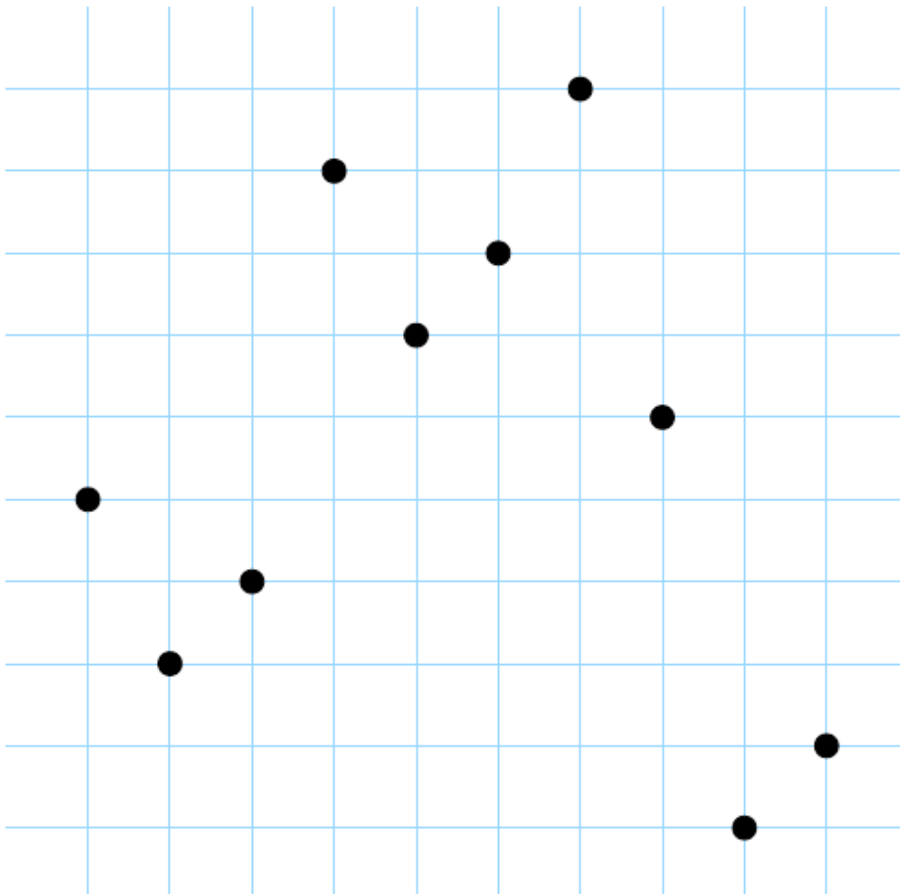
Bijective counting of involutive Baxter permutations

Eric Fusy (LIX, Ecole Polytechnique)

Part 1: Baxter families

Baxter permutations

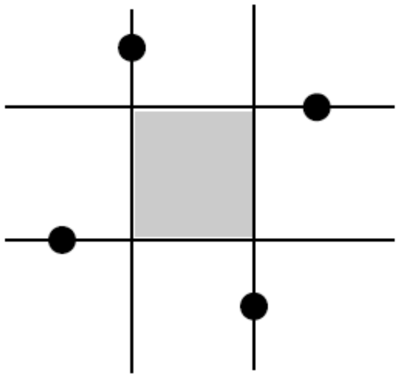
- We adopt the diagram-representation of a permutation



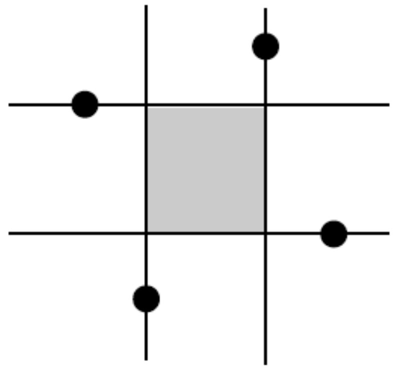
$$\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$$

Baxter permutations

- **Def:** Whenever there are 4 points in position

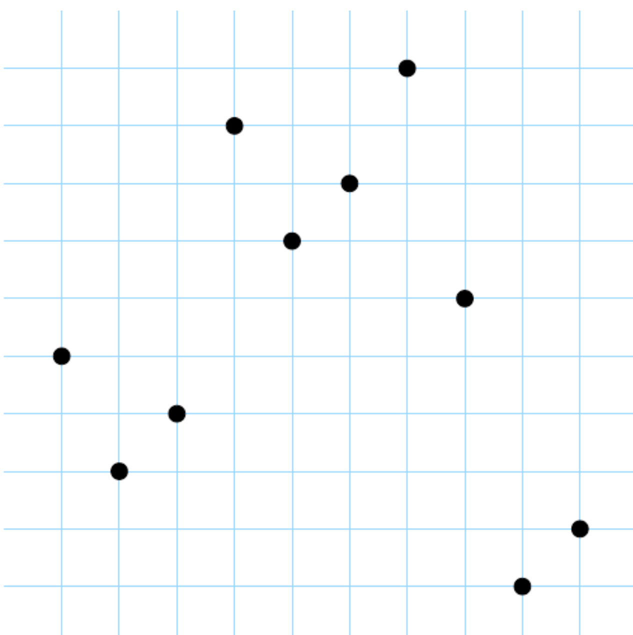


or



then the dashed square is not empty.

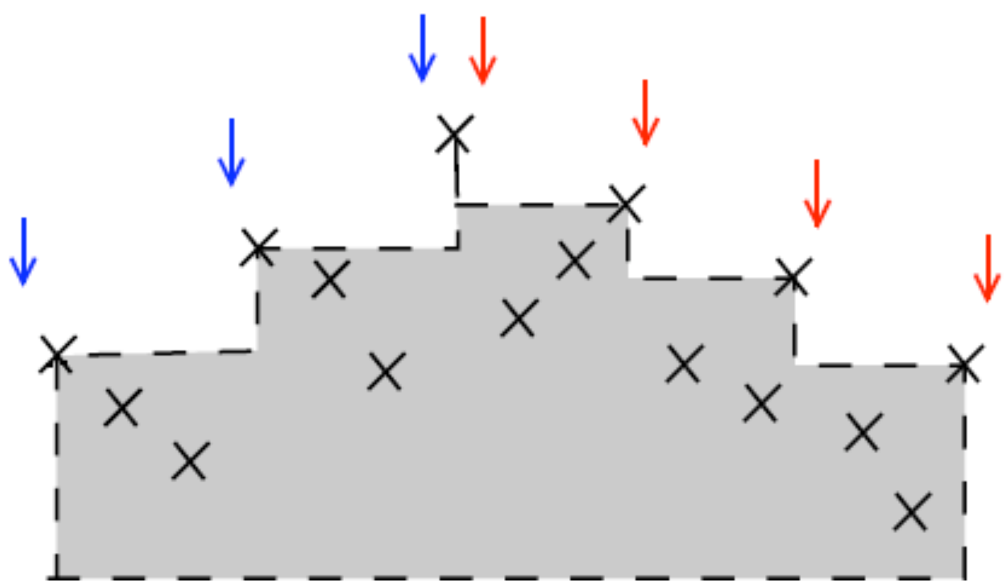
(i.e., no pattern $2\,5\,\bar{3}\,1\,4$ nor $4\,1\,\bar{3}\,5\,2$)



$$\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$$

Characterisation

- **Inductive construction:** at each step, insert *n* either
  - just before a left-to-right maximum (among i of them)
  - just after a right-to-left maximum (among j of them)



Insertion at left-to-right min:

