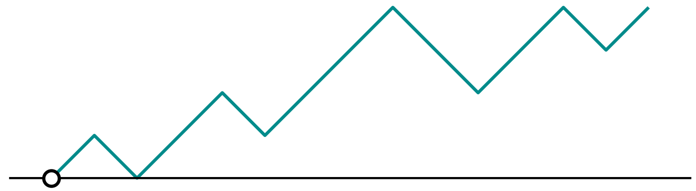
$$a_{2n} = \binom{2n}{n}$$

$\mathcal{A}$



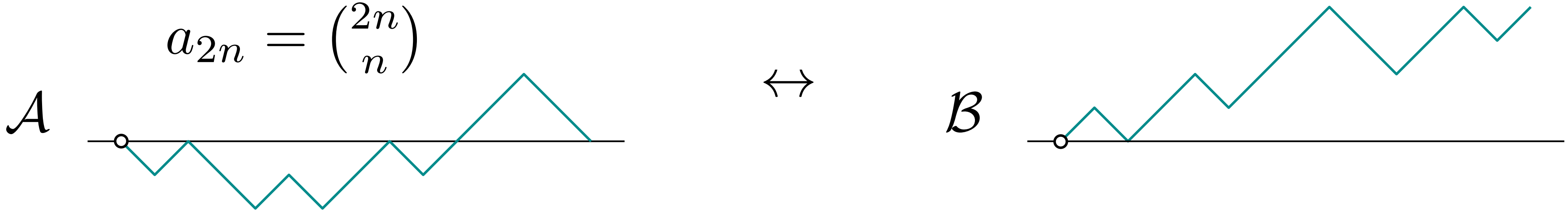
$\leftrightarrow$

$\mathcal{B}$

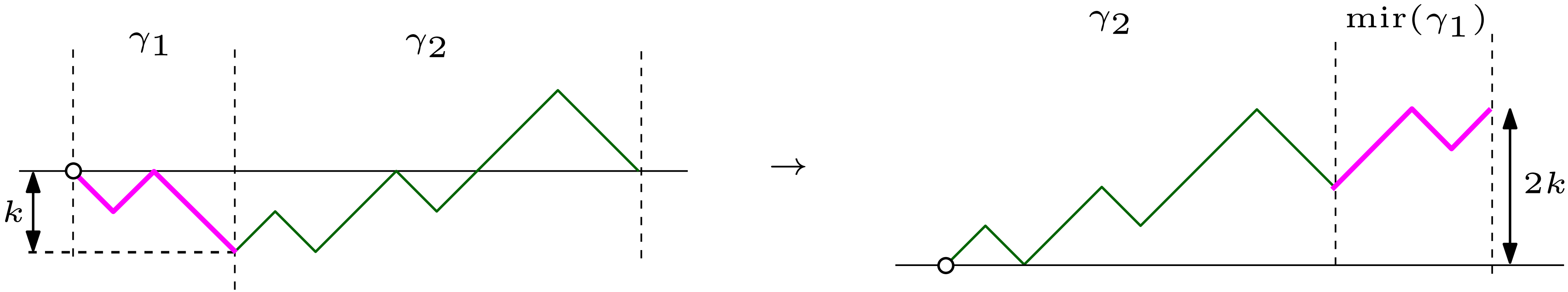


Classical 1D example

$$a_{2n} = \binom{2n}{n}$$

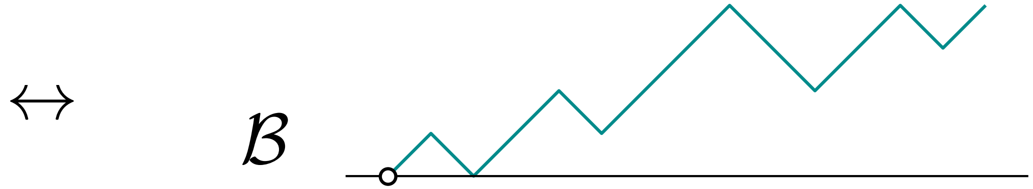


1st bijection:

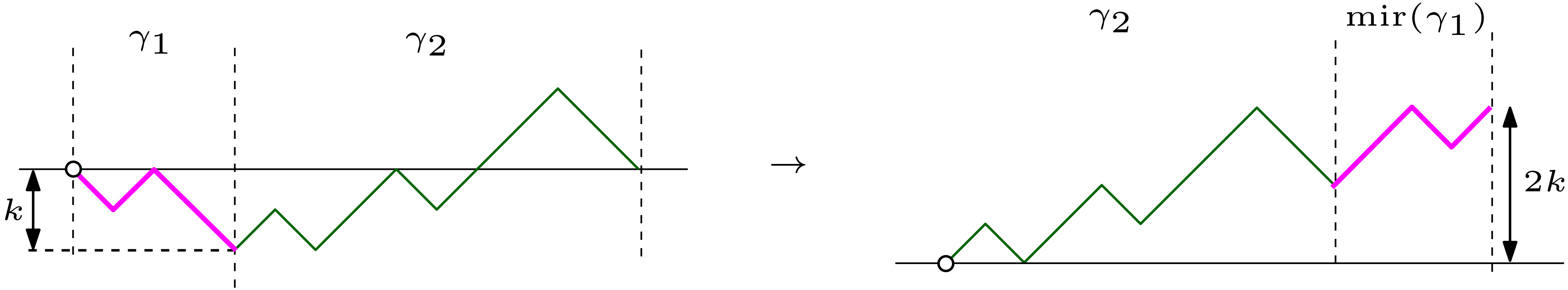


Classical 1D example

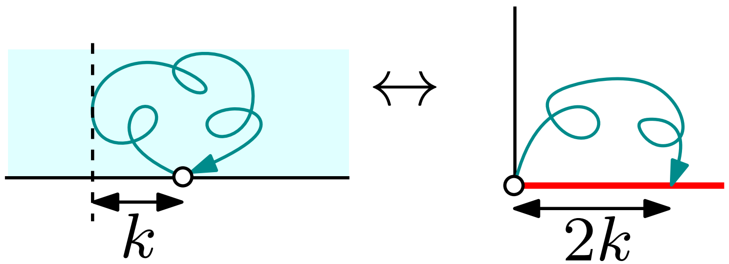
$$a_{2n} = \binom{2n}{n}$$



1st bijection:



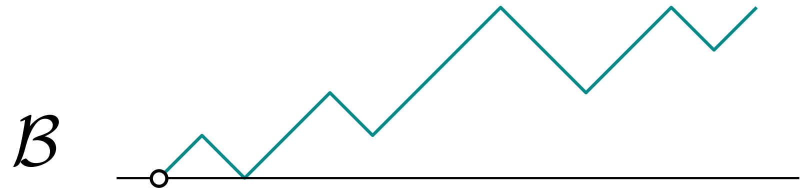
Rk: implies



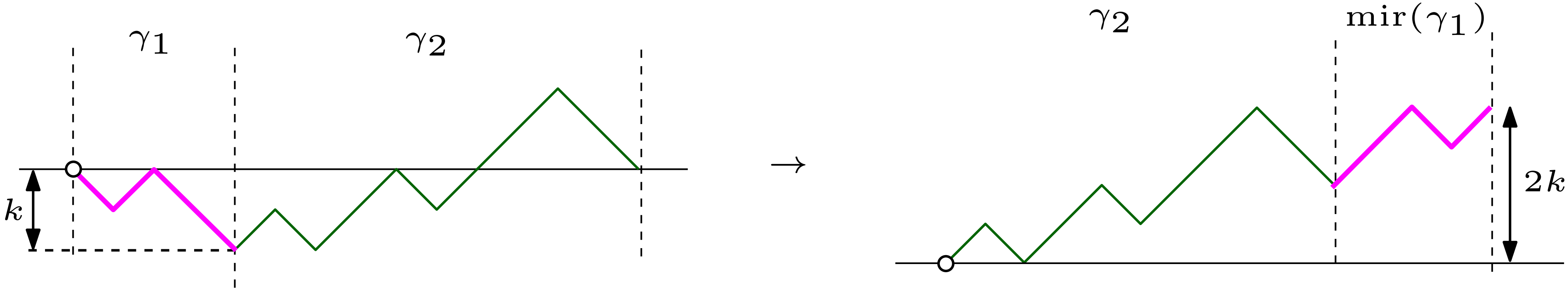
Classical 1D example

$$a_{2n} = \binom{2n}{n}$$

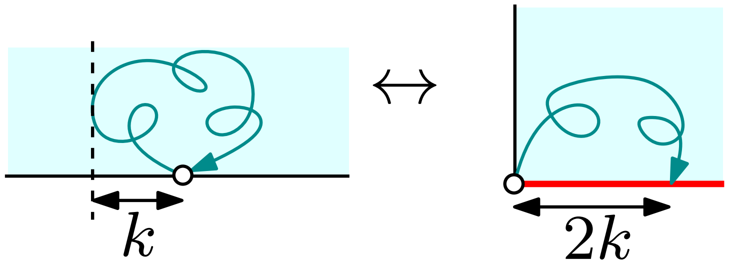
$\leftrightarrow$



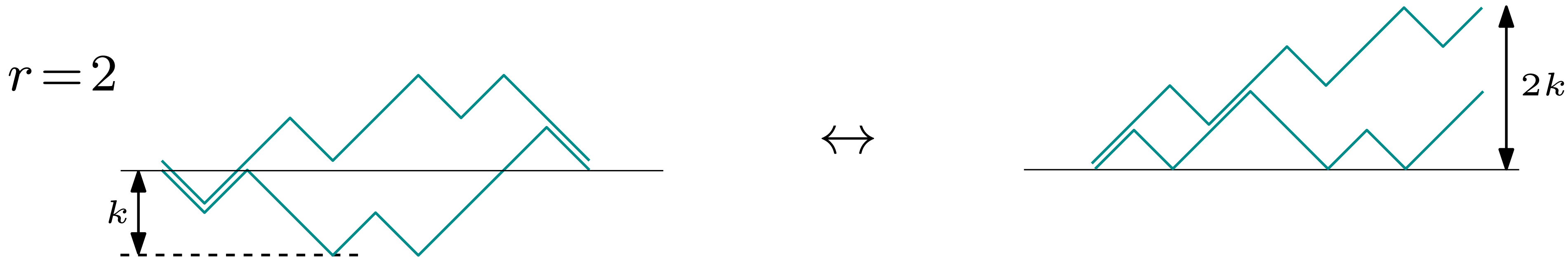
1st bijection:



Rk: implies



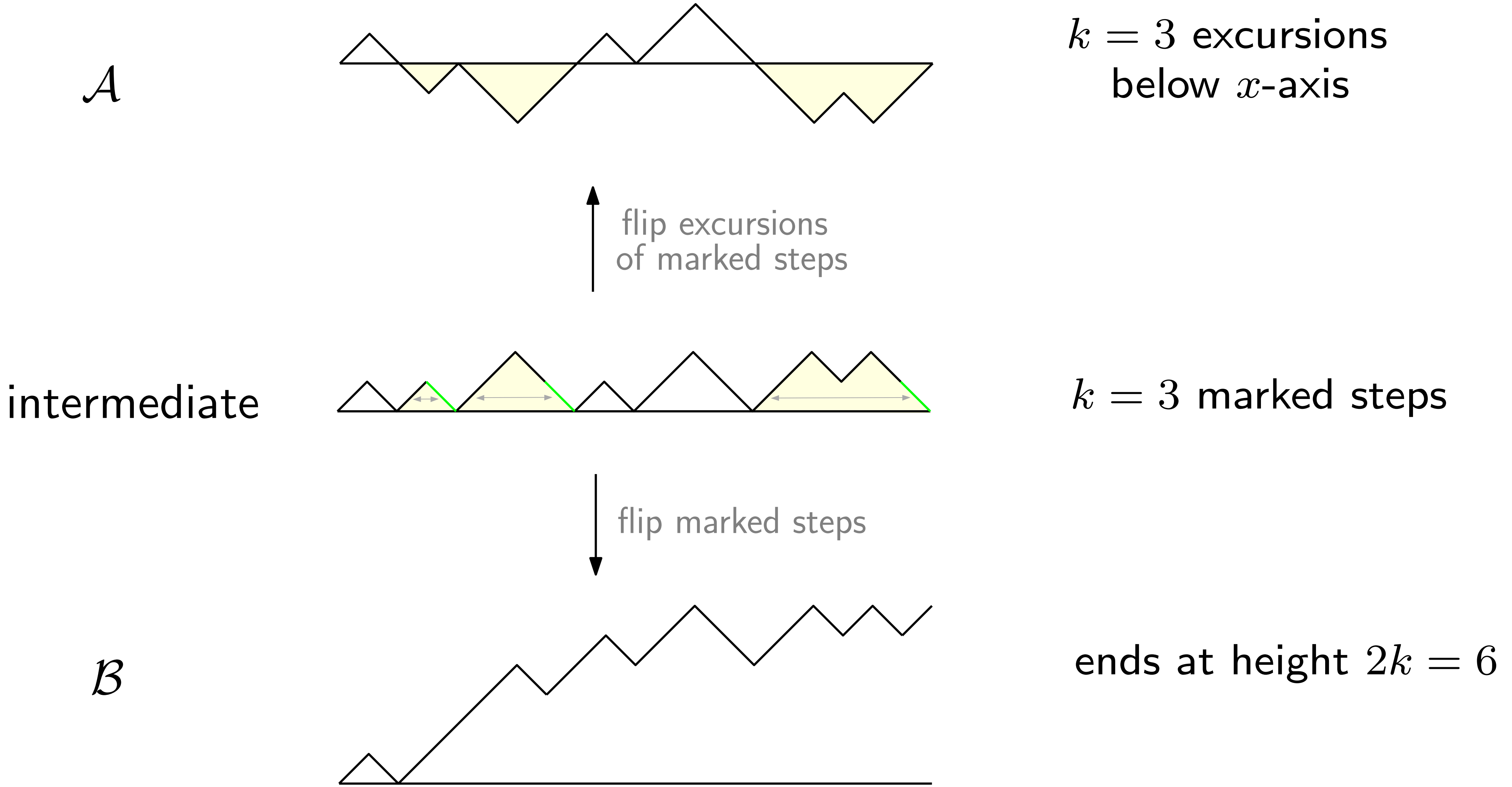
Rk: extends to $r \geq 1$ paths [Proctor'83, Elizalde'15, Hanaker et al.'17]



Classical 1D example

2nd bijection:

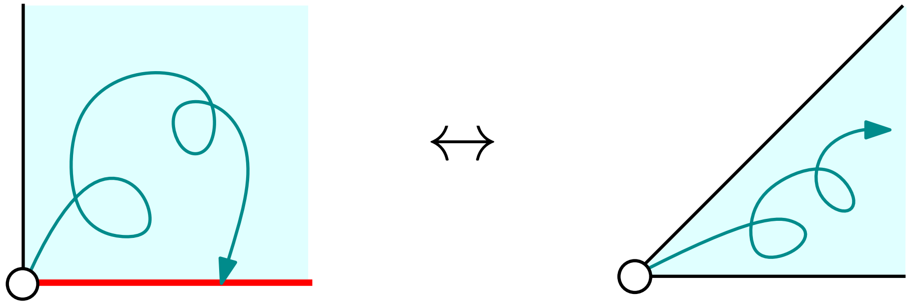
via Dyck paths with marked down-steps ending on x-axis



Outline of the talk

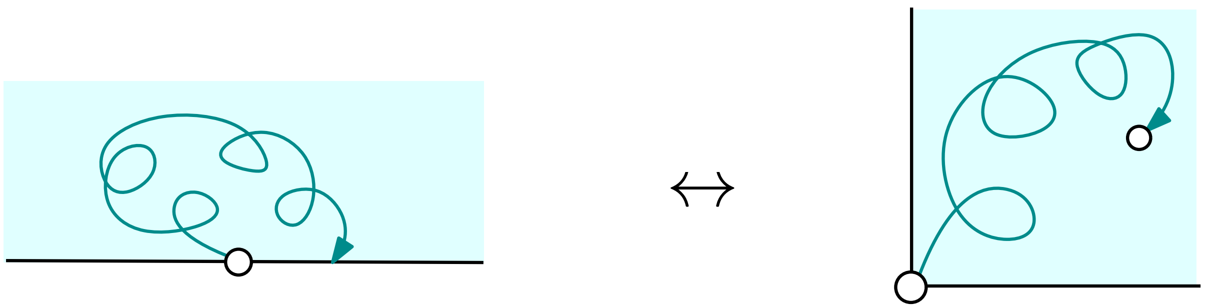
Duality relations for 2D walks using bijections to oriented maps

- Simple walks: $\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$



using Bernardi-Bonichon bijection for Schnyder woods

- Tandem walks: $\{\leftarrow, \uparrow, \searrow\}$ (and extension)

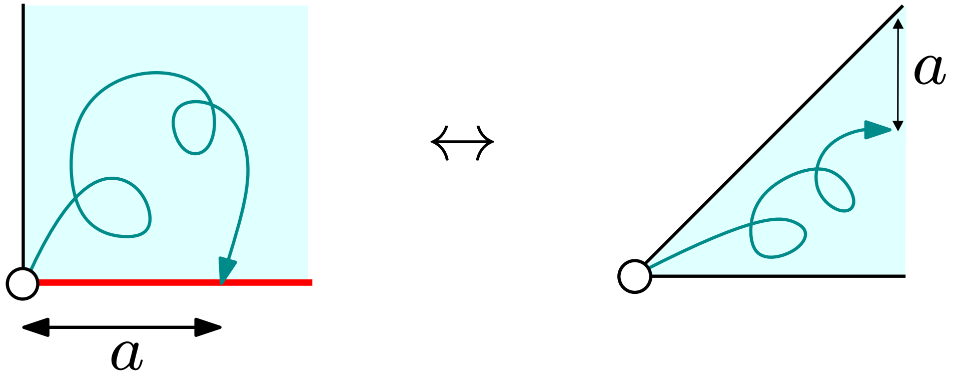


using Kenyon et al. bijection for bipolar orientations

Outline of the talk

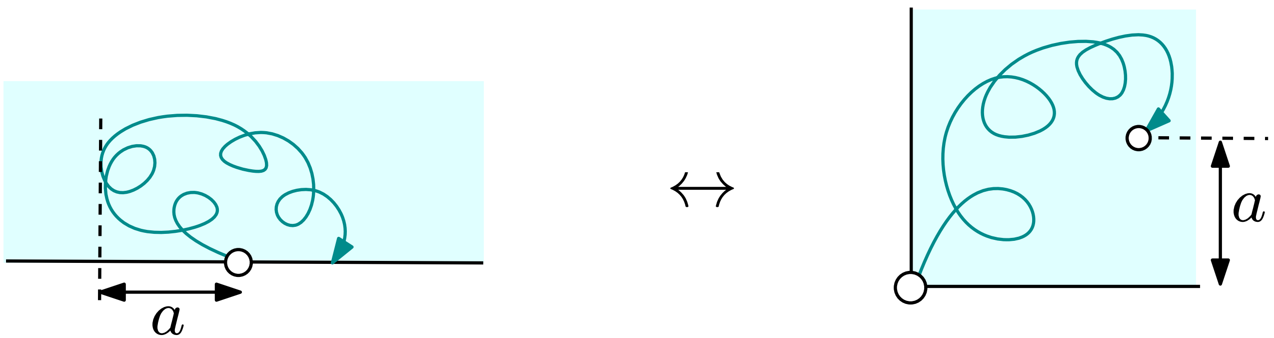
Duality relations for 2D walks using bijections to oriented maps

- Simple walks: $\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$



using Bernardi-Bonichon bijection for Schnyder woods

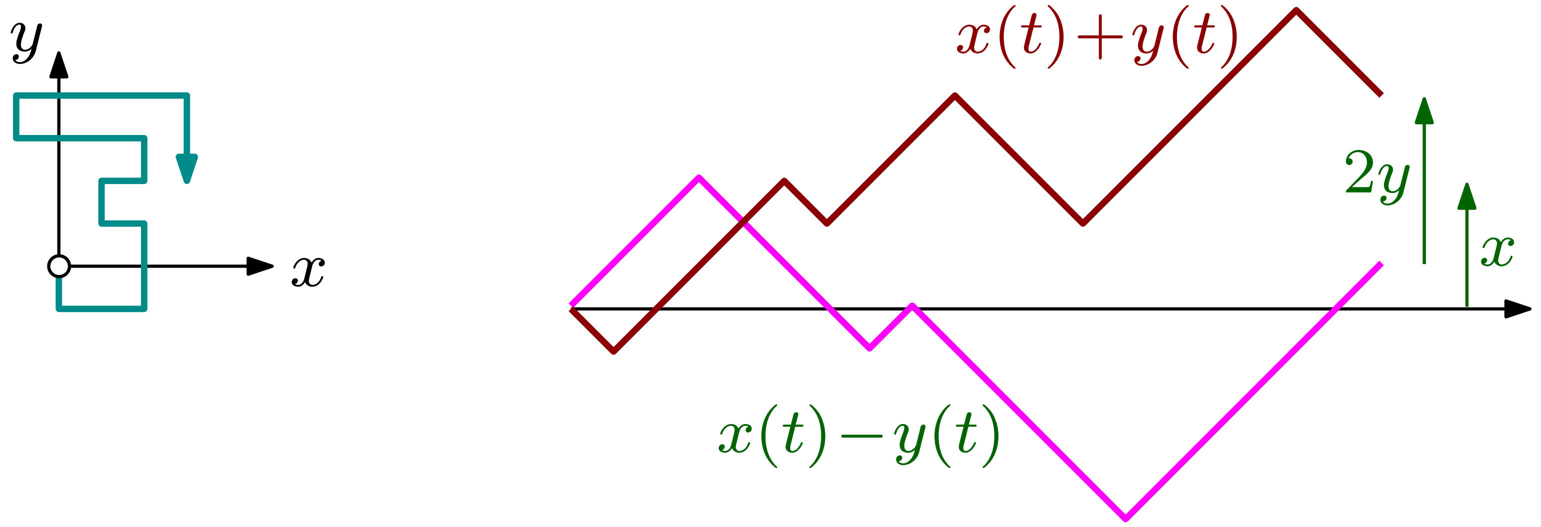
- Tandem walks: $\{\leftarrow, \uparrow, \searrow\}$ (and extension)



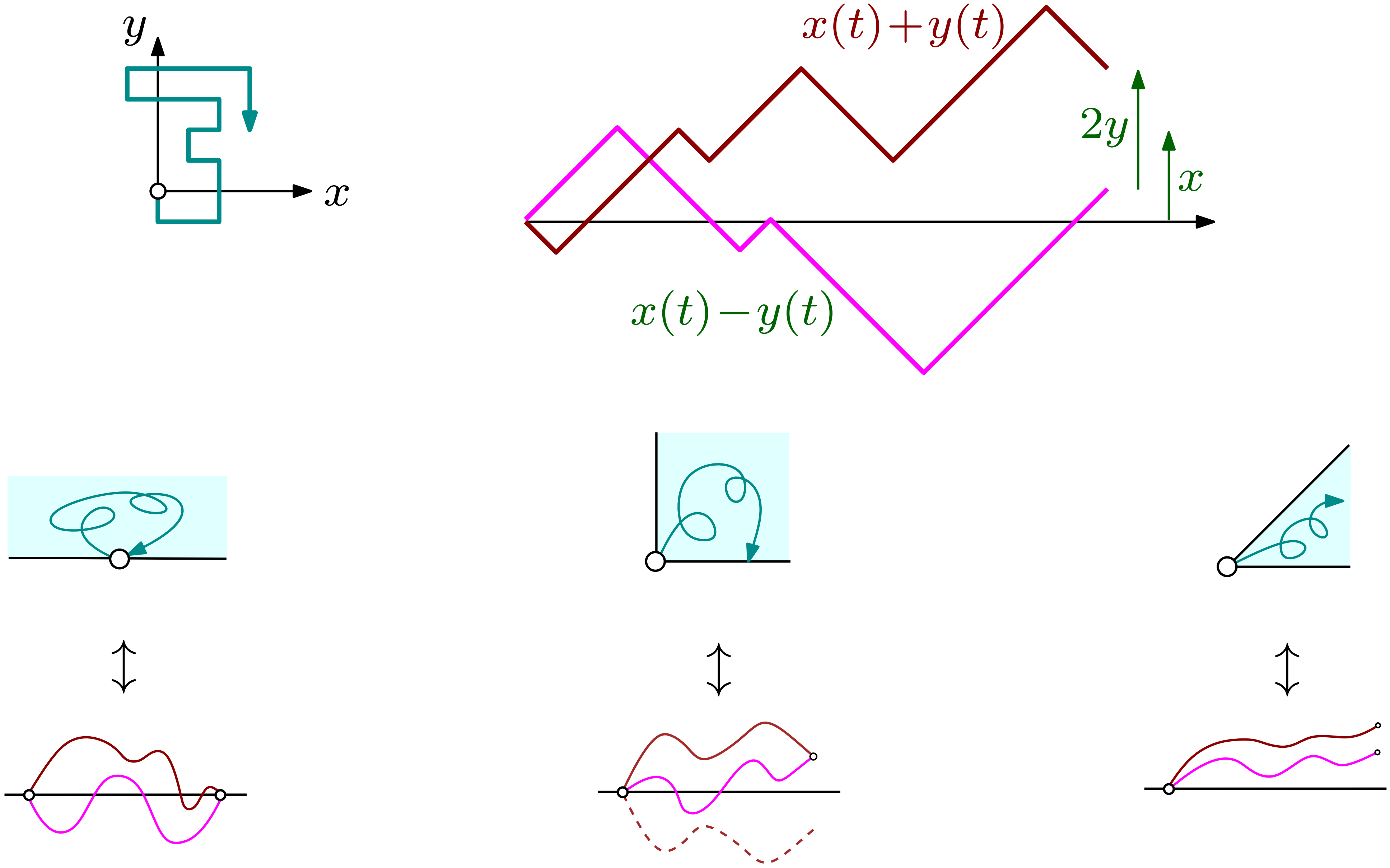
using Kenyon et al. bijection for bipolar orientations

Simple walks

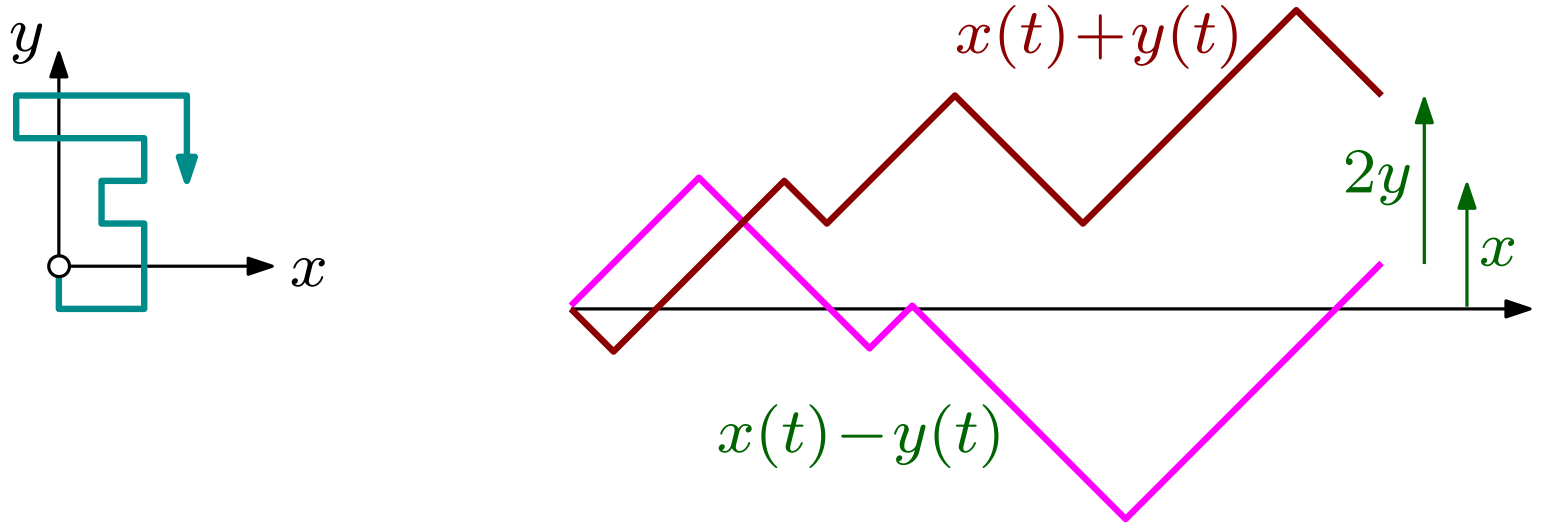
2D simple walk $\leftrightarrow$ pair of directed walks

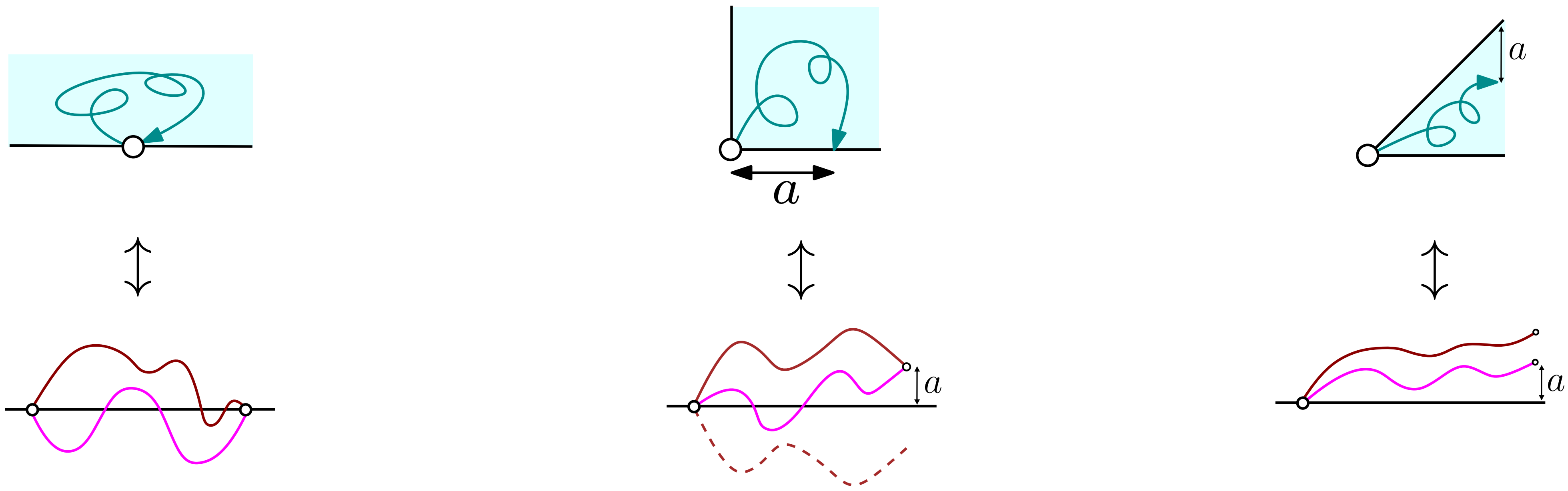


2D simple walk $\leftrightarrow$ pair of directed walks

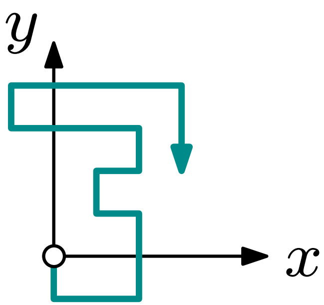


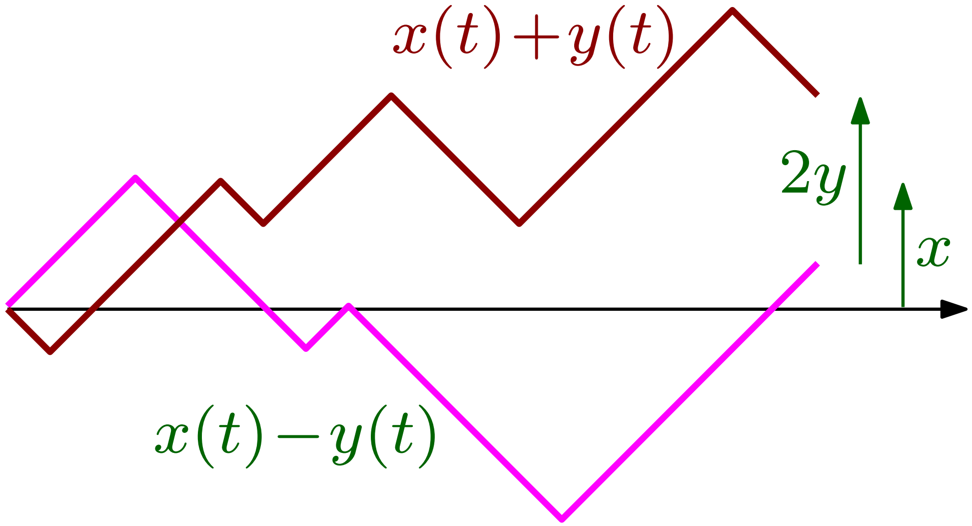
2D simple walk $\leftrightarrow$ pair of directed walks

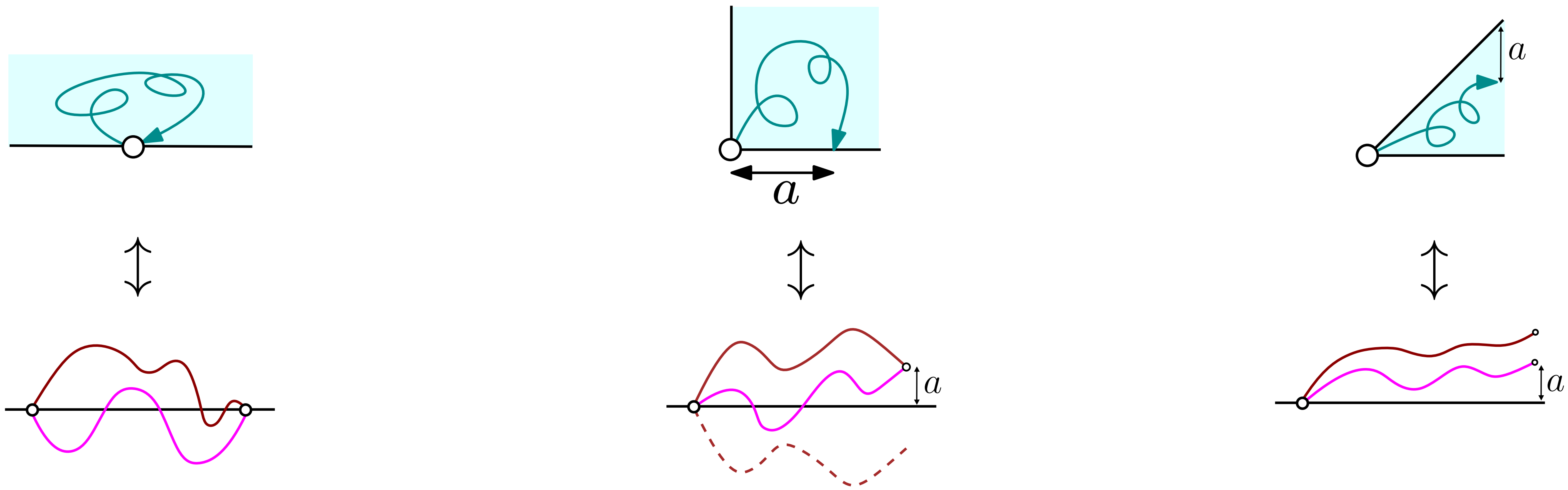


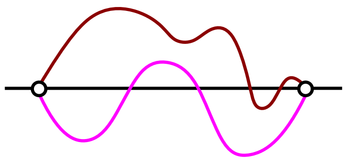
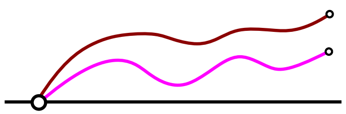
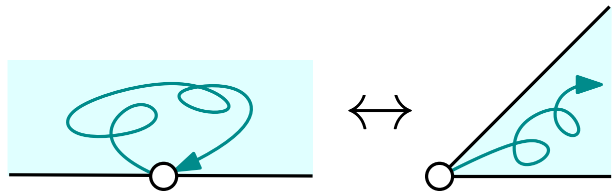