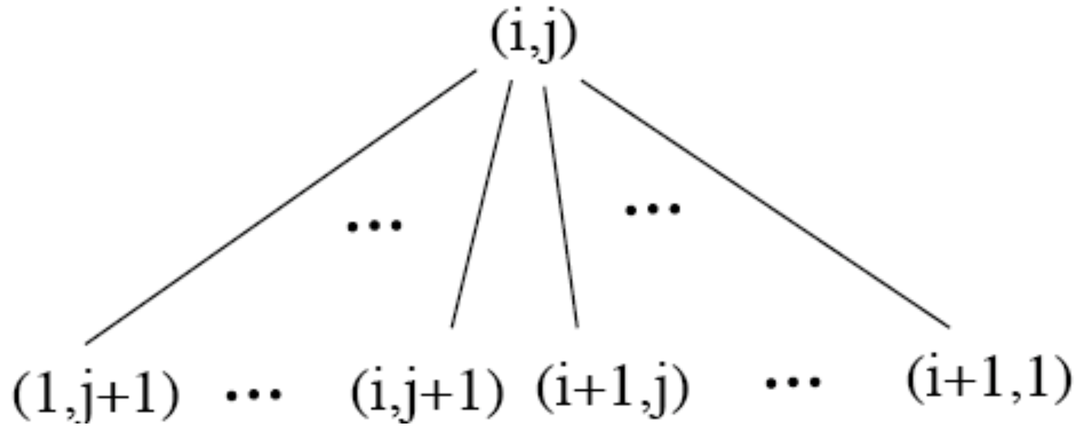
- choose k in [1..i]
- update: i:=k, j:=j+1

Insertion at right-to-left min:

- choose k in [1..j]
- update: j:=k, i:=i+1

Baxter families

Def: Any combinatorial family with generating tree isomorphic to the generating tree T with root (1,1) and children rule

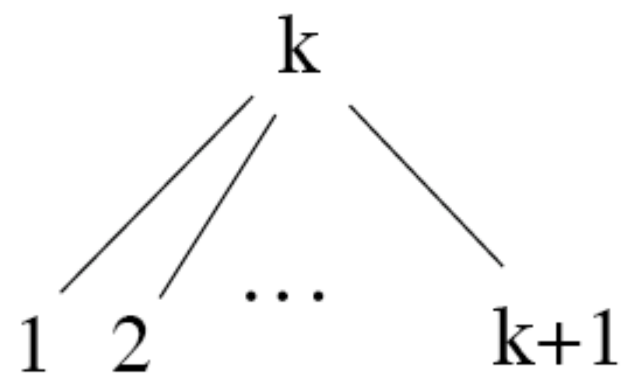


is called a Baxter family

- Parallel with Catalan families: one catalytic parameter

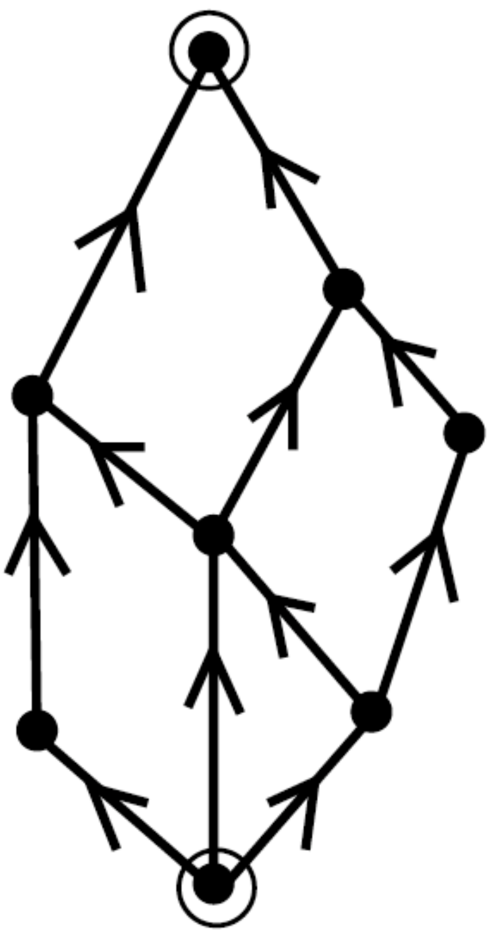
Dyck paths

Children rule is:



Other Baxter family: plane bipolar ori.

- Bipolar orientation = acyclic orientation with unique source and unique sink
- Planar map = graph embedded in the plane, no edge-crossing