2D simple walk $\leftrightarrow$ pair of directed walks

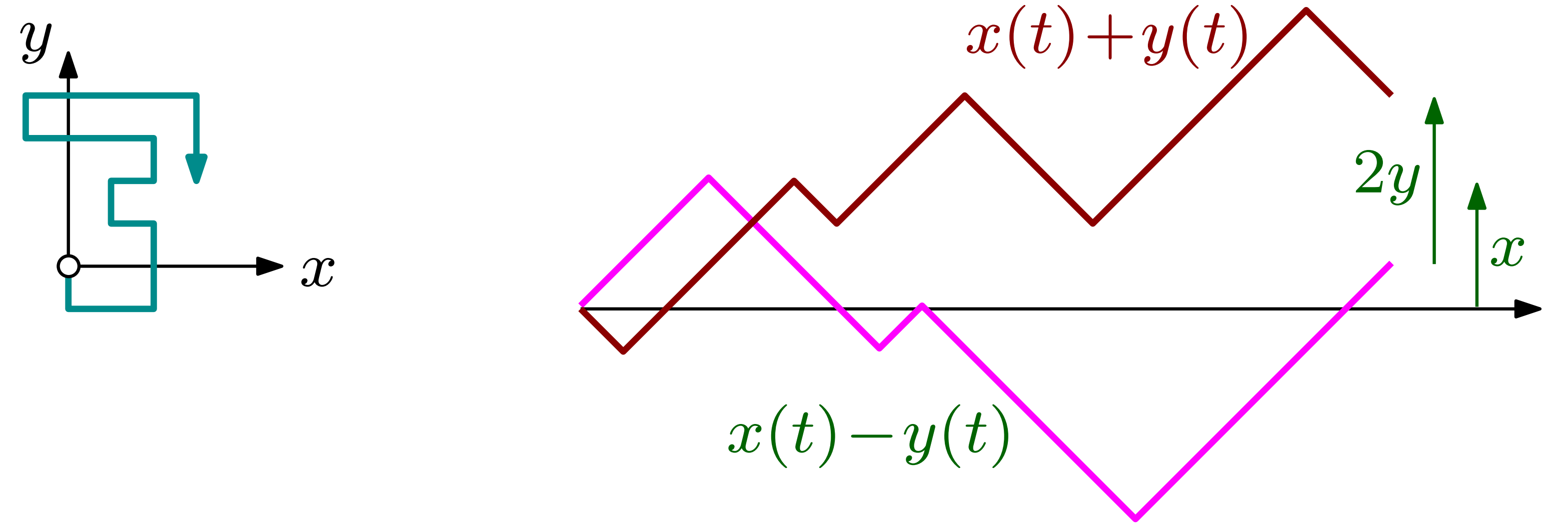


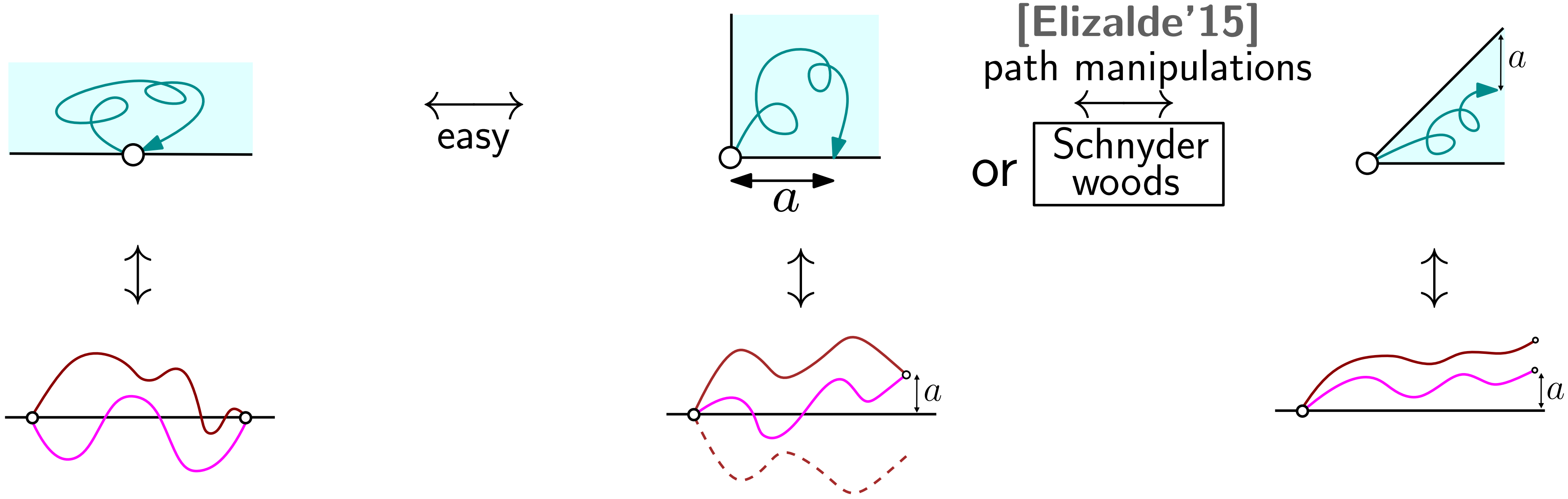




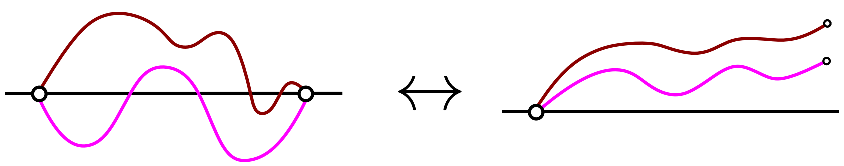
Rk:  $\leftrightarrow$  is the same as 

2D simple walk ↔ pair of directed walks

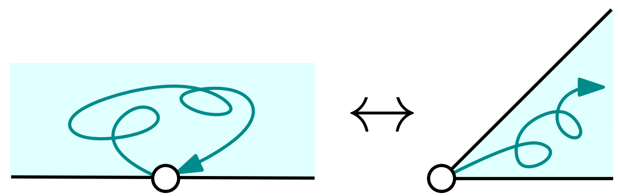




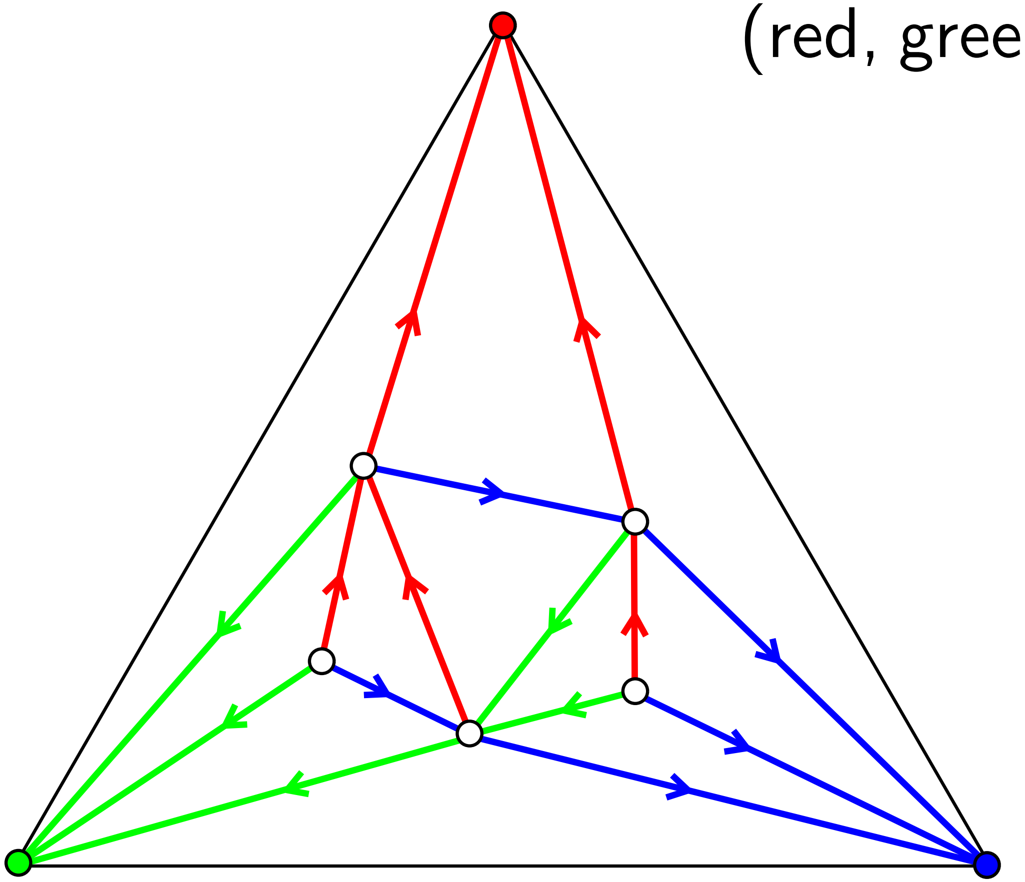
Rk:



is the same as



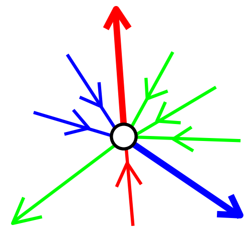
Schnyder woods on triangulations [Schnyder'89]



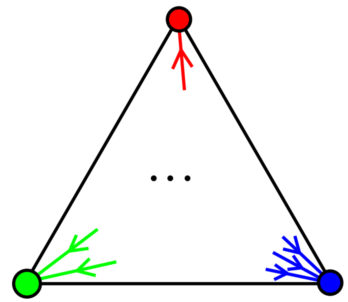
Schnyder wood = choice of a direction and color (red, green, or blue) for each inner edge, such that:

Local conditions:

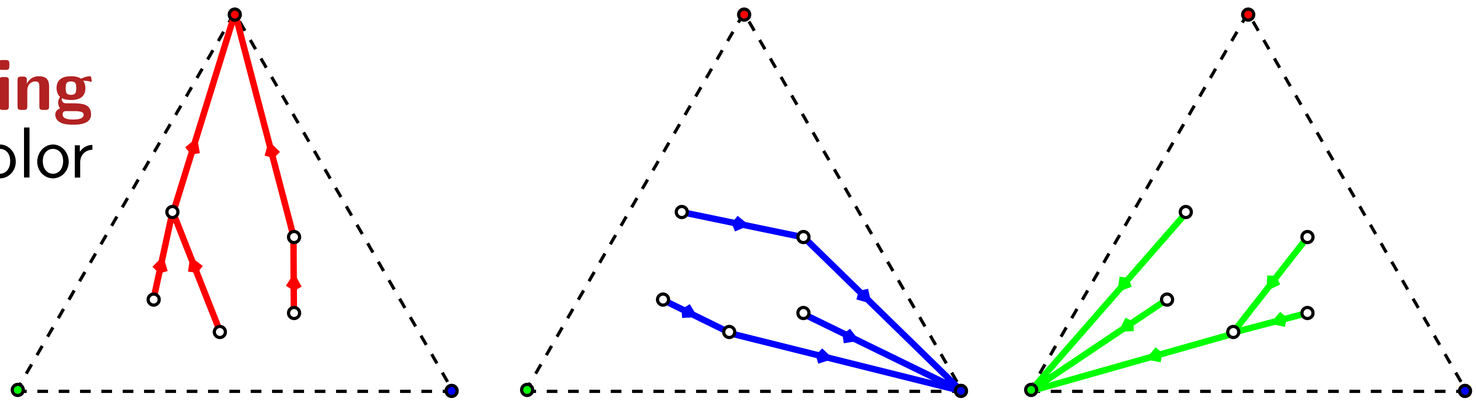
at each inner vertex



at the outer vertices



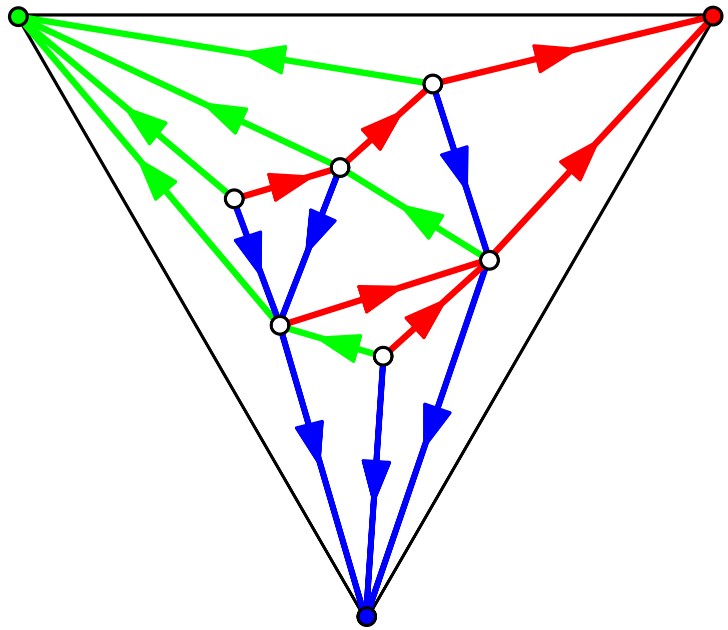
yields a **spanning tree** in each color

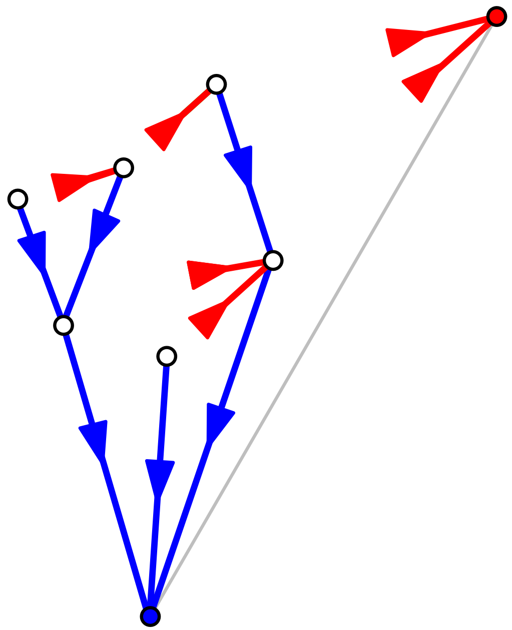


Bijection for Schnyder woods [Bernardi, Bonichon'07]

Some information is redundant:

just need the blue tree and positions of the ingoing red edges

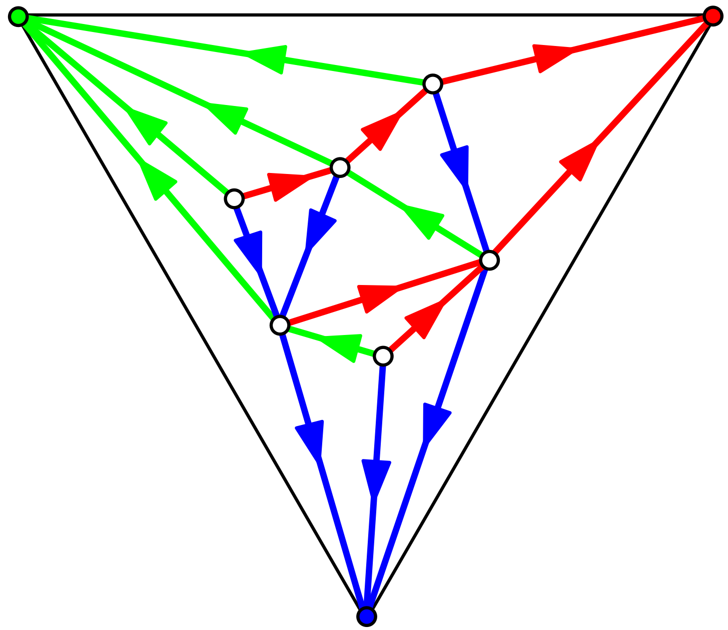


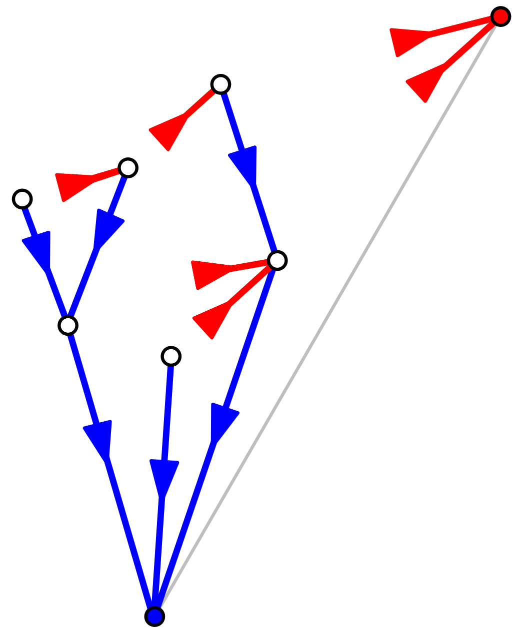


Bijection for Schnyder woods [Bernardi, Bonichon'07]

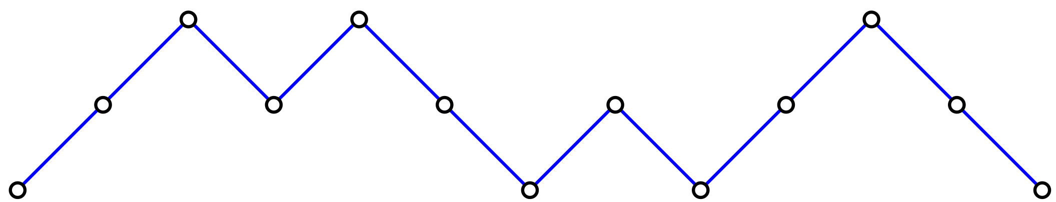
Some information is redundant:

just need the blue tree and positions of the ingoing red edges





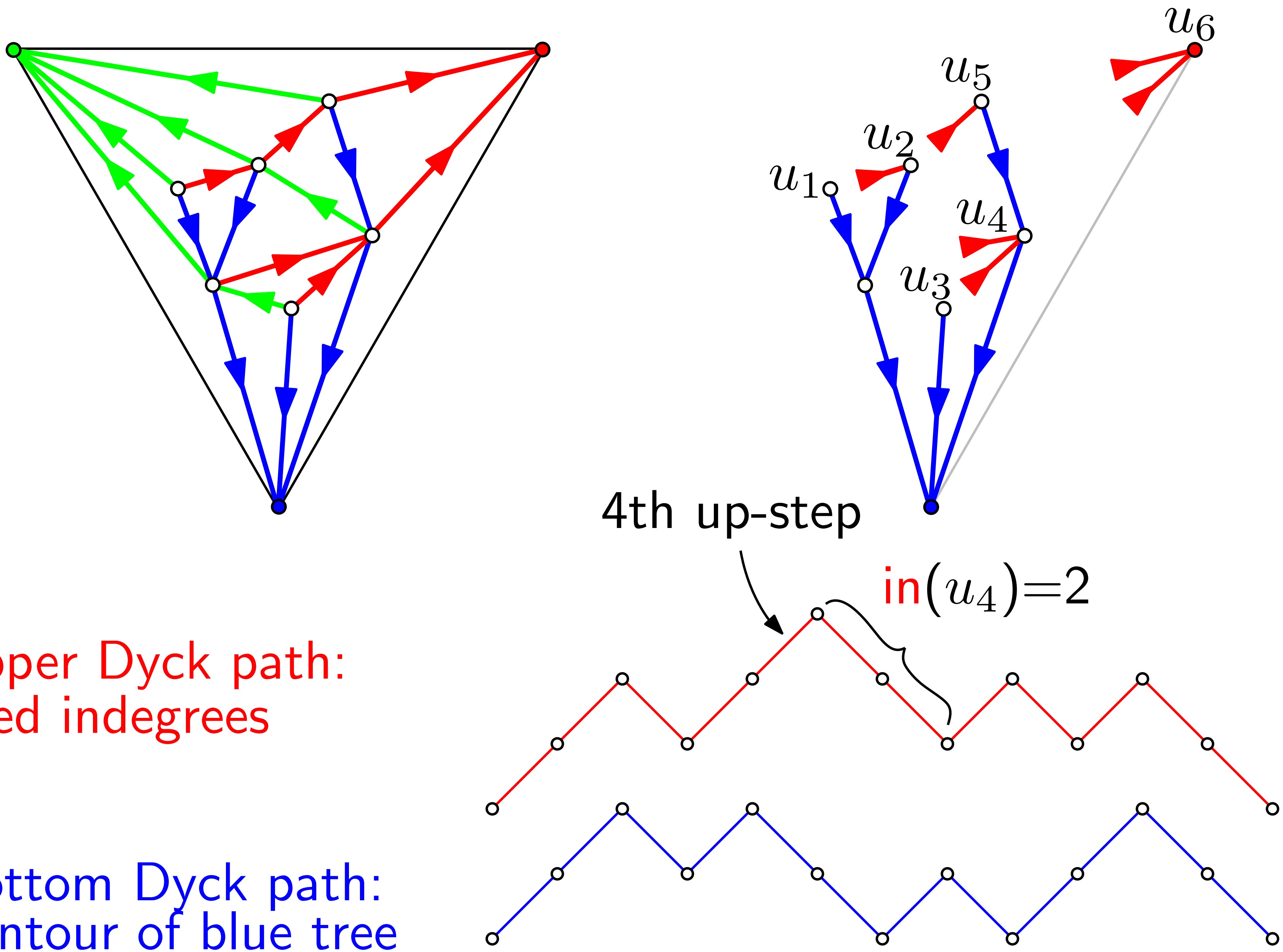
Bottom Dyck path:
contour of blue tree



Bijection for Schnyder woods [Bernardi, Bonichon'07]

Some information is redundant:

just need the blue tree and positions of the ingoing red edges

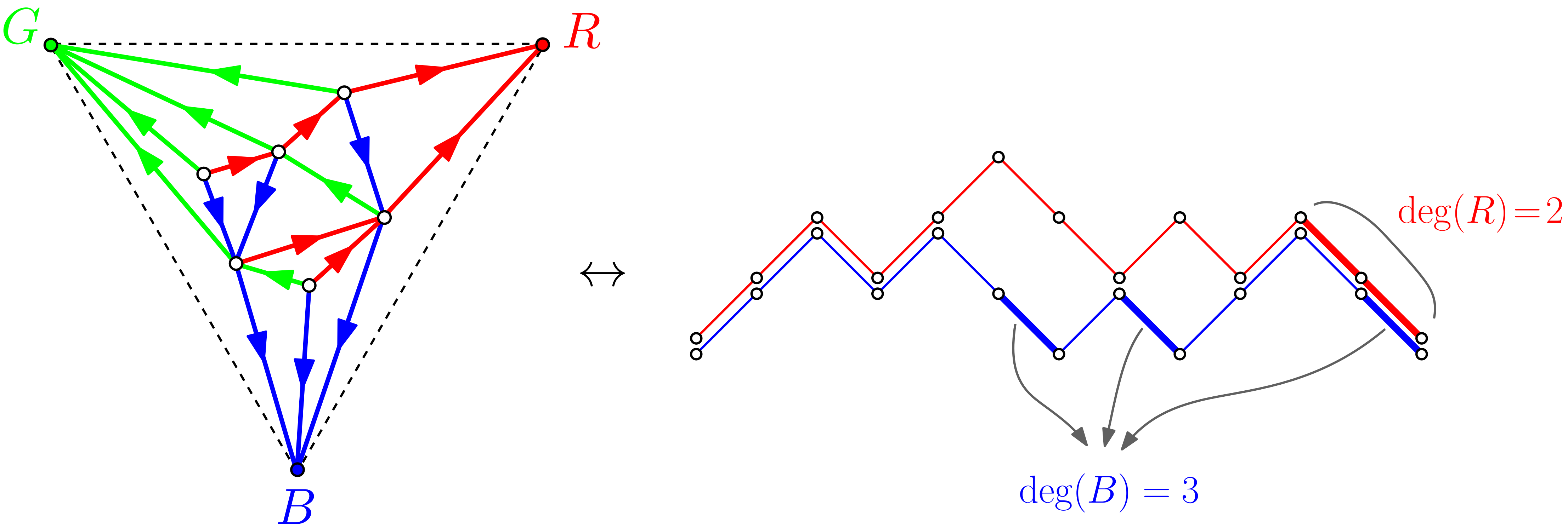


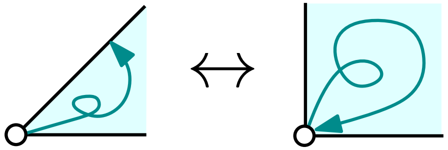
Upper Dyck path:
red indegrees

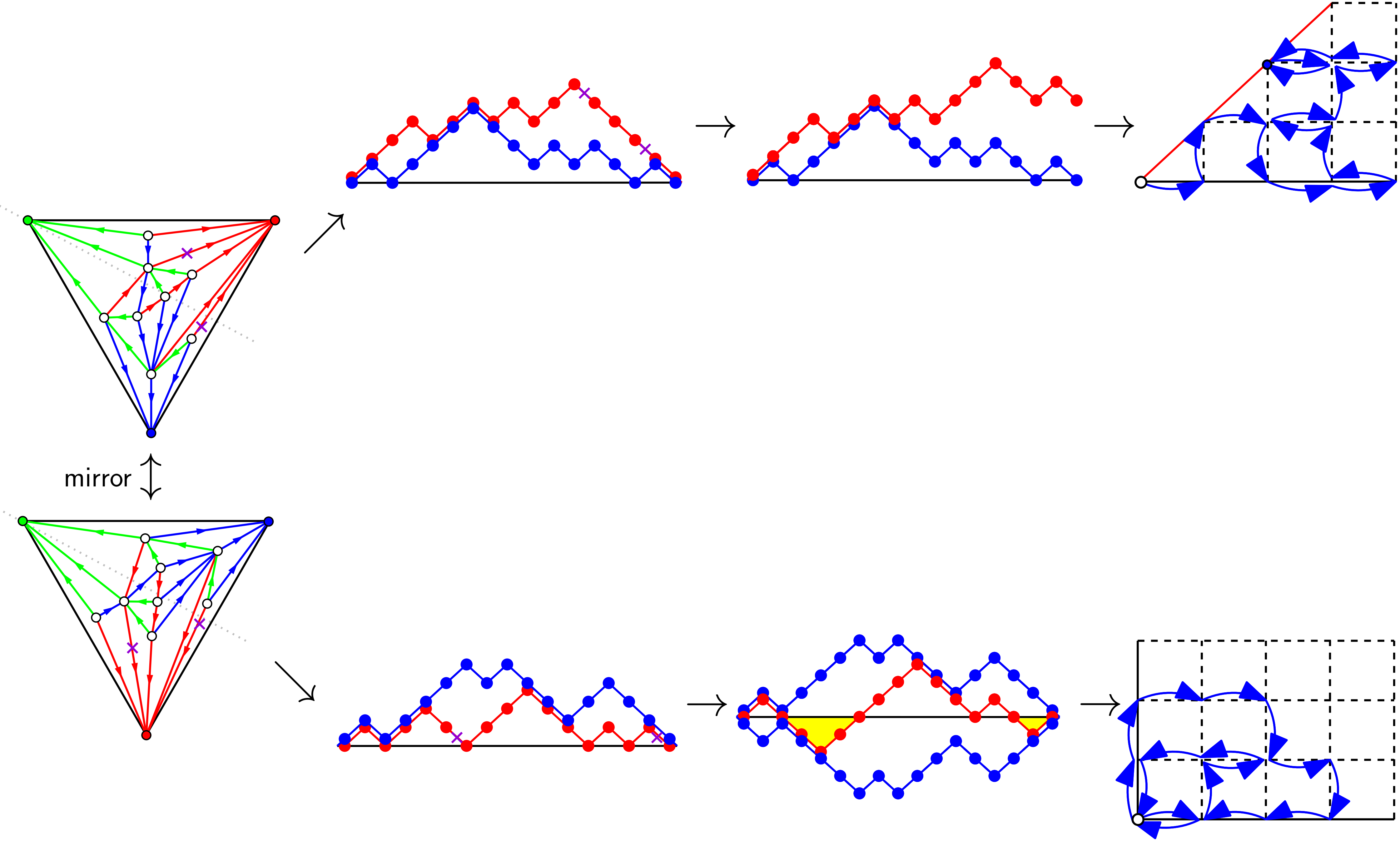
Bottom Dyck path:
contour of blue tree

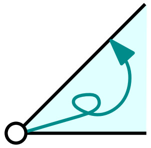
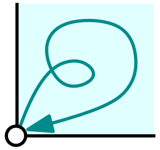
Bijection for Schnyder woods [Bernardi, Bonichon'07]

The mapping is a bijection from Schnyder woods with $n+3$ vertices to non-crossing pairs of Dyck paths of lengths $2n$

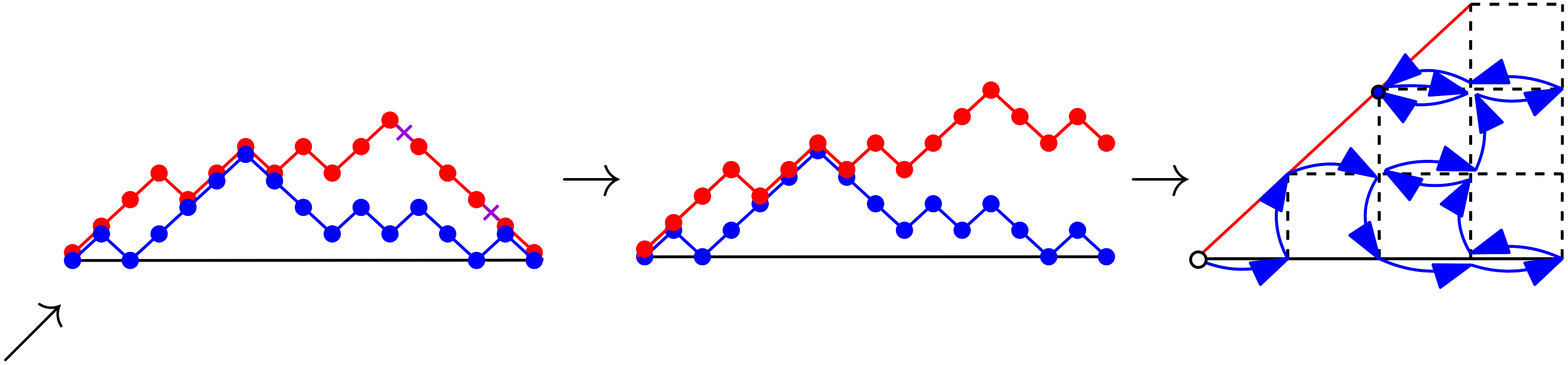


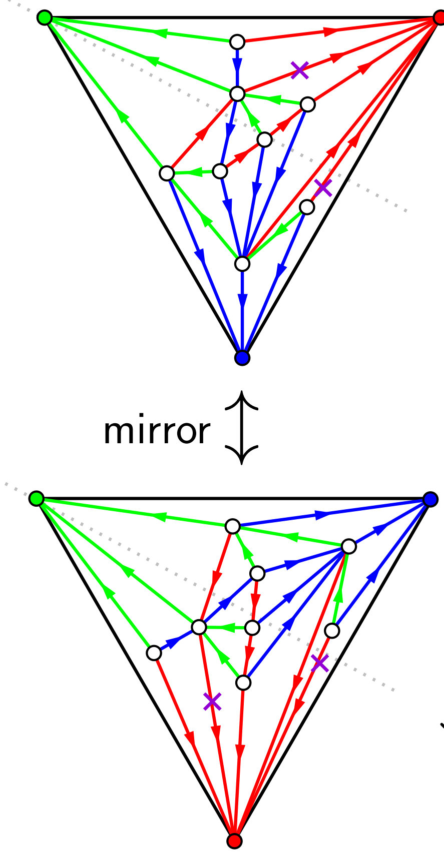
Proof of 

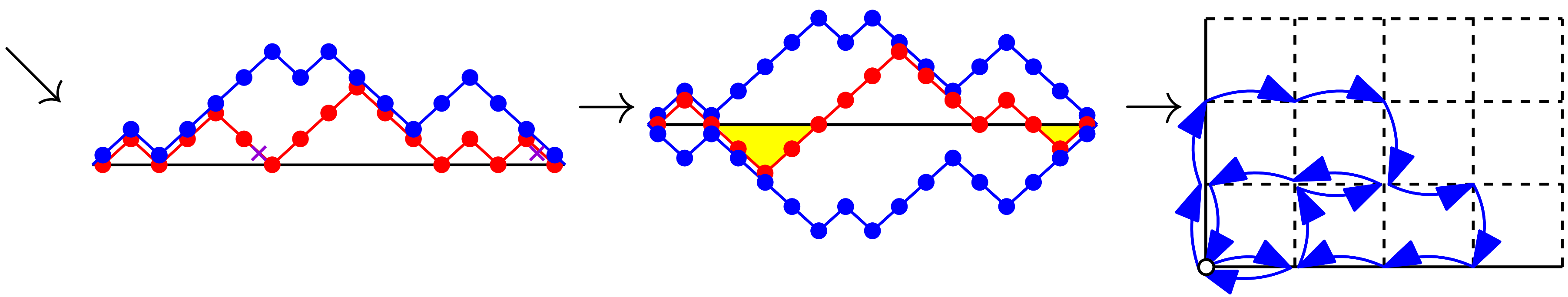


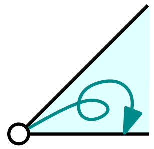
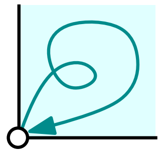
Proof of  ↔ 

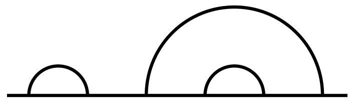
[Courtiel,F,Lepoutre,Mishna'18]





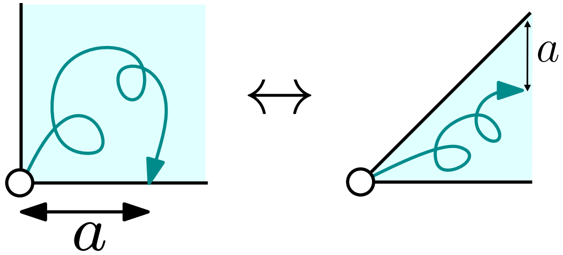


Rk:  ↔  also holds

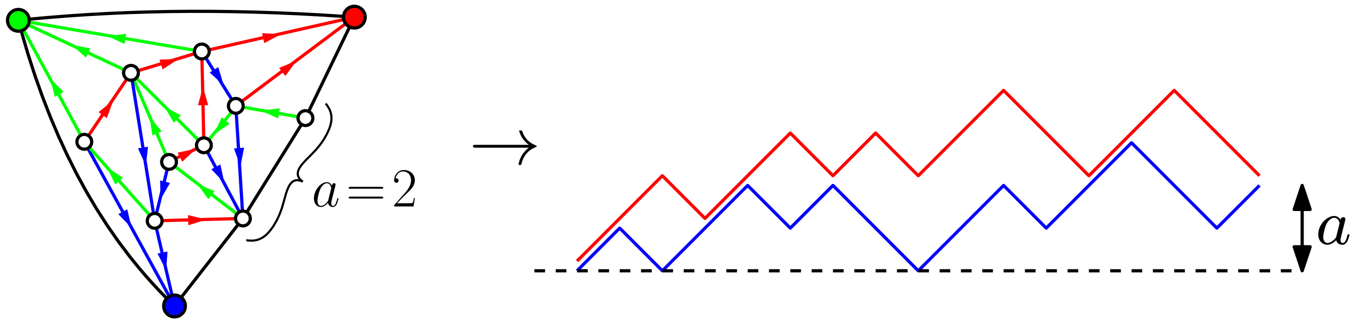
proof via arc diagrams 

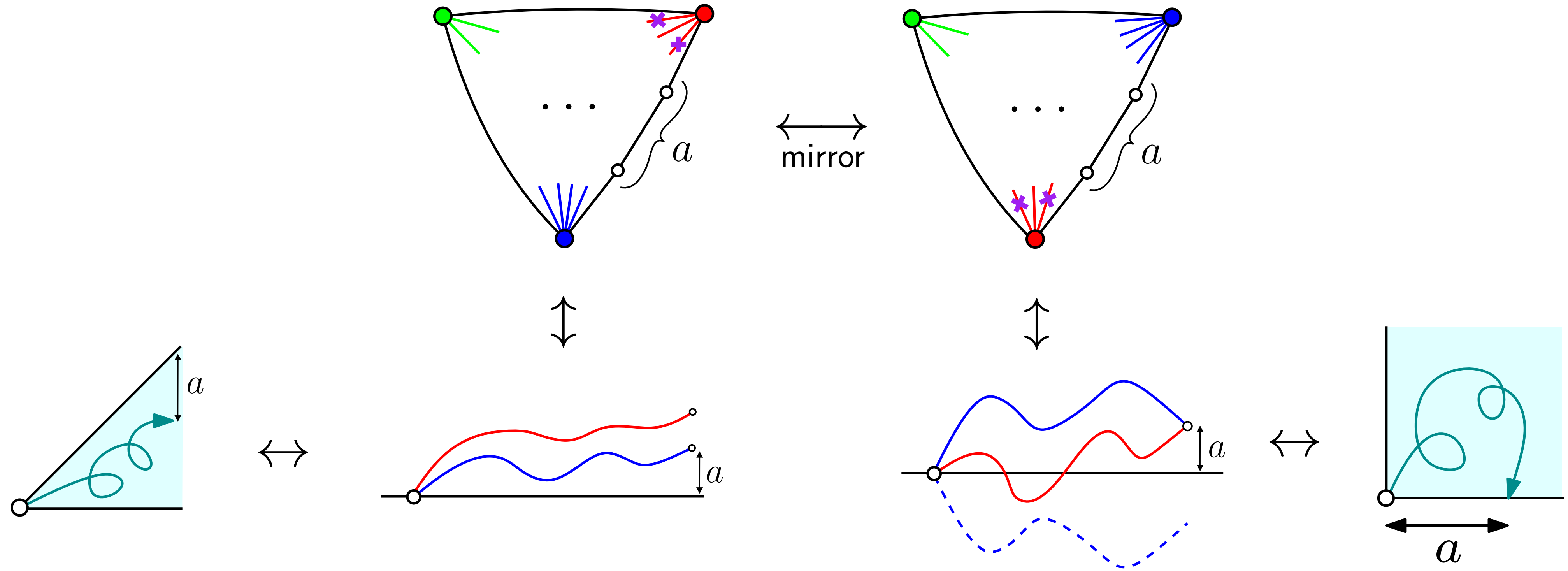
[Courtiel,F,Lepoutre,Mishna'18]