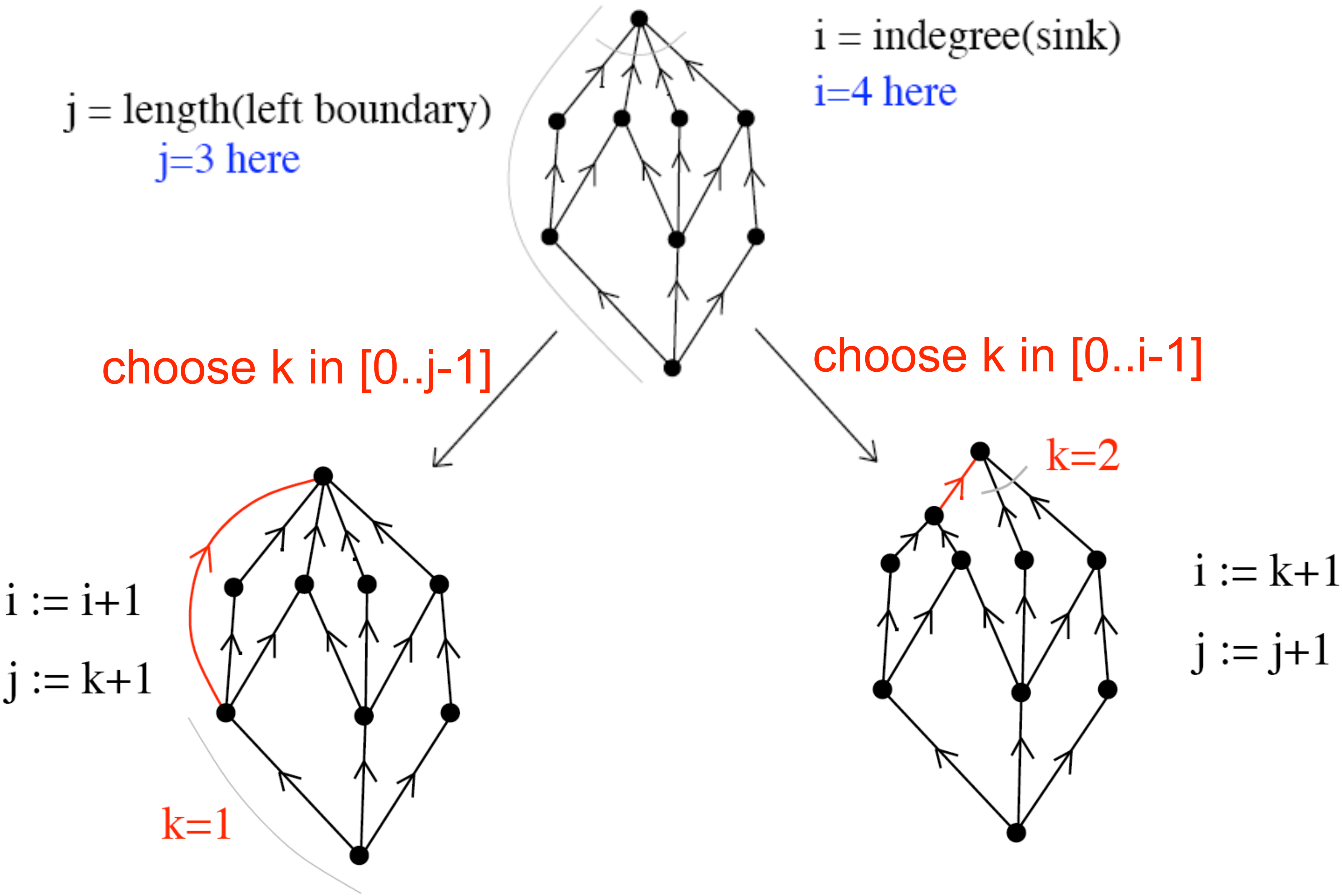


Plane bipolar orientation =

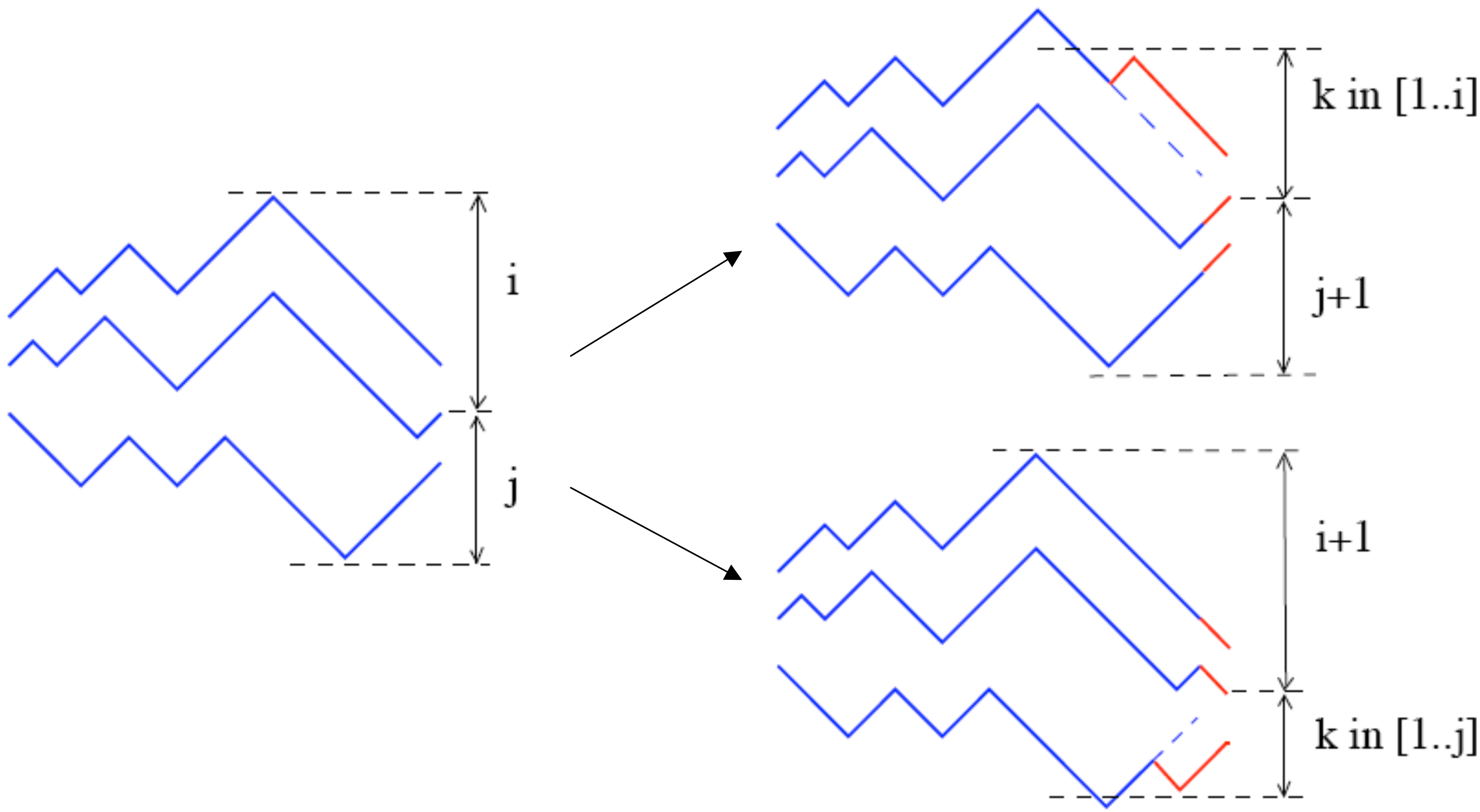
- bipolar orientation on a planar map
- the source and the sink are incident to the outer face

Other Baxter family: plane bipolar ori.

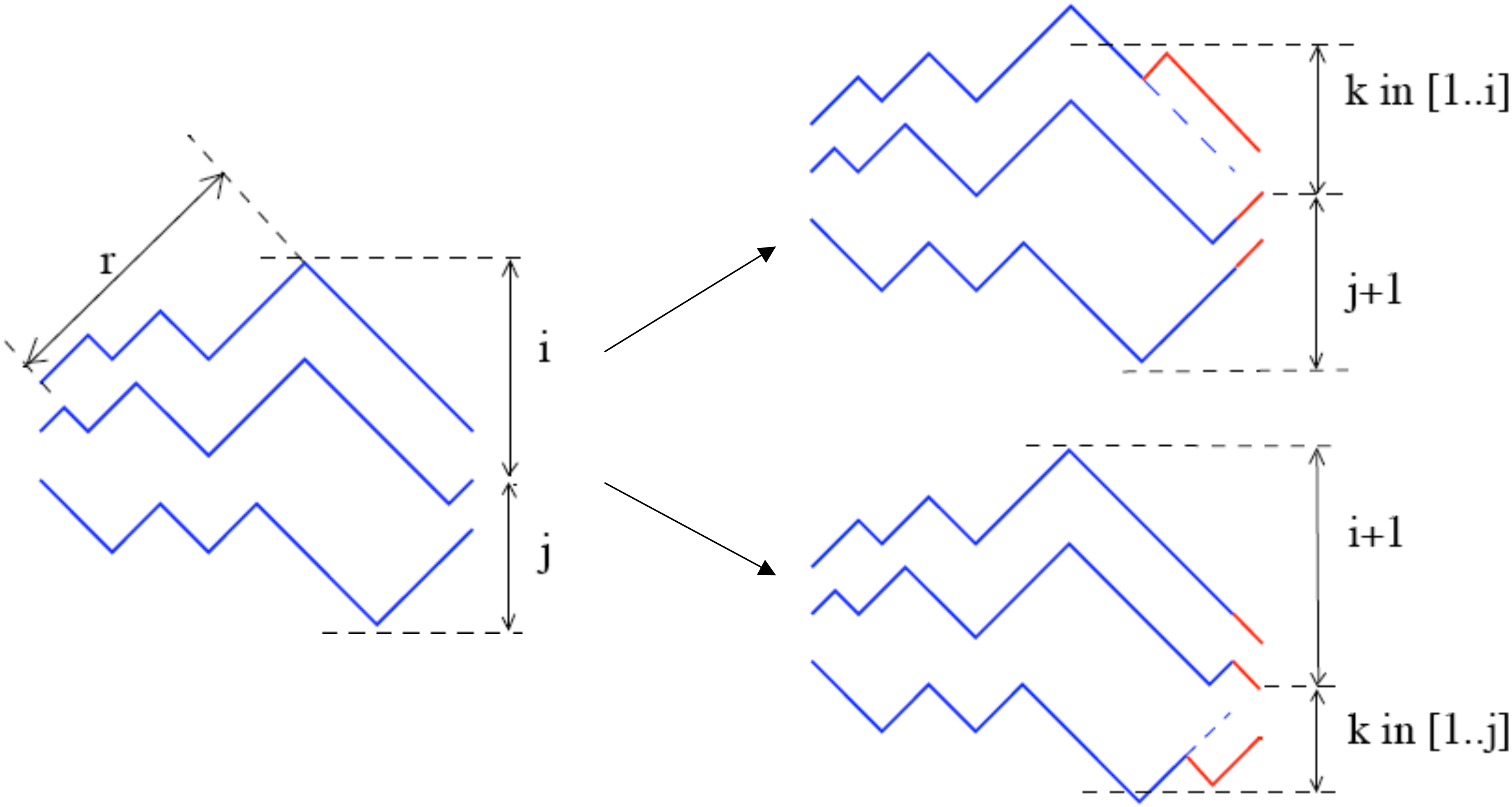
- Two possibilites for inserting the topleft edge:



A countable Baxter family: triples of paths



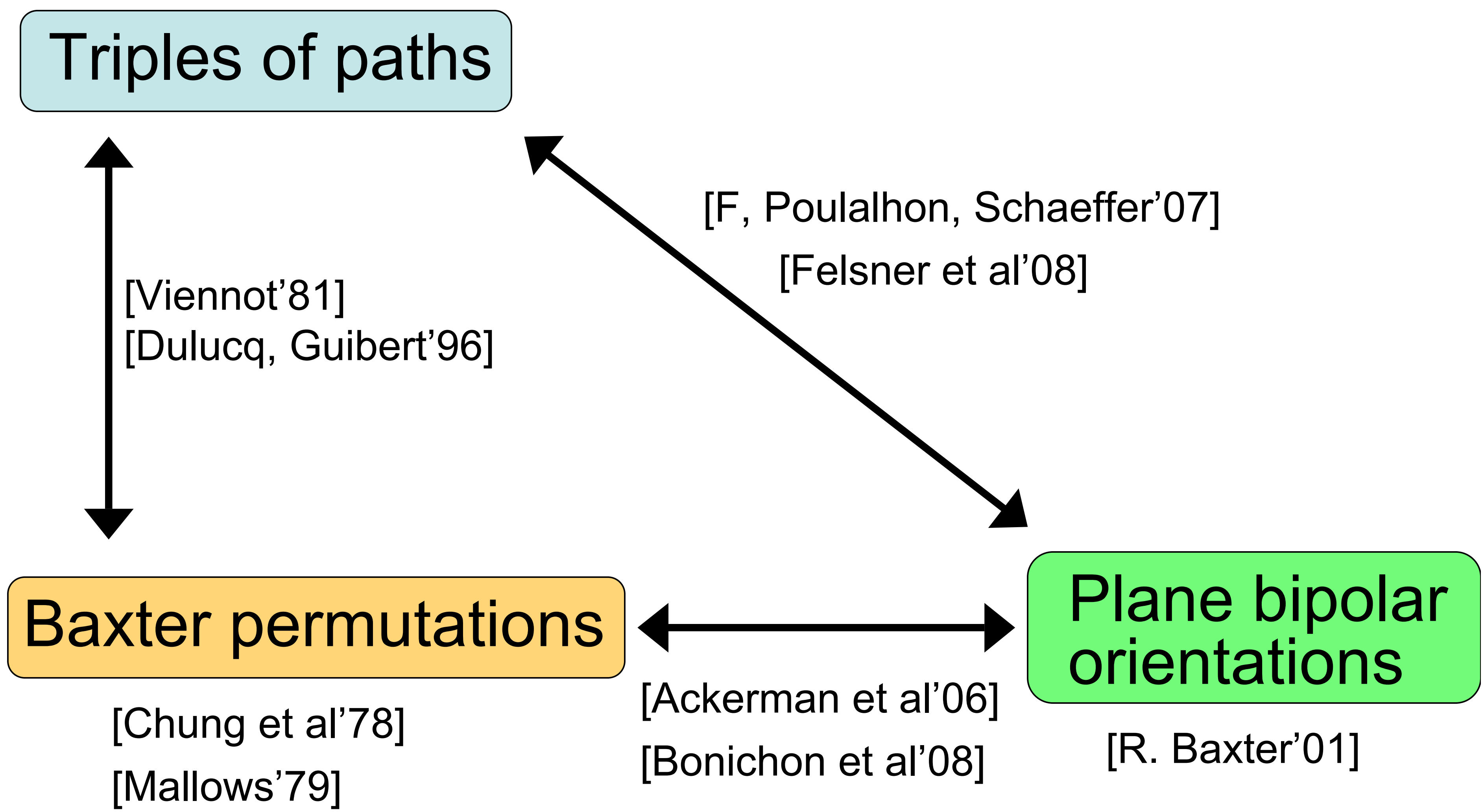
A countable Baxter family: triples of paths



- Counting (by Gessel-Viennot's lemma):

$$q_n = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{r=0}^{n-1} \binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}$$

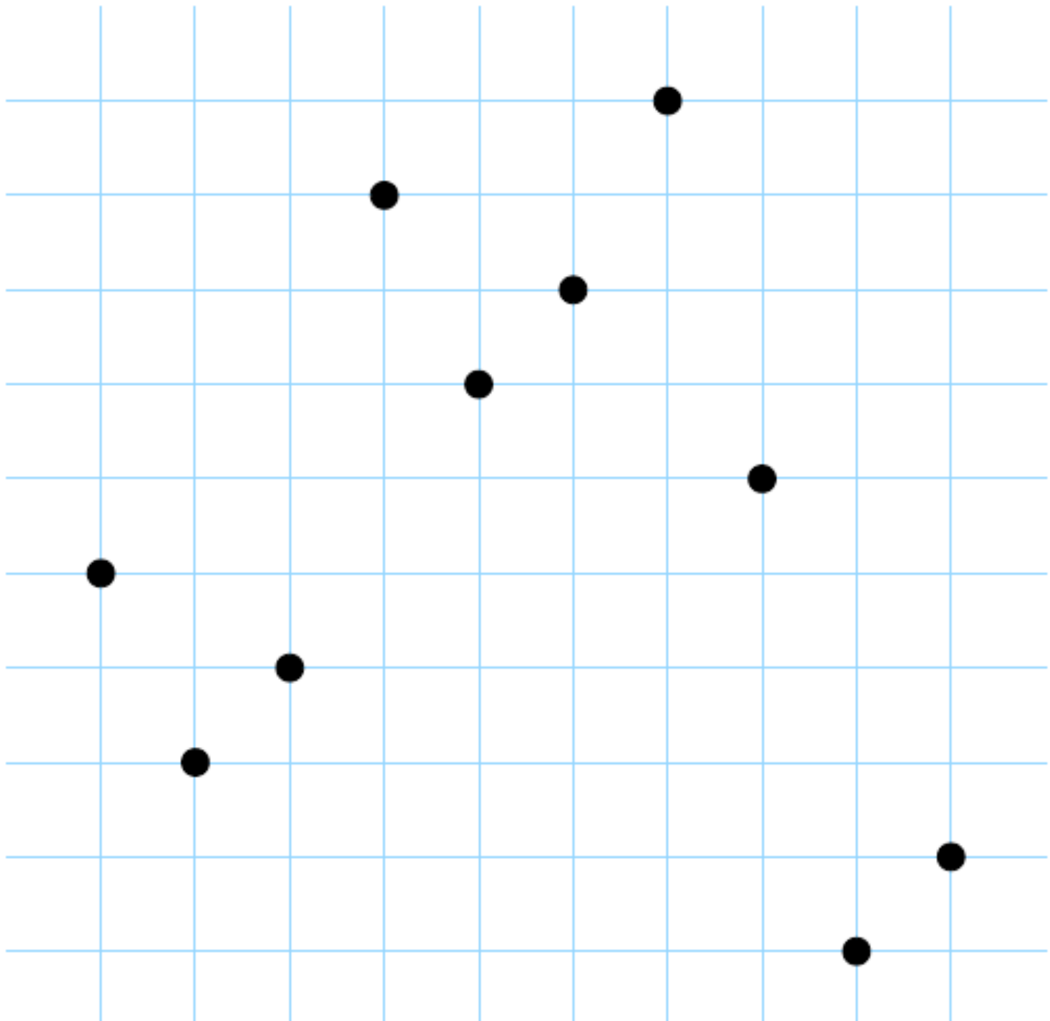
Bijective links and bibliography



Part 2: Baxter permutations and plane bipolar orientations

Baxter permutation -> plane bipolar orientation

(hint: #ascents is distributed like #vertices)