Extension to prove



Bijection extended to

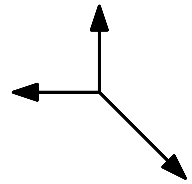


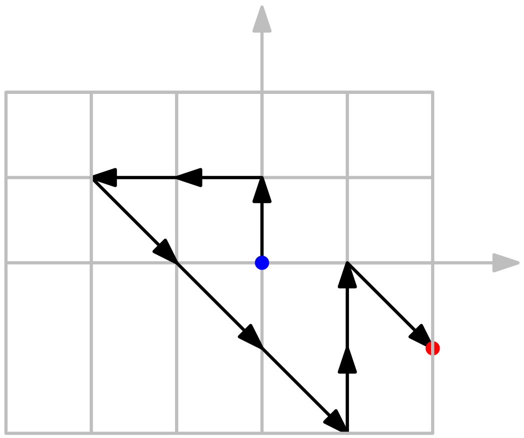


Tandem walks

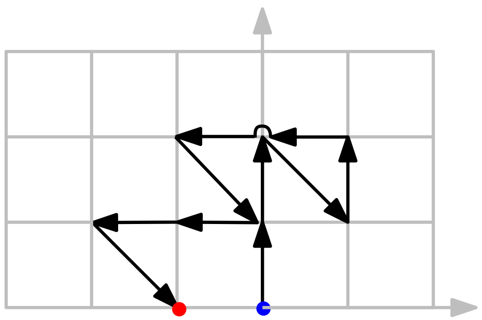
Tandem walks

A **tandem-walk** is a walk in $\mathbb{Z}^2$ with step-set $\{N, W, SE\}$

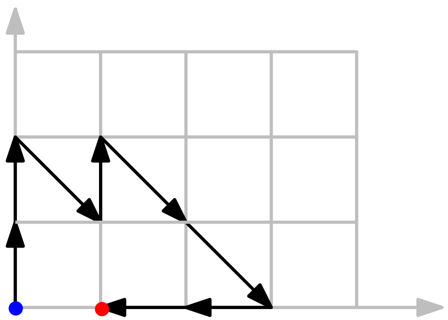




in the plane $\mathbb{Z}^2$

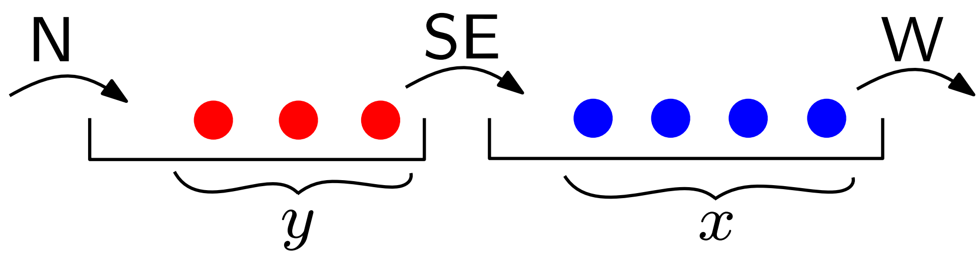


in the half-plane $\{y \geq 0\}$



in the quarter plane $\mathbb{N}^2$

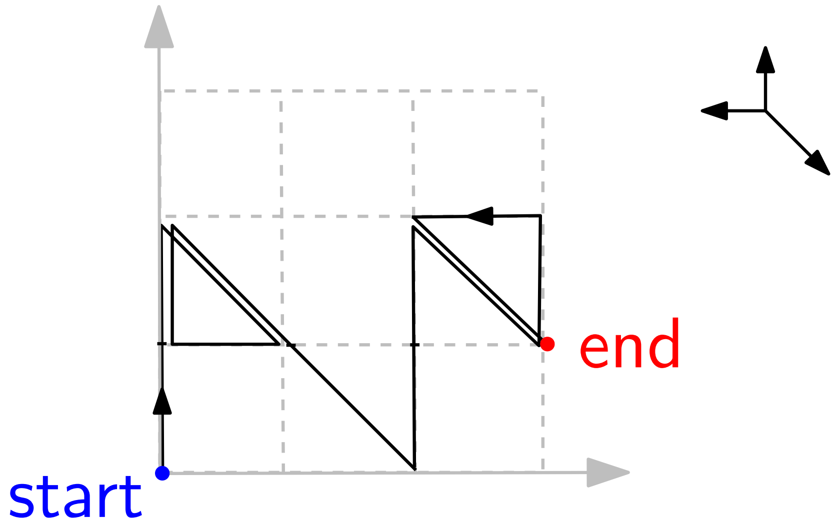
cf 2 queues in series



Duality relation for tandem walks

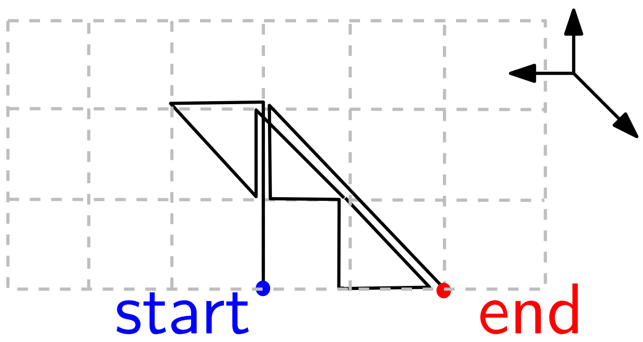
There is a bijection between:

- tandem walks of length n
  staying in the quarter plane $\mathbb{N}^2$

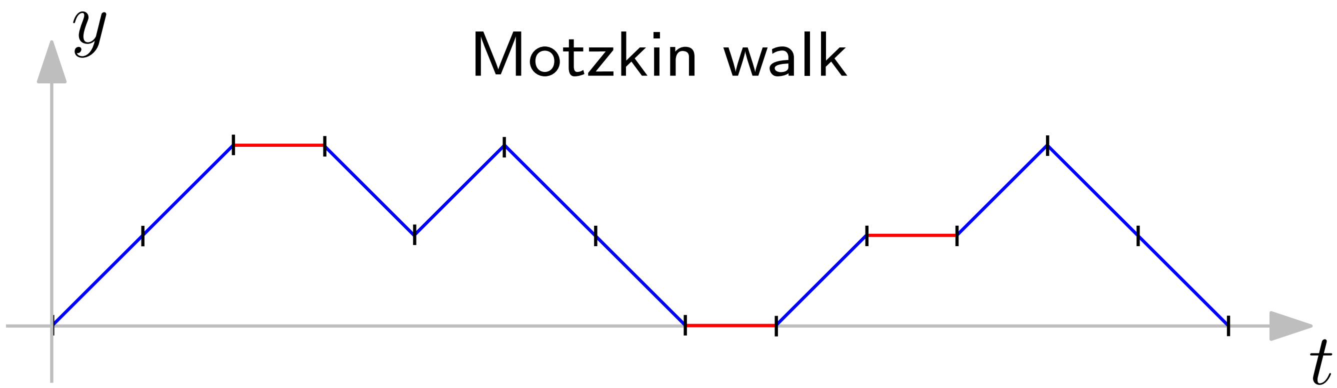


$\Updownarrow$

- tandem walks of length n
  staying in the half-plane $\{y \geq 0\}$
  and ending at $y = 0$



$\Leftrightarrow$



Rk: The bijection **preserves the number of SE steps**

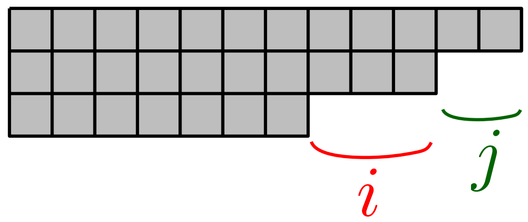
Link to Young tableaux of height ≤ 3

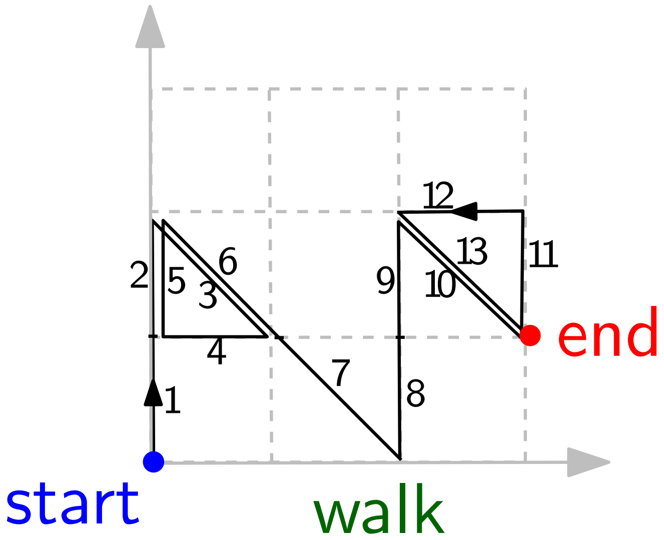
- There is a bijection between:

tandem walks of length n **staying in the quadrant** $\mathbb{N}^2$, ending at (i, j)

$$\Updownarrow$$

Young tableaux of size n and height ≤ 3, of shape





$\longleftrightarrow$

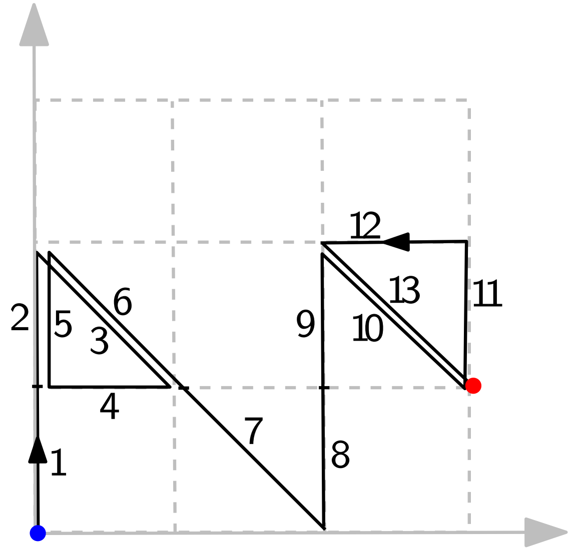
N	1	2	5	8	9	11
SE	3	6	7	10	13	
W	4	12				

tableau

(after s steps, current $y = \#N - \#SE$, current $x = \#SE - \#W$)

Bijection with Motzkin walks

[Gouyou-Beauchamps'89]



tandem walk in $\mathbb{N}^2$

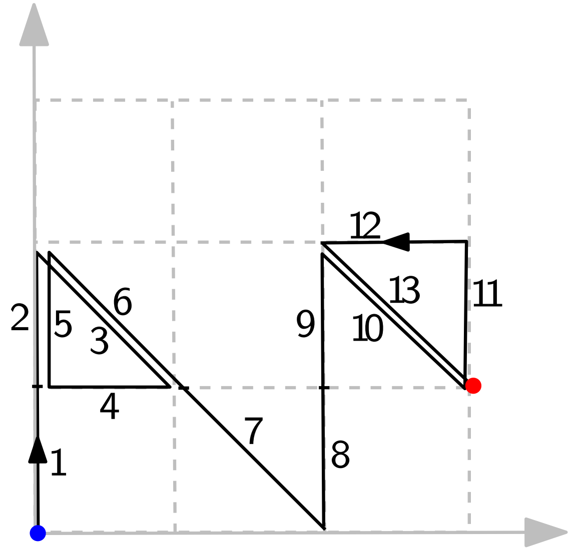
$\longleftrightarrow$

1	2	5	8	9	11
3	6	7	10	13	
4	12				

Young tableau of height ≤ 3

Bijection with Motzkin walks

[Gouyou-Beauchamps'89]

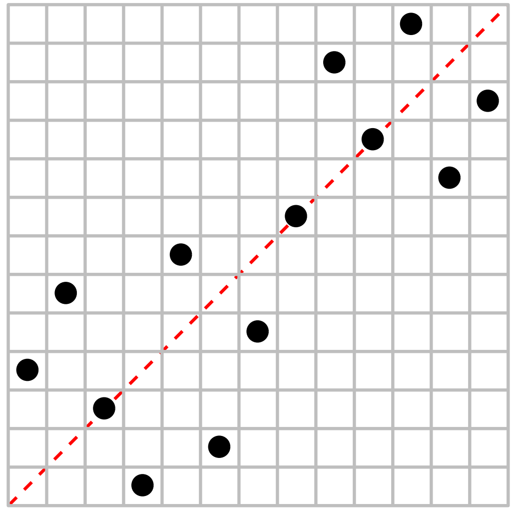


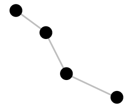
tandem walk in $\mathbb{N}^2$

1	2	5	8	9	11
3	6	7	10	13	
4	12				

Young tableau of height ≤ 3

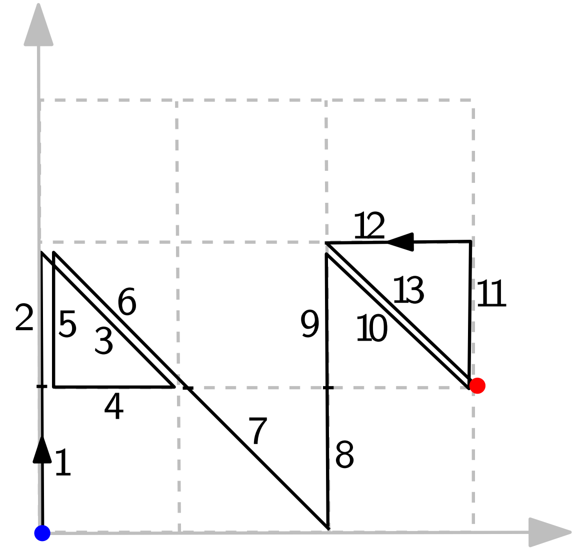
Robinson Schensted



involution with no 

Bijection with Motzkin walks

[Gouyou-Beauchamps'89]