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Baxter permutation -> plane bipolar orientation

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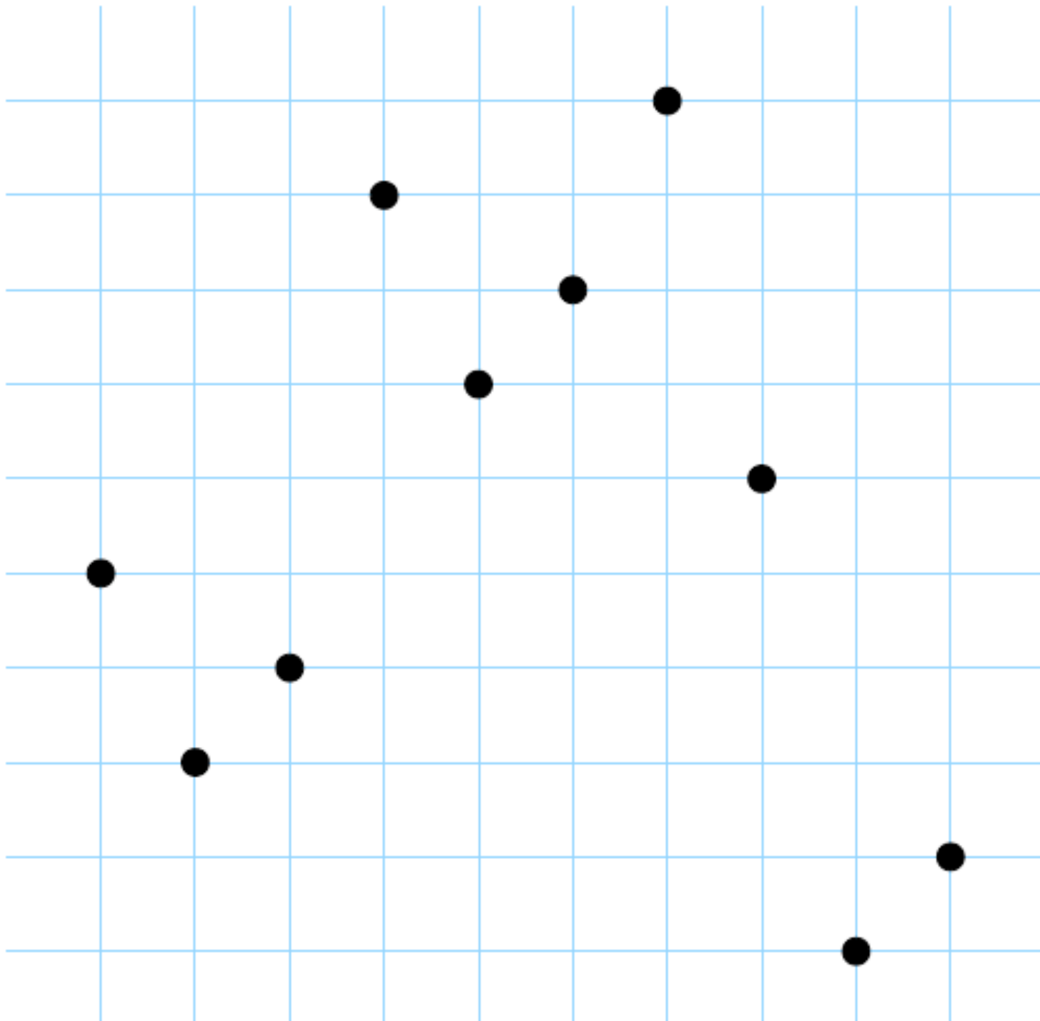
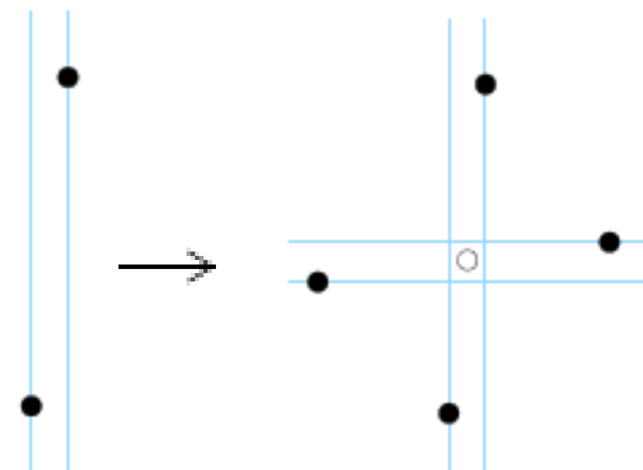


$\pi = 5\ 3\ 4\ 9\ 7\ 8\ 10\ 6\ 1\ 2$

- Ascents of π are in 1-to-1 correspondence with ascents of π^{-1}
- Place a white vertex at the intersection



Baxter permutation -> plane bipolar orientation

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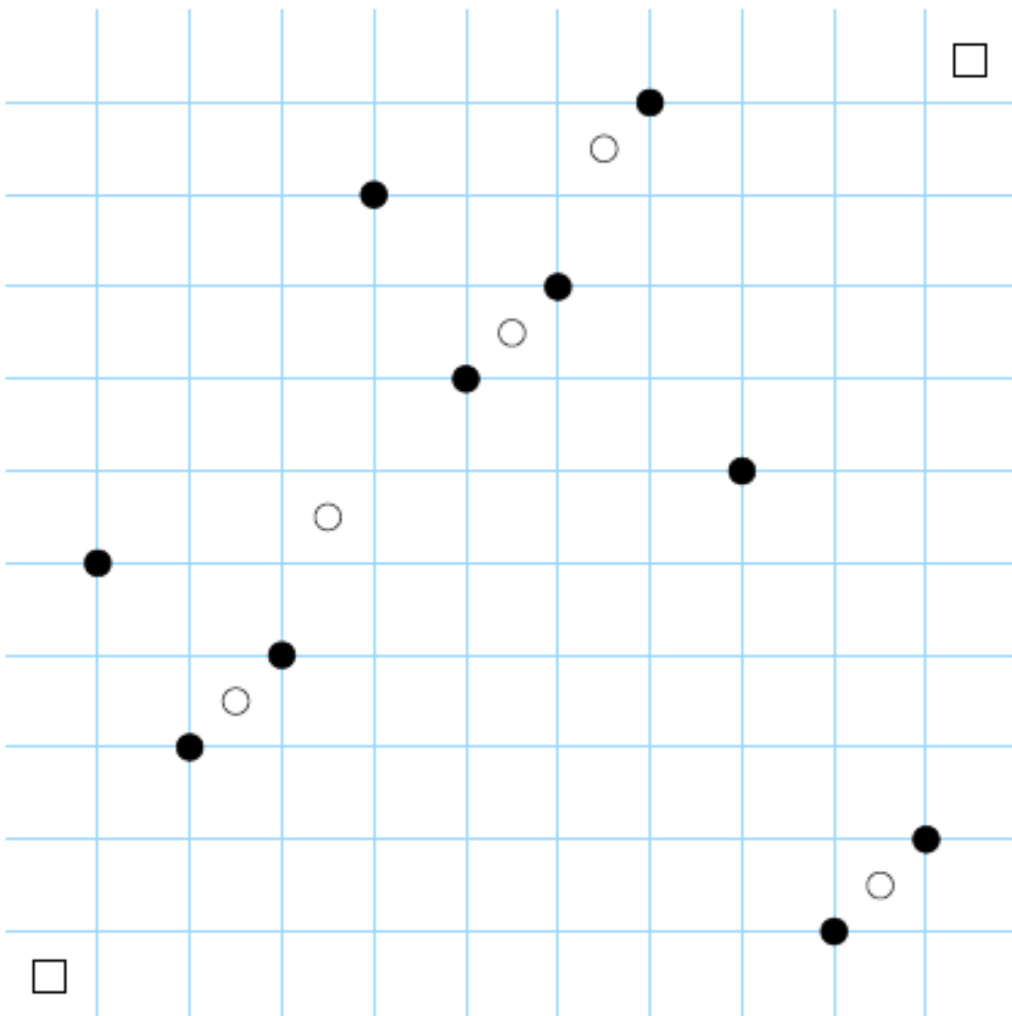

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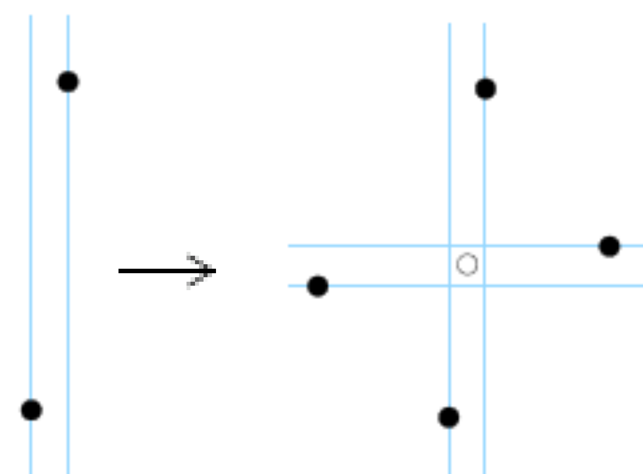
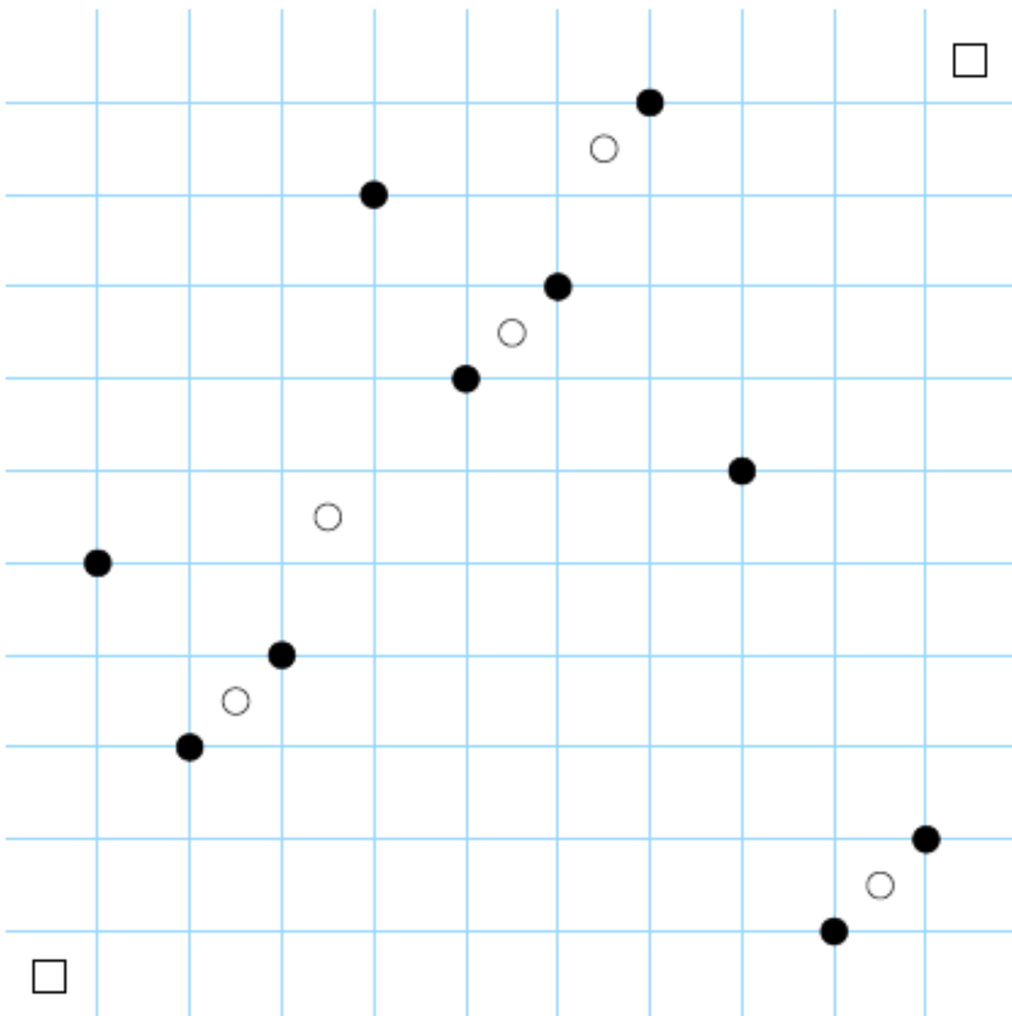


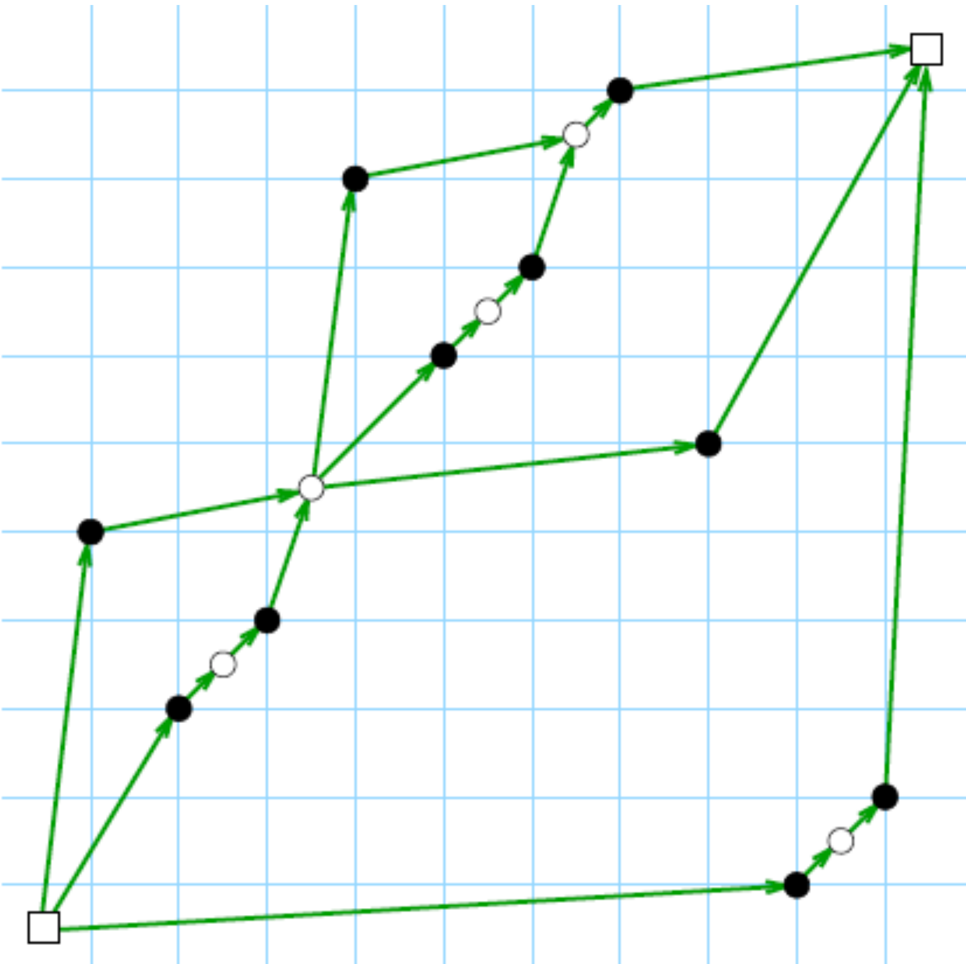
Baxter permutation -> plane bipolar orientation