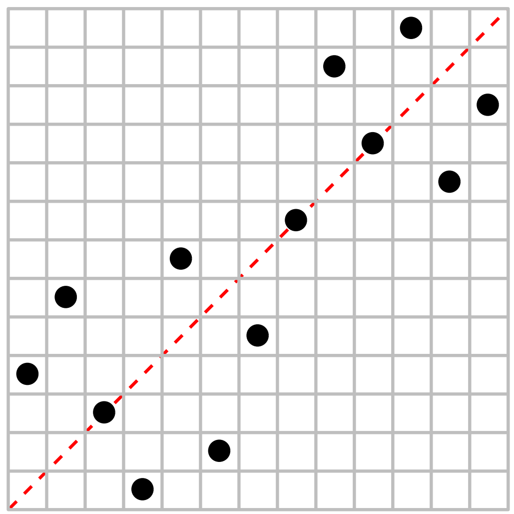
tandem walk in $\mathbb{N}^2$

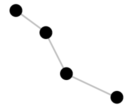
↔

1	2	5	8	9	11
3	6	7	10	13	
4	12				

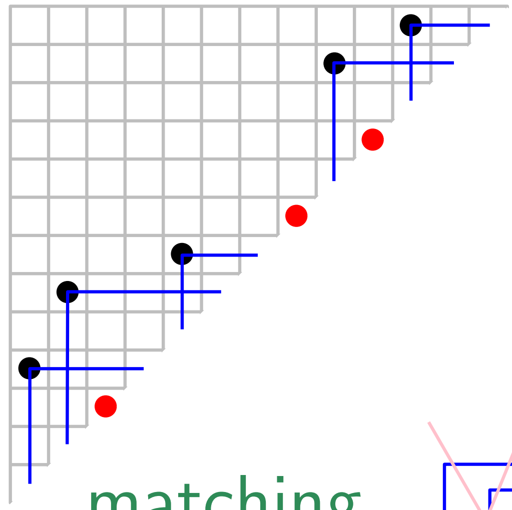
Young tableau
of height ≤ 3

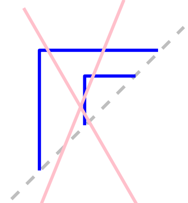
↔ Robinson Schensted



involution
with no 

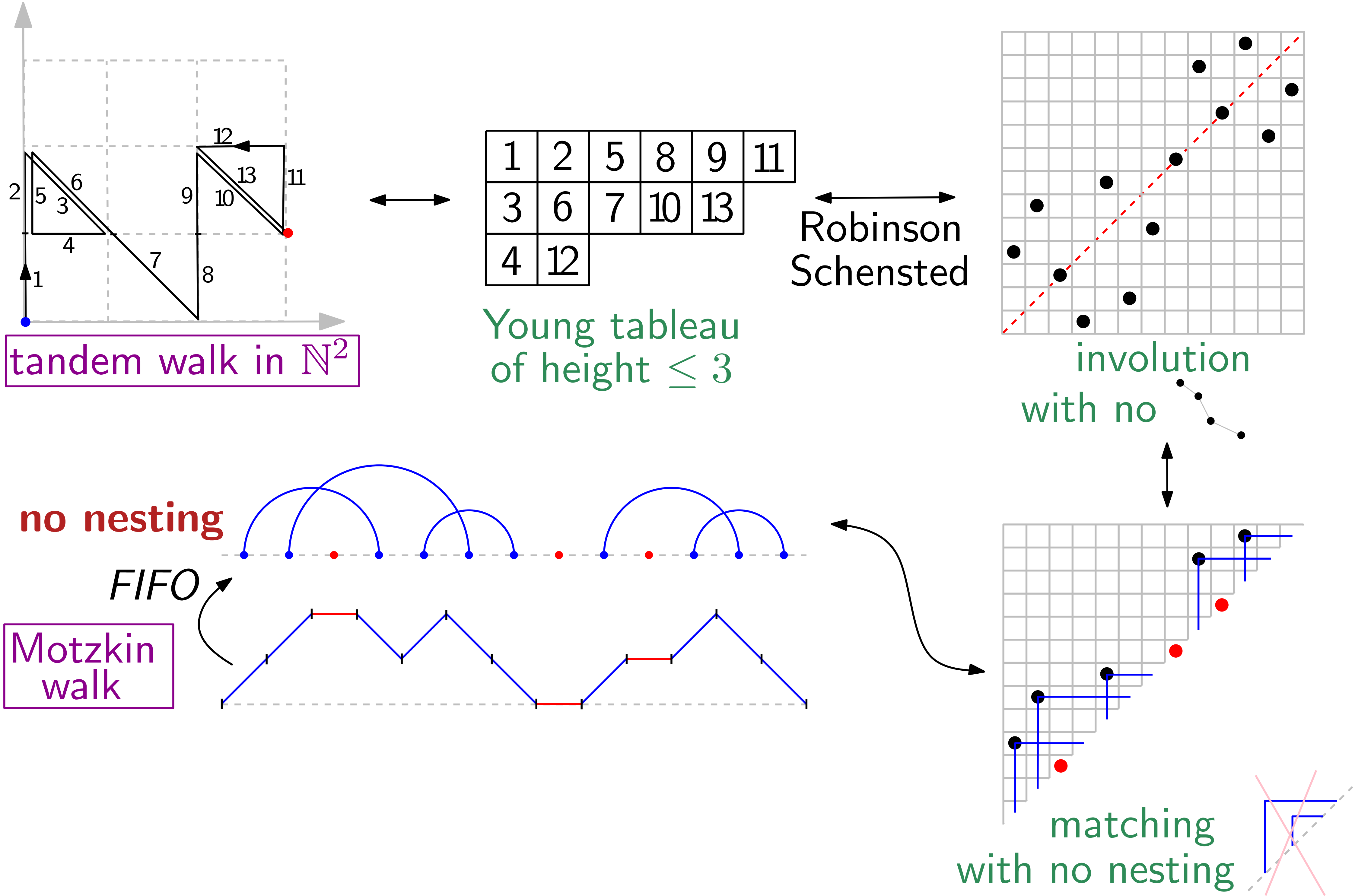
↕



matching
with no nesting 

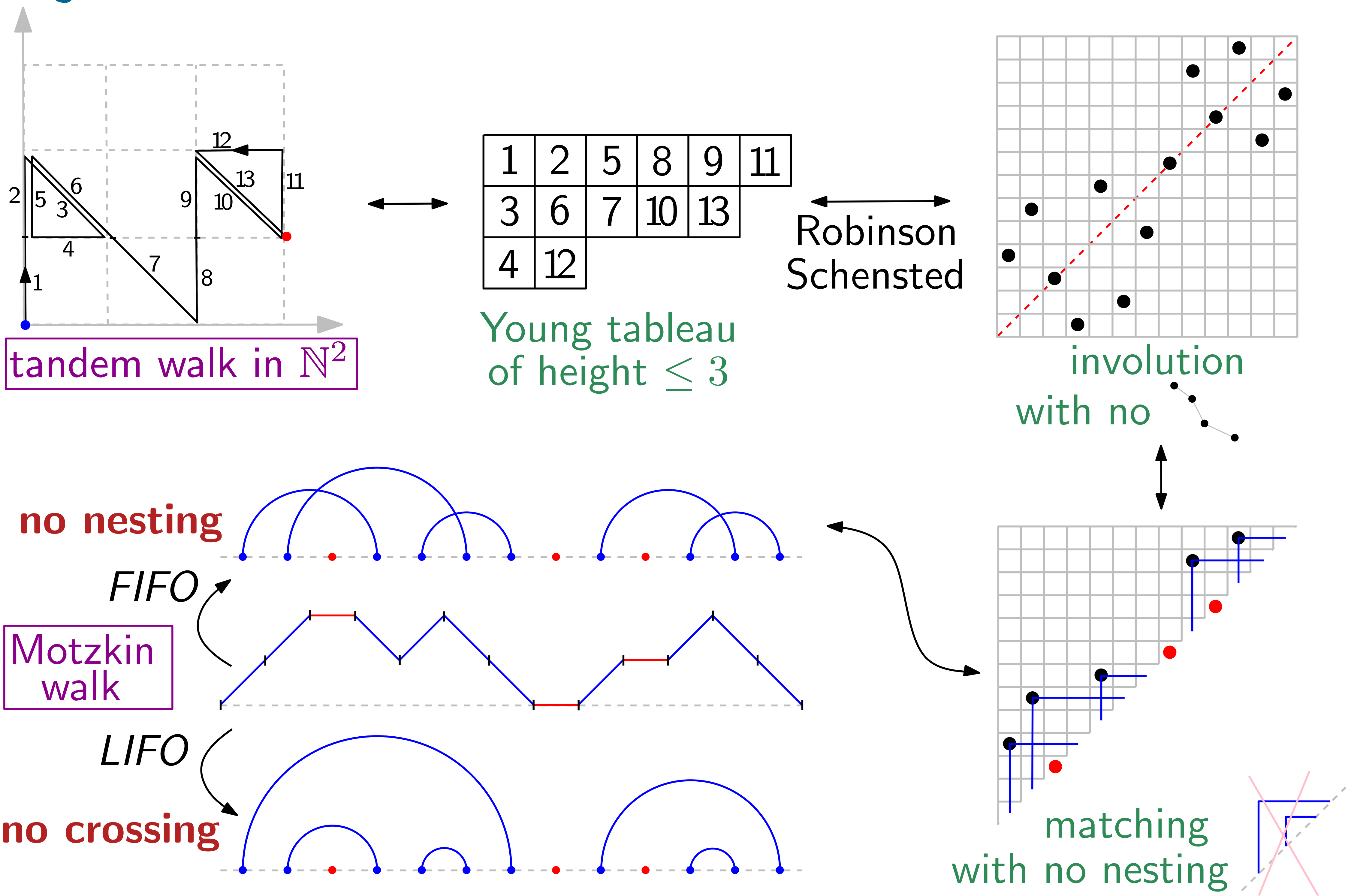
Bijection with Motzkin walks

[Gouyou-Beauchamps'89]



Bijection with Motzkin walks

[Gouyou-Beauchamps'89]

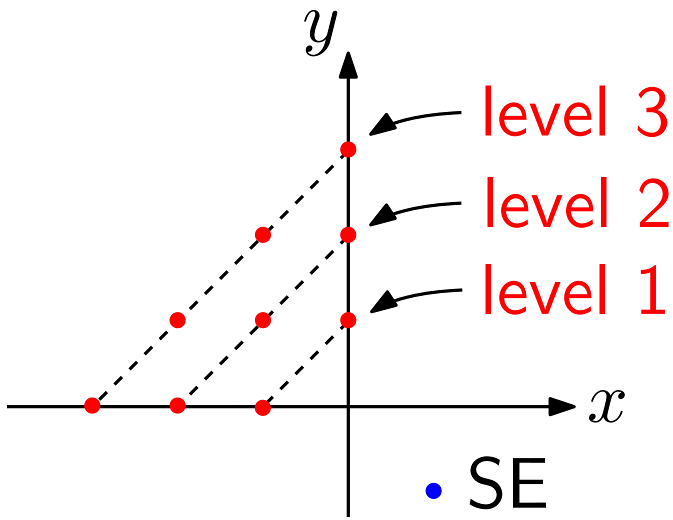


An extension of the walk model

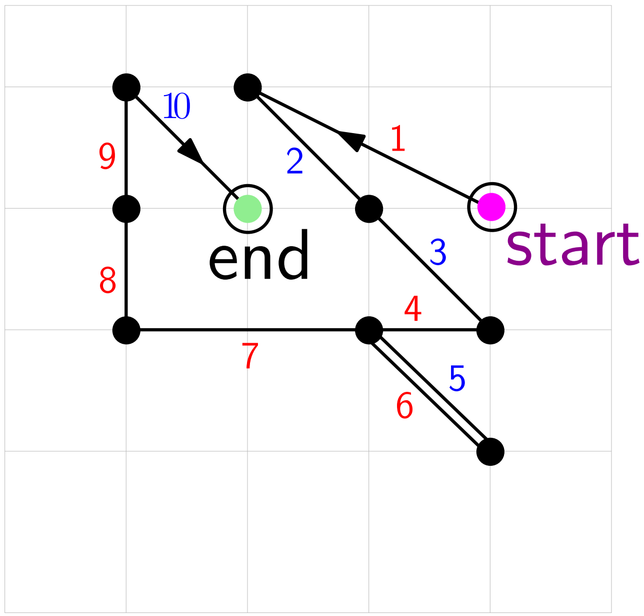
General model:

step-set: • the SE step
• every step $(-i, j)$ (with $i, j \geq 0$)

level:$= i + j$



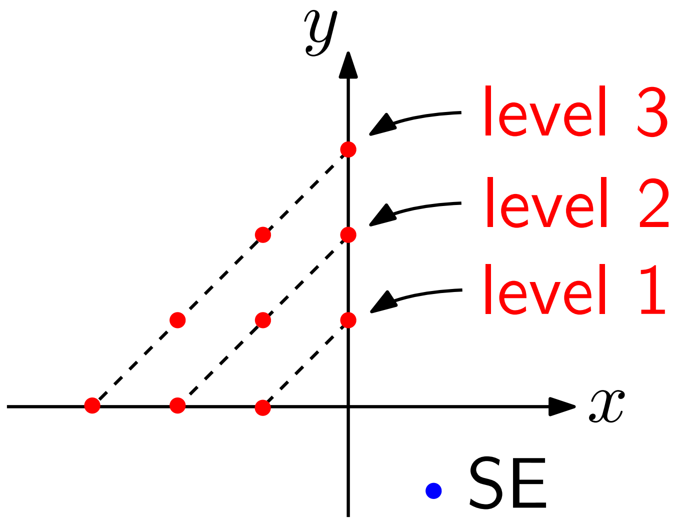
Example:



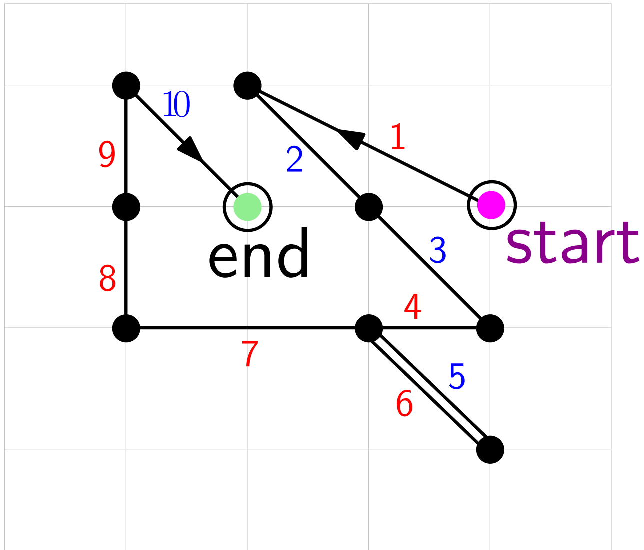
An extension of the walk model

General model:

step-set: • the SE step
• every step $(-i, j)$ (with $i, j \geq 0$)
level:$= i + j$

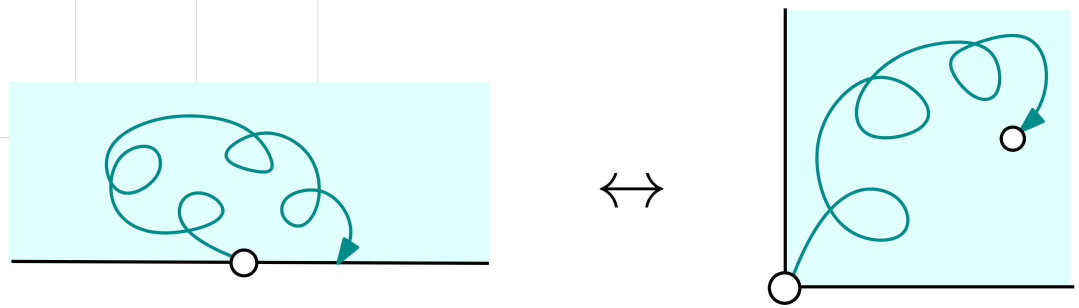


Example:



We still have

[Bousquet-Mélou,F,Raschel'19]



The bijection (using bipolar orientations) **preserves** the number of SE-steps and the number of steps in each level $p \geq 1$

different bijection using automata rules **[Chyzak-Yeats'18]**

Bipolar and marked bipolar orientations

bipolar orientation:

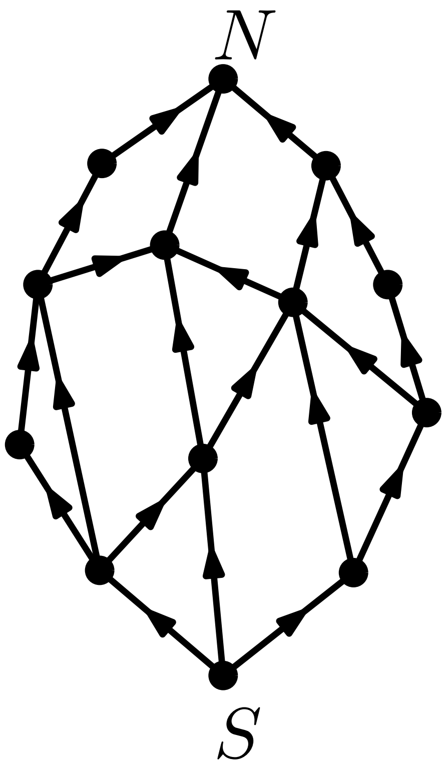
(on planar maps)

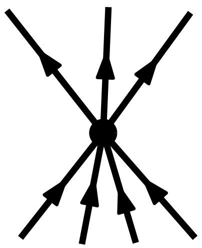
= acyclic orientation

with a unique source S

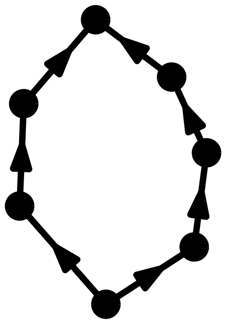
and a unique sink N

with S, N incident to the outer face





inner vertex



inner face

Bipolar and marked bipolar orientations

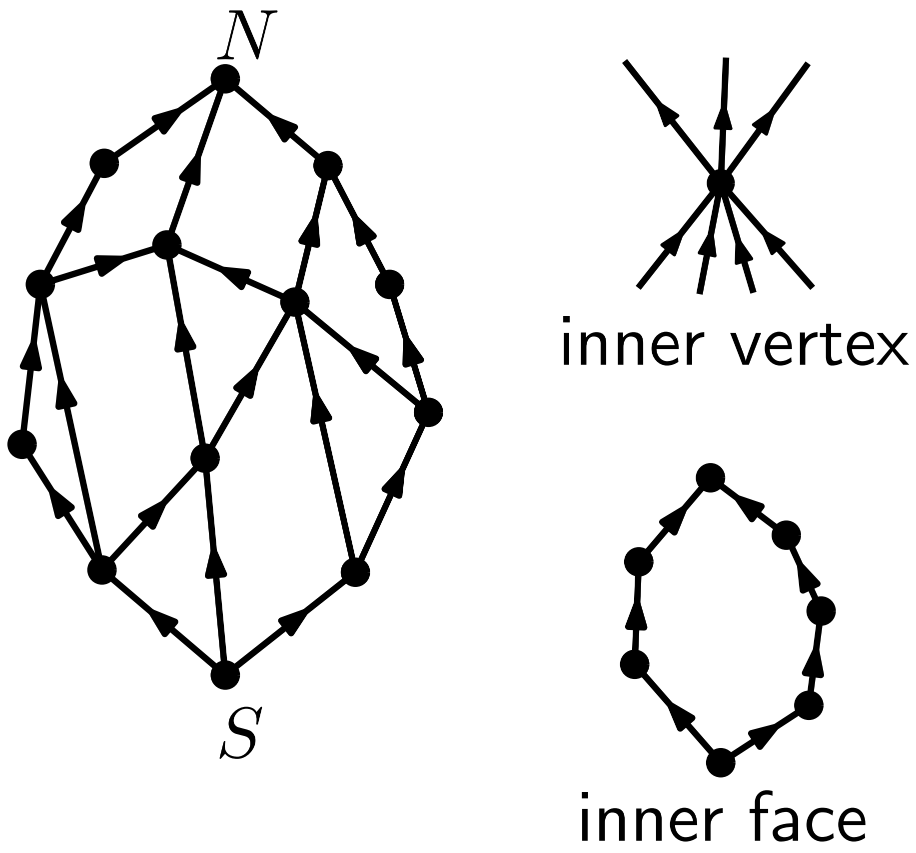
bipolar orientation:

(on planar maps)

$=$ acyclic orientation

with a unique source S
and a unique sink N

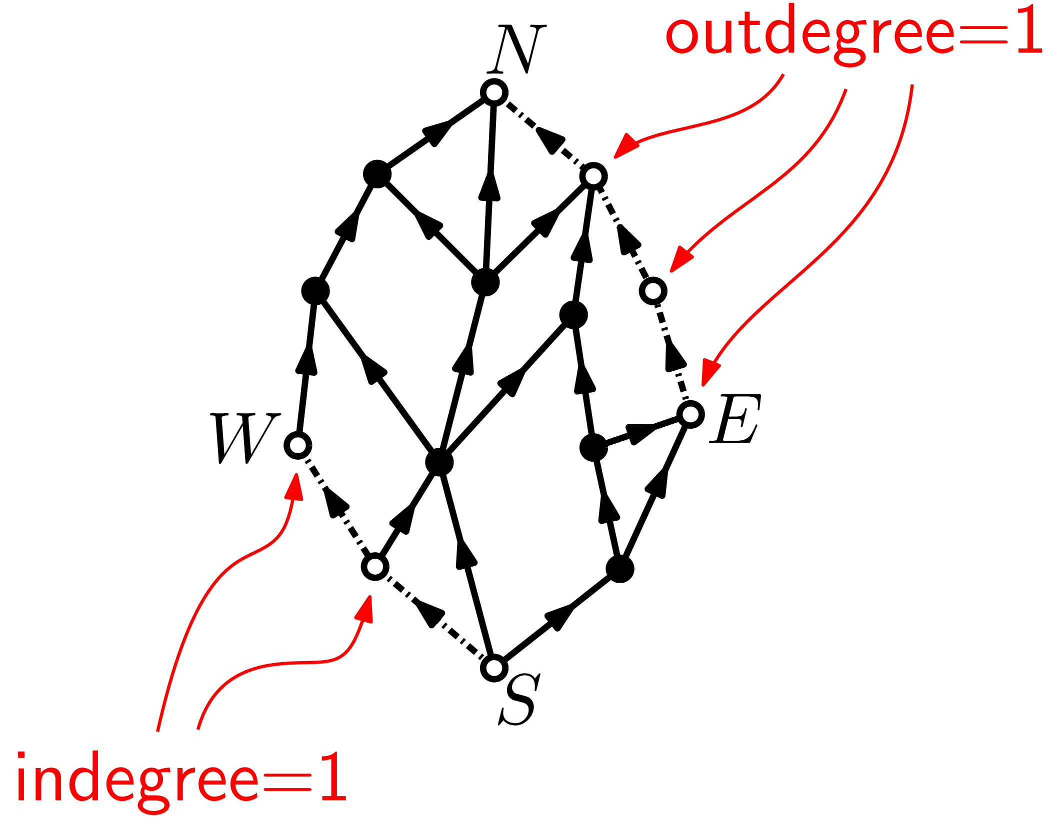
with S, N incident to the outer face



marked bipolar orientation:

a marked vertex $W \neq N$ on left boundary

a marked vertex $E \neq S$ on right boundary



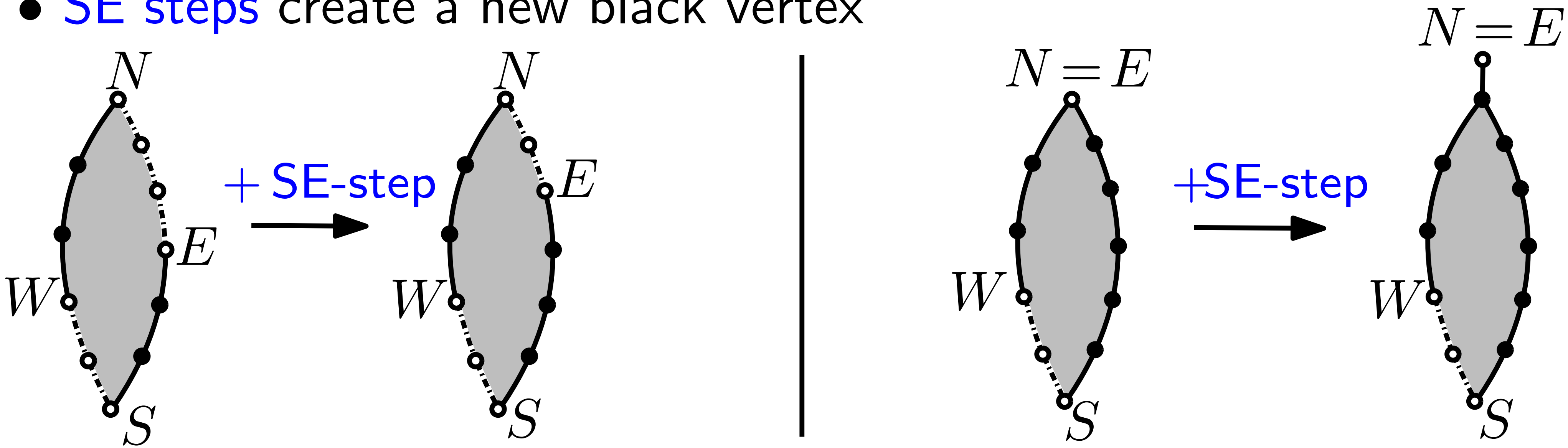
The Kenyon et al. bijection

[Kenyon, Miller, Sheffield, Wilson'16]

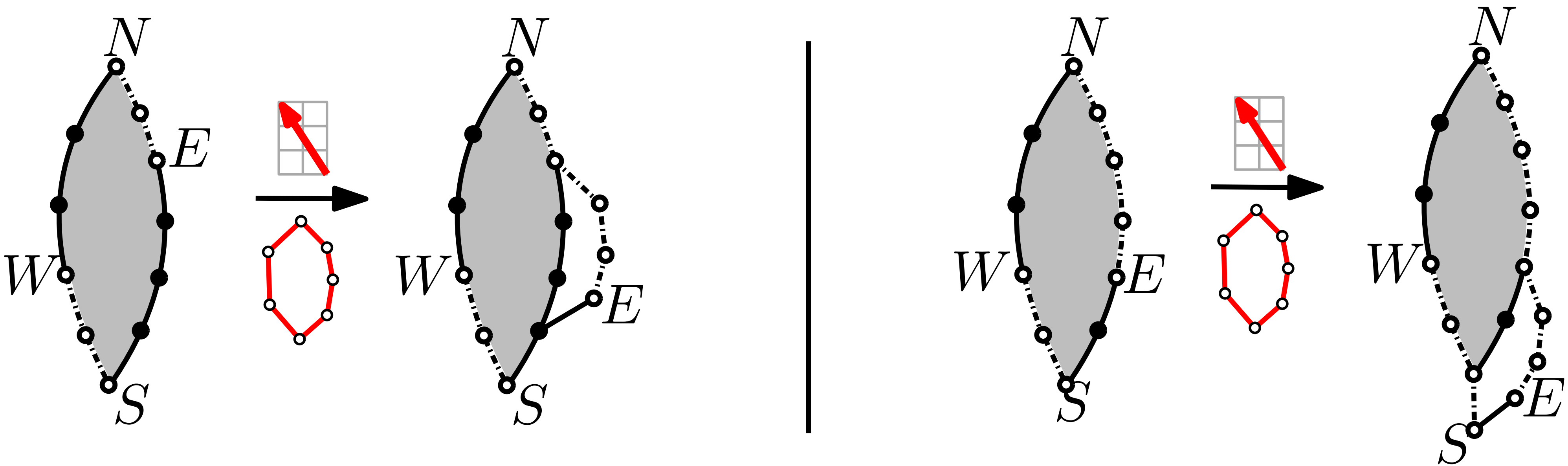
start with 

and read the walk step by step

- SE steps create a new black vertex



- steps $(-i, j)$ create a new inner face (of degree $i + j + 2$)



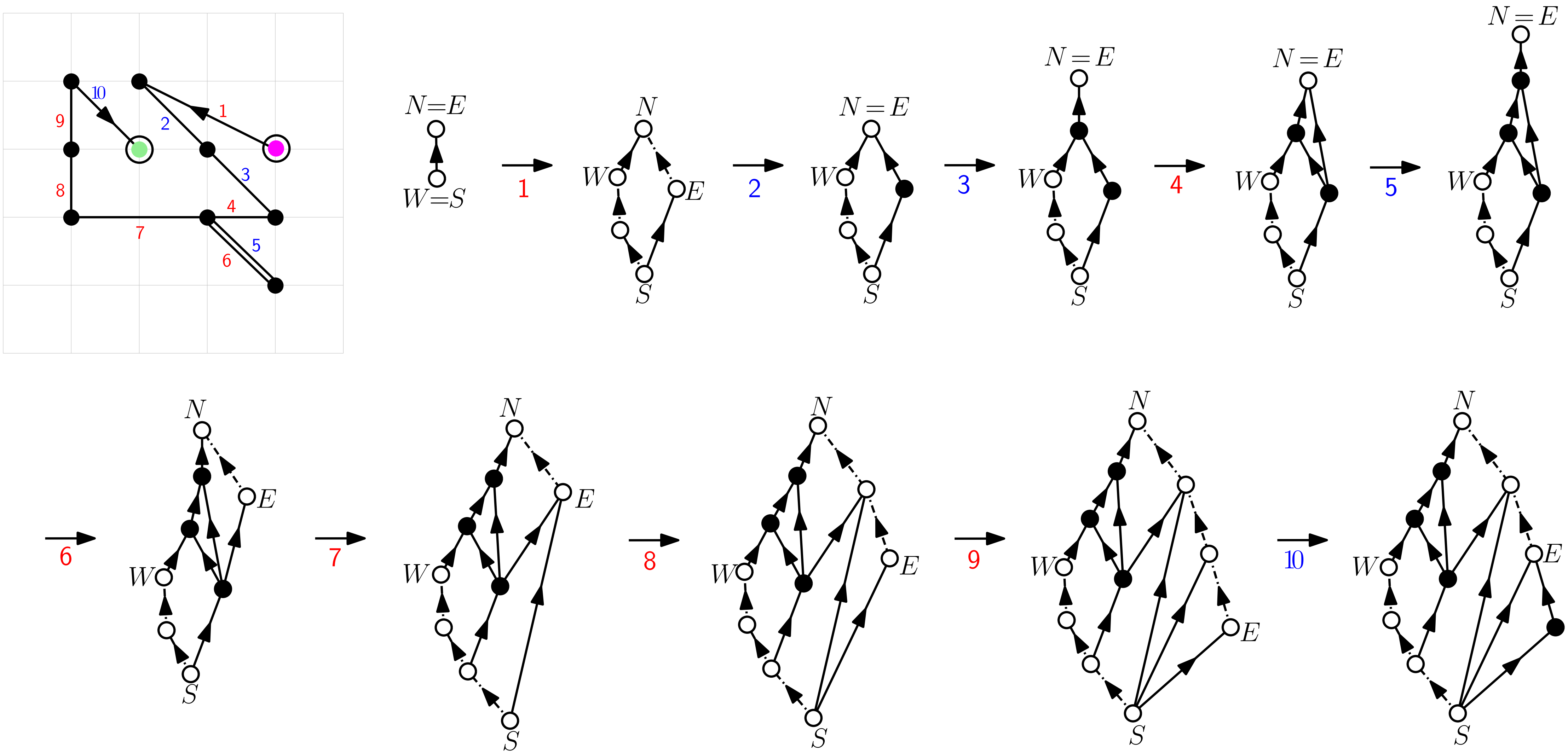
The Kenyon et al. bijection

[Kenyon, Miller, Sheffield, Wilson'16]

general tandem-walk (in $\mathbb{Z}^2$) $\xleftrightarrow{\textit{bijection}}$ marked bipolar orientation

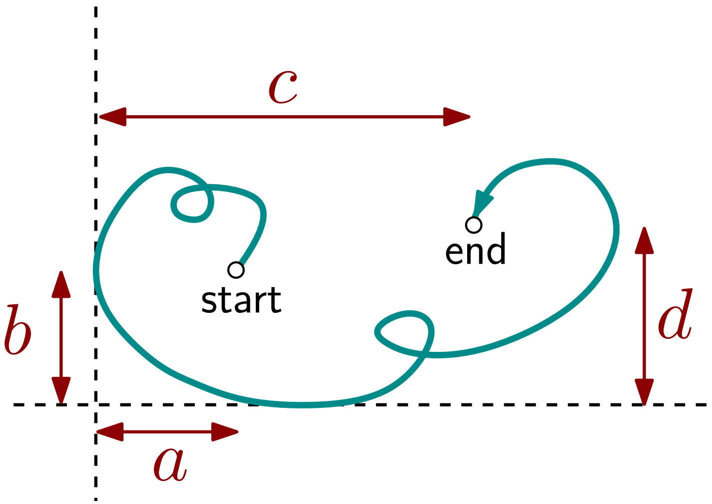
SE step $\longleftrightarrow$ black vertex

step $(-i, j)$ $\longleftrightarrow$ inner face of degree $i+j+2$

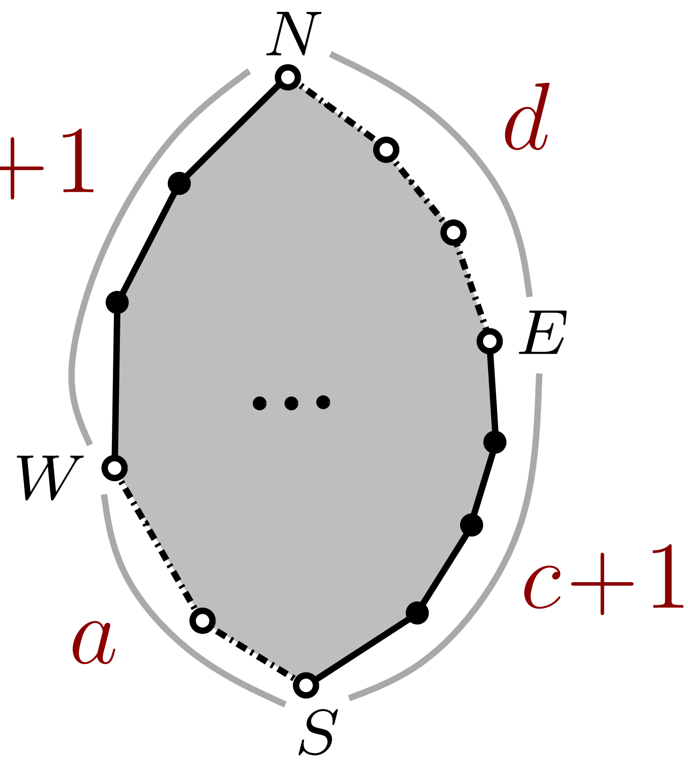


Parameter-correspondence in the bijection

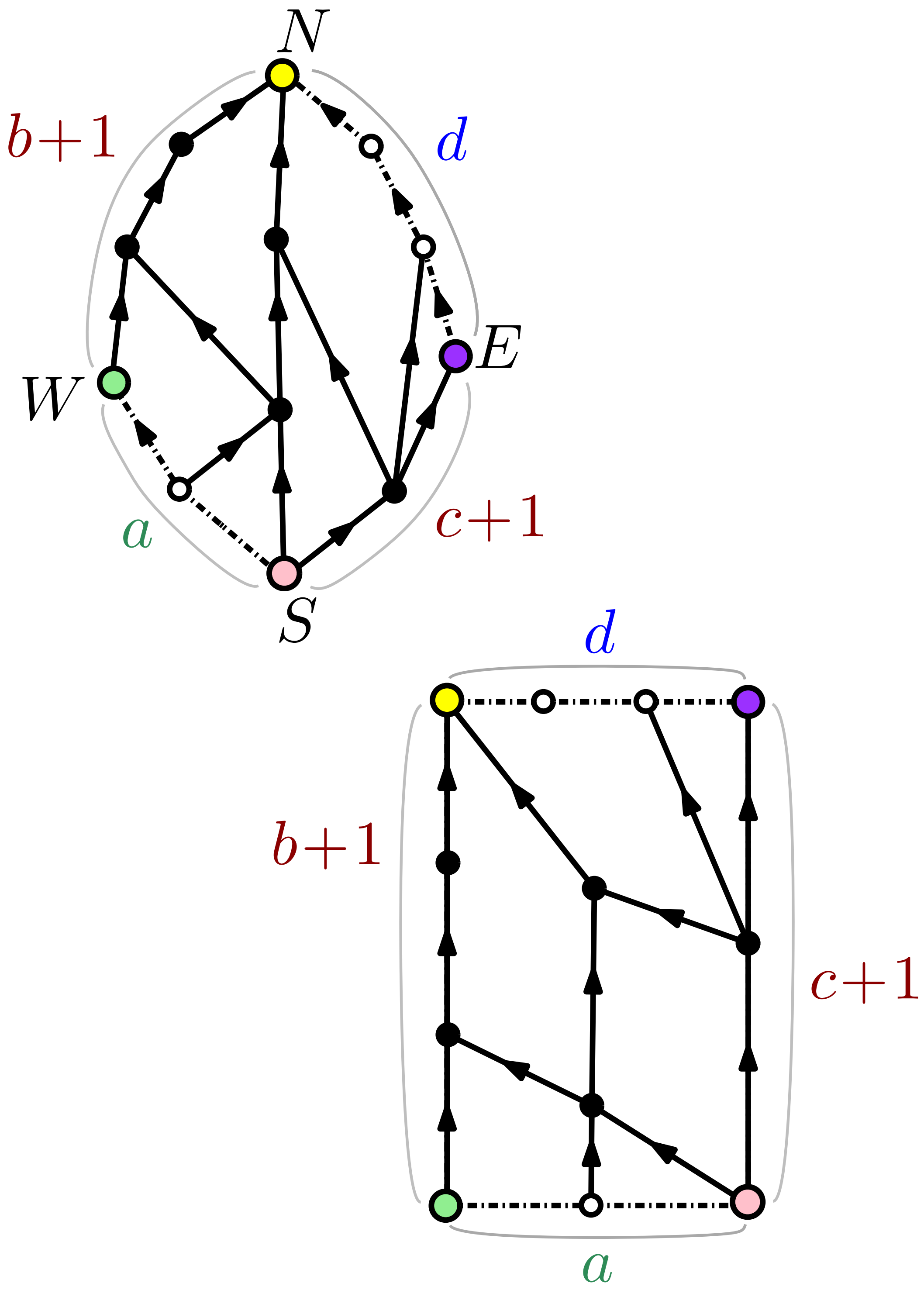
# "face-steps" of level p	$\longleftrightarrow$	# inner faces of degree $p+2$
# SE-steps	$\longleftrightarrow$	# black vertices
$1 + \#$ steps	$\longleftrightarrow$	# plain edges (not dashed)