$\pi = 5\ 3\ 4\ 9\ 7\ 8\ 10\ 6\ 1\ 2$

Dominance drawing:
draw segment (x,y) -> (x’,y’)
whenever x<x’ and y<y’

Baxter permutation -> plane bipolar orientation



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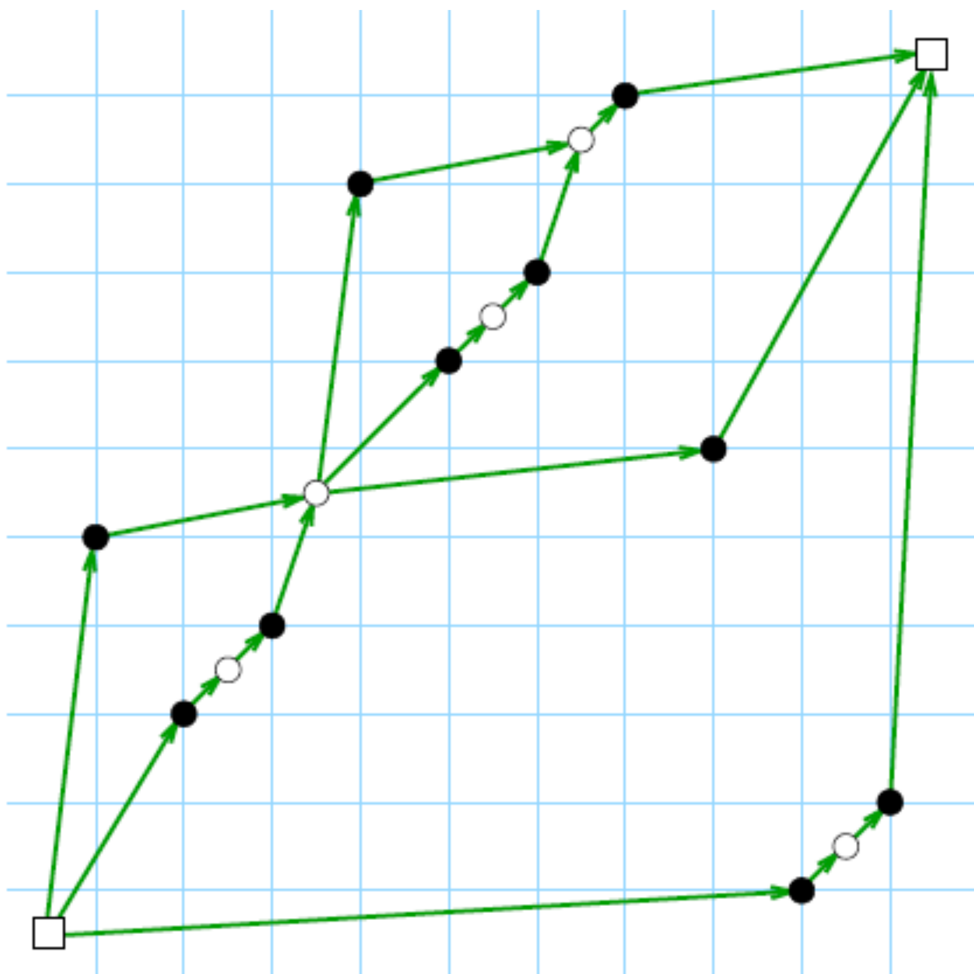
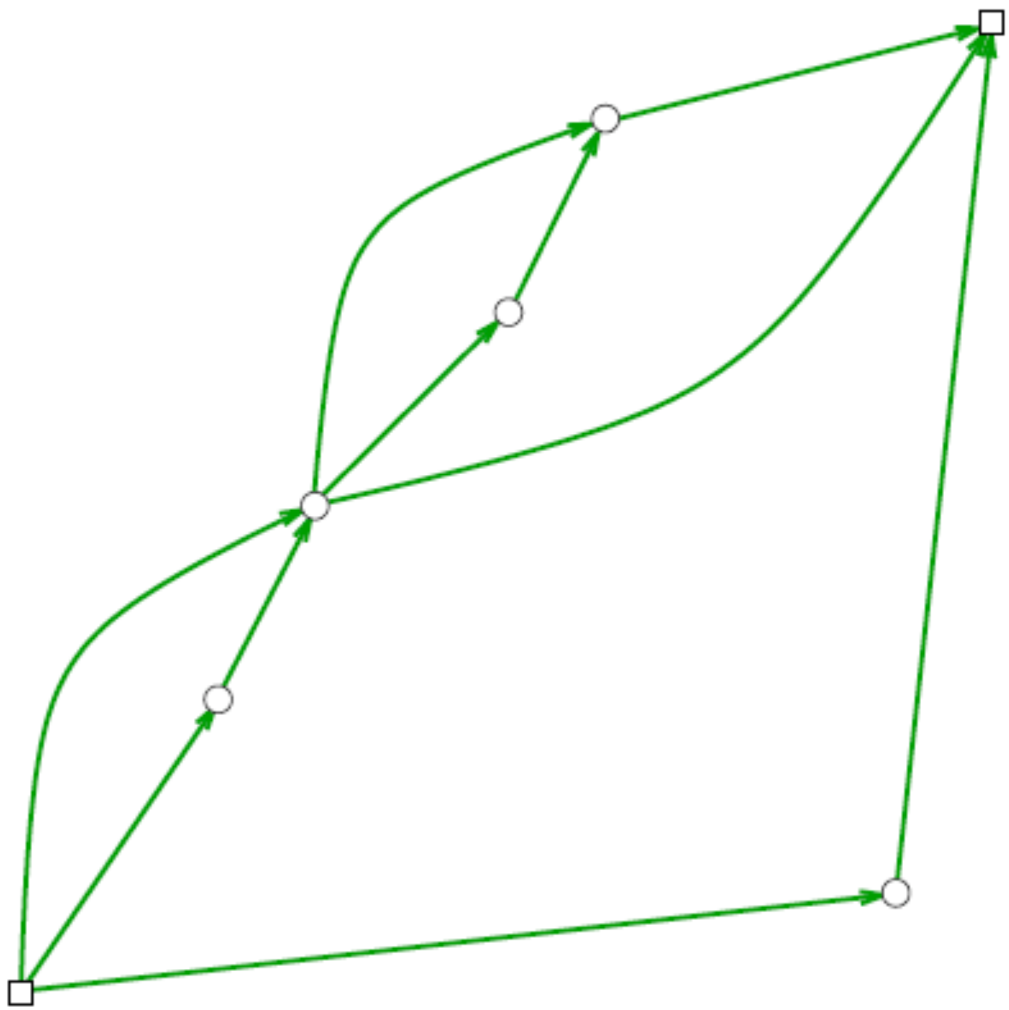


$$\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$$

Erase the black vertices
(all have degree 2)