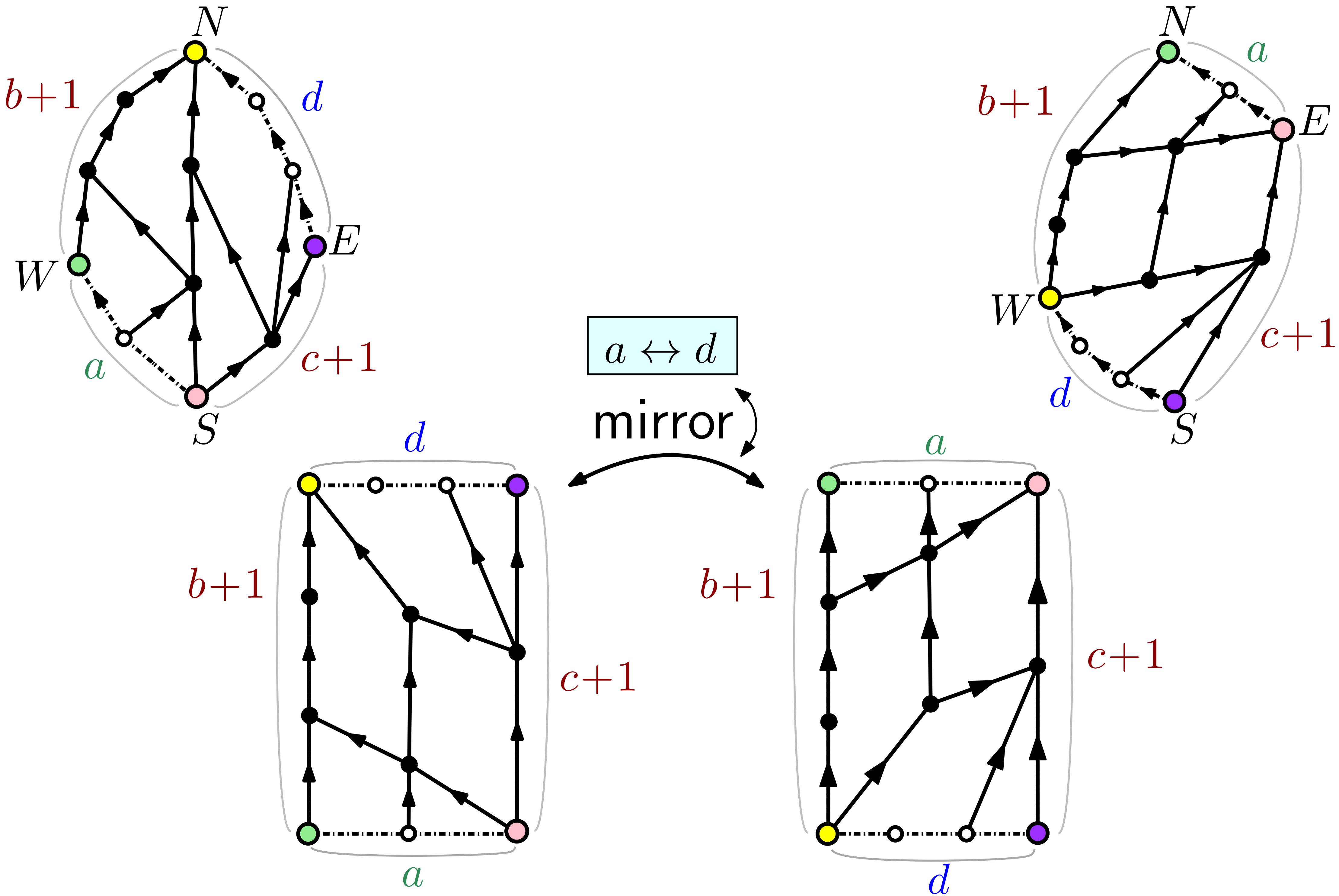
$\longleftrightarrow$



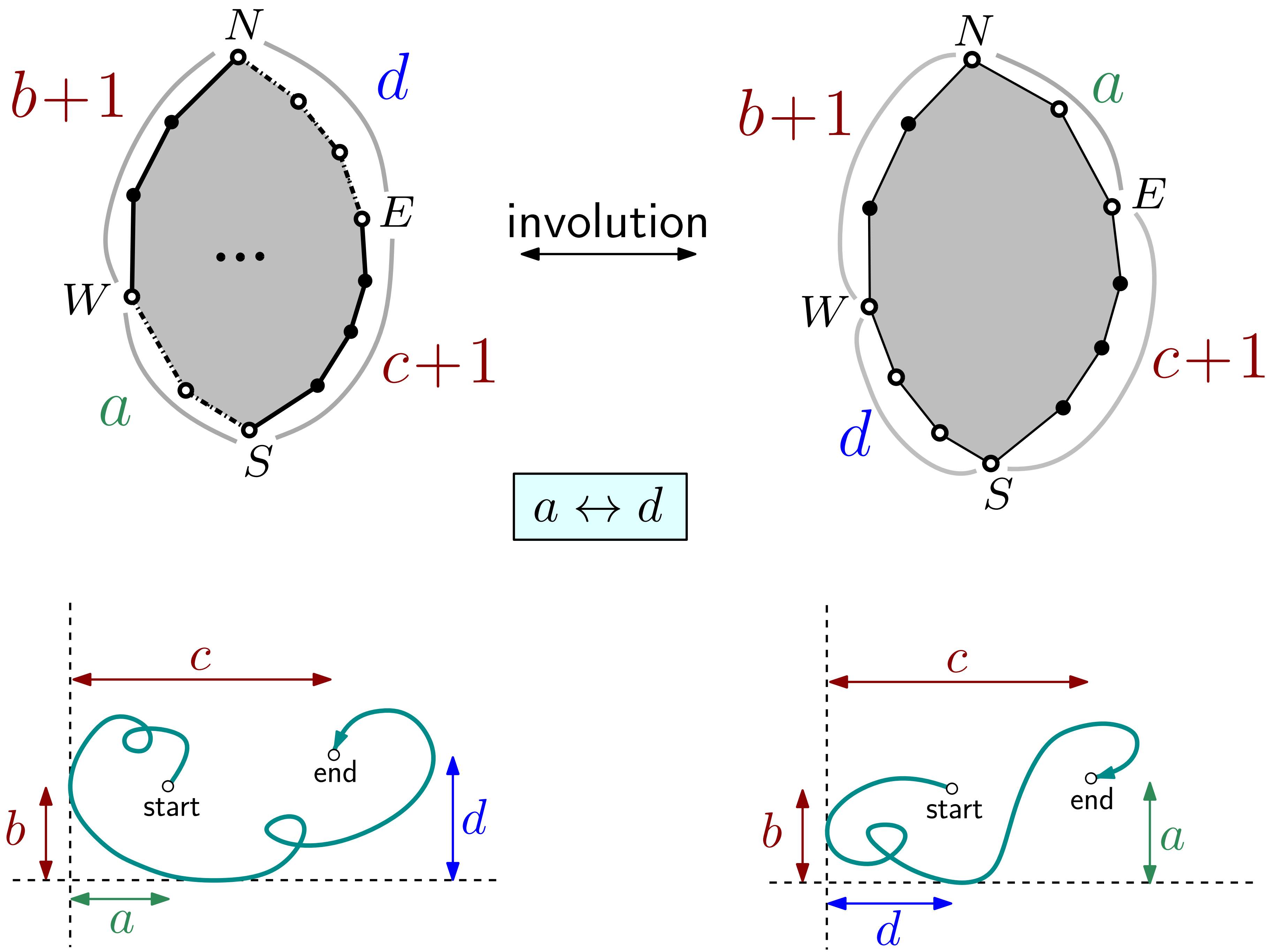
An involution on marked bipolar orientations

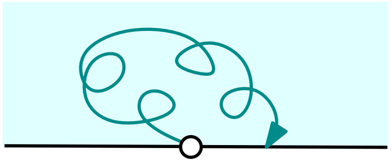
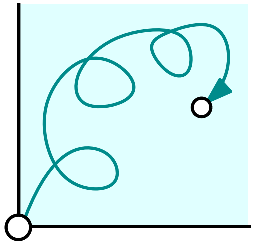


An involution on marked bipolar orientations

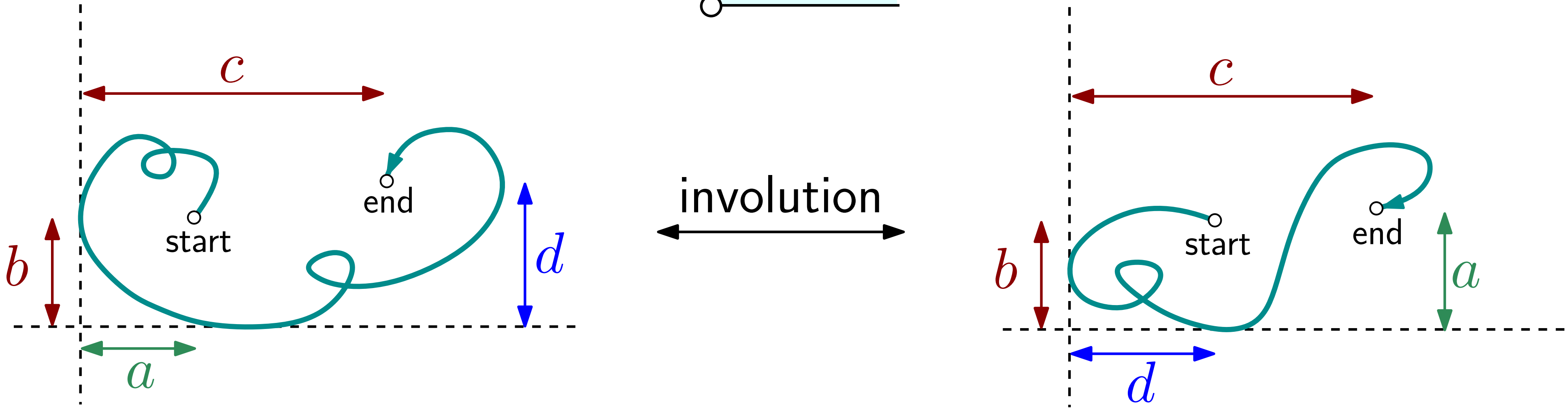


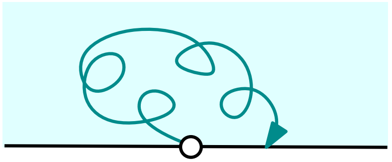
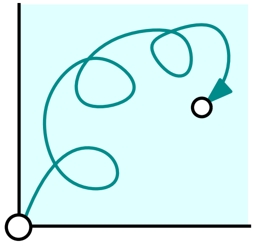
Effect of the involution on walks



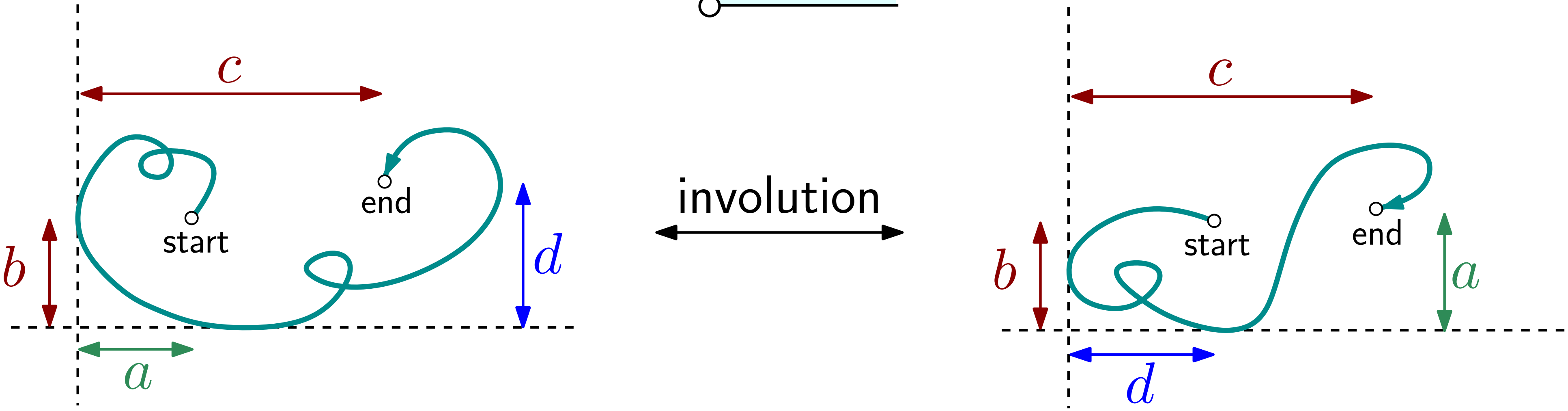
Proof of  ↔ 

[Bousquet-Mélou,F,Raschel'19]

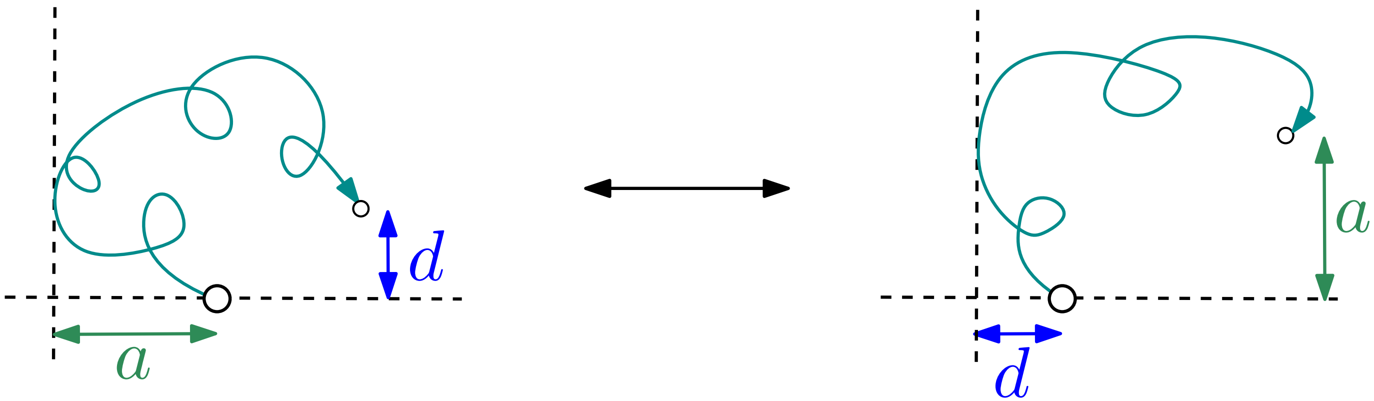


Proof of  ↔ 

[Bousquet-Mélou,F,Raschel'19]



- Specialize the involution at $b = 0$



& specialize further at $d = 0$

General situation in duality bijections

Two families $\mathcal{A}, \mathcal{B}$ of walks $\quad A(t) = \sum_n a_n t^n \quad B(t) = \sum_n b_n t^n$

want to prove bijectively that $A(t) = B(t)$

General situation in duality bijections

Two families $\mathcal{A}, \mathcal{B}$ of walks $\quad A(t) = \sum_n a_n t^n \quad B(t) = \sum_n b_n t^n$

want to prove bijectively that $A(t) = B(t)$

There is a superfamily $\mathcal{C} \supset \mathcal{A}, \mathcal{B}$ and an involution on $\mathcal{C}$ exchanging two parameters i, j such that, with $C(t; u, v) = \sum c_{n,i,j} t^n u^i v^j$, we have

$$A(t) = C(t; 1, 0) \qquad B(t) = C(t; 0, 1)$$

General situation in duality bijections

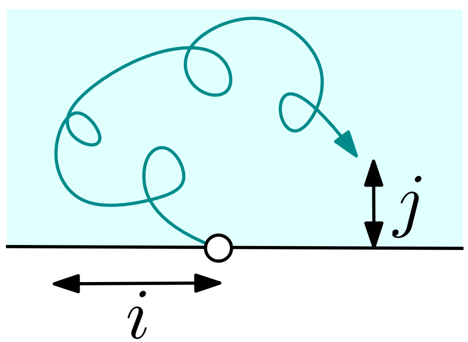
Two families $\mathcal{A}, \mathcal{B}$ of walks $\quad A(t) = \sum_n a_n t^n \quad B(t) = \sum_n b_n t^n$

want to prove bijectively that $A(t) = B(t)$

There is a superfamily $\mathcal{C} \supset \mathcal{A}, \mathcal{B}$ and an involution on $\mathcal{C}$ exchanging two parameters i, j such that, with $C(t; u, v) = \sum c_{n,i,j} t^n u^i v^j$, we have

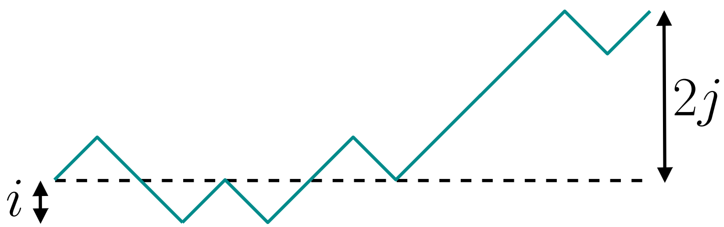
$$A(t) = C(t; 1, 0) \qquad B(t) = C(t; 0, 1)$$

Ex: for tandem walks

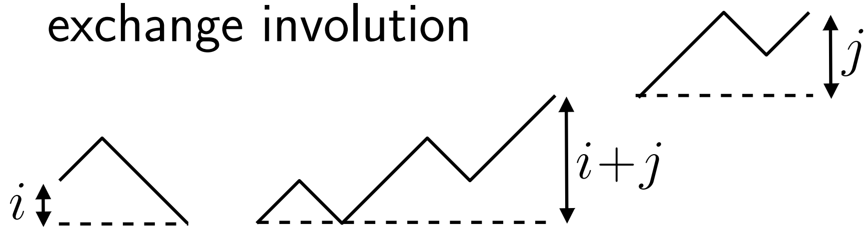


mirror-involution
via bipolar orientations

Ex: for 1D walks of even length

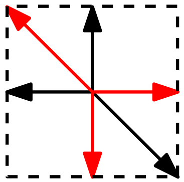


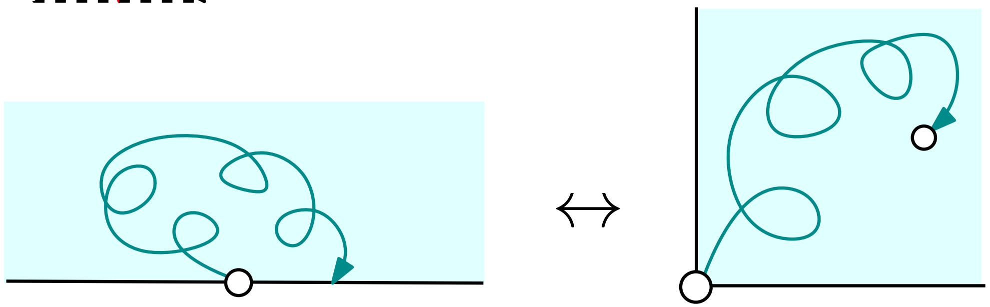
exchange involution



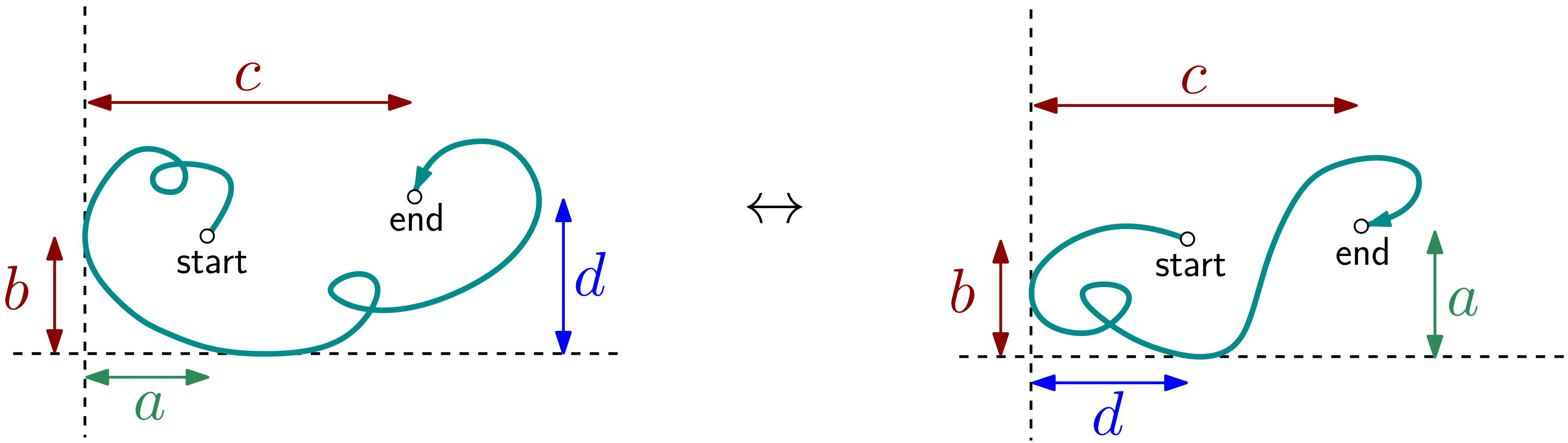
extension for $r \geq 1$ walks: involutivity of jeu de taquin [Hanaker et al.'17]

Conjecture for double-tandem walks

Step-set 

Known:  [Yeats'14, Chyzak-Yeats'18]

Conjecture: There is an involution that realizes



and preserves the length and the number of steps in $\{\rightarrow, \downarrow, \nwarrow\}$