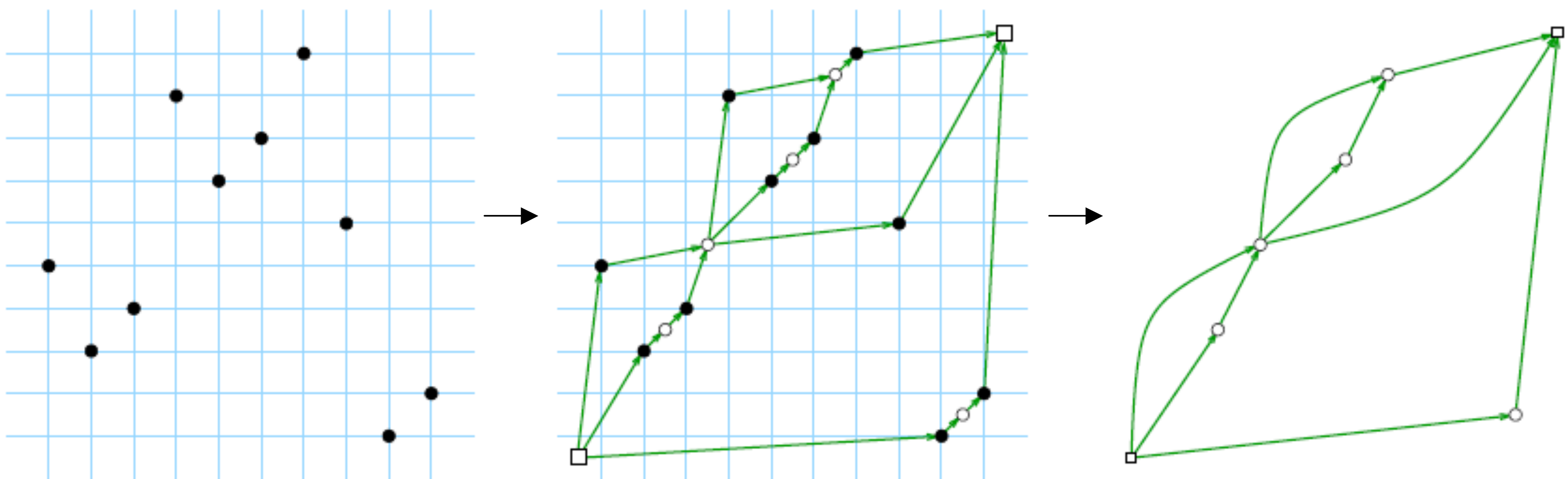
Baxter permutation -> plane bipolar orientation



Erase the black vertices
(all have degree 2)

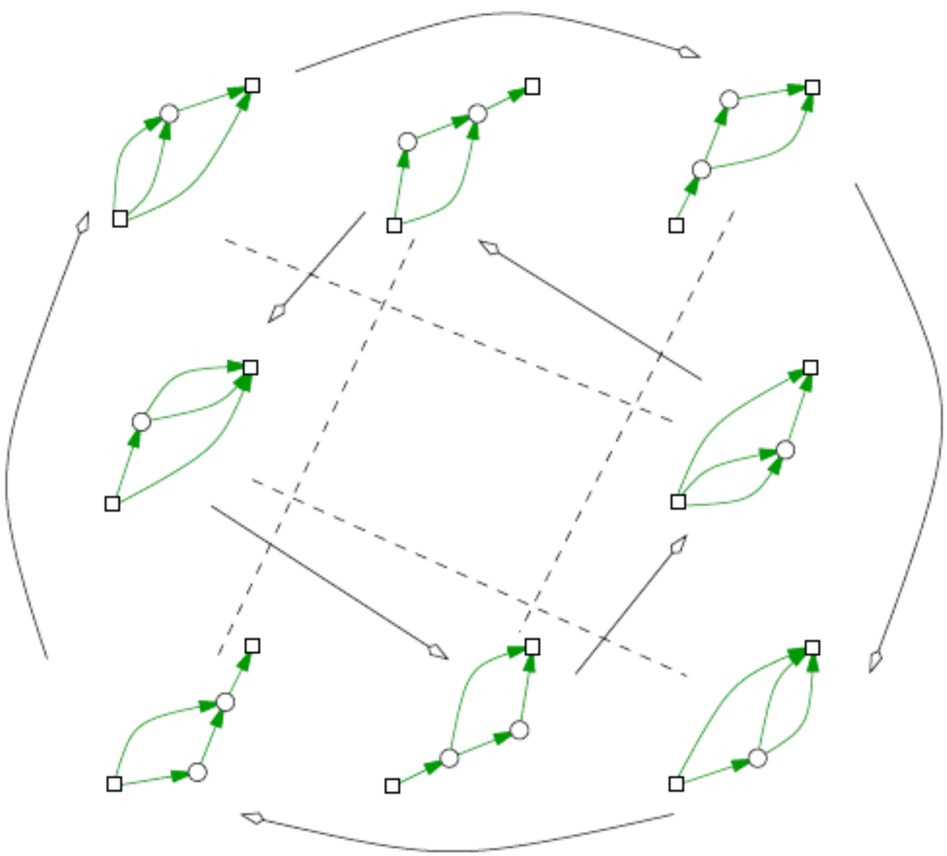
Baxter permutation -> plane bipolar orientation

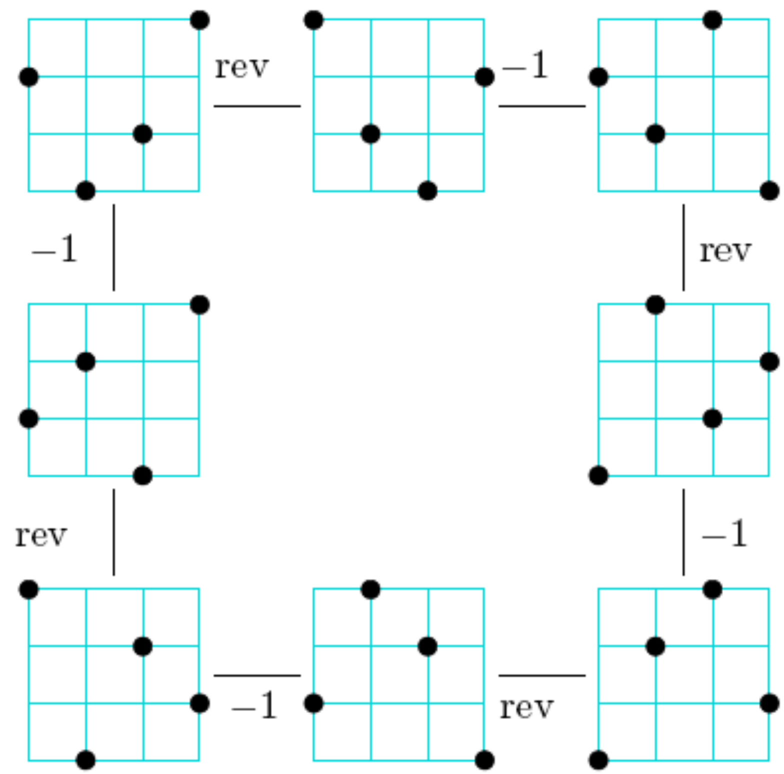
Theorem [Bonichon, Bousquet-Mélou, F'08]:
The mapping is the canonical bijection (implements the isomorphism between generating trees)



Symmetry properties of the bijection

- The bijection ``commutes'' with transformations in the dihedral group D_4





Part 3: bijective counting of involutive Baxter permutations

Results

- **Univariate formula** (bijective proof of formula by M. Bousquet-Mélou): The number of involutive Baxter perm. with no fixed point and with 2n elements is

$$\frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}$$

- **Multivariate formula**: number of involutive Baxter perm. with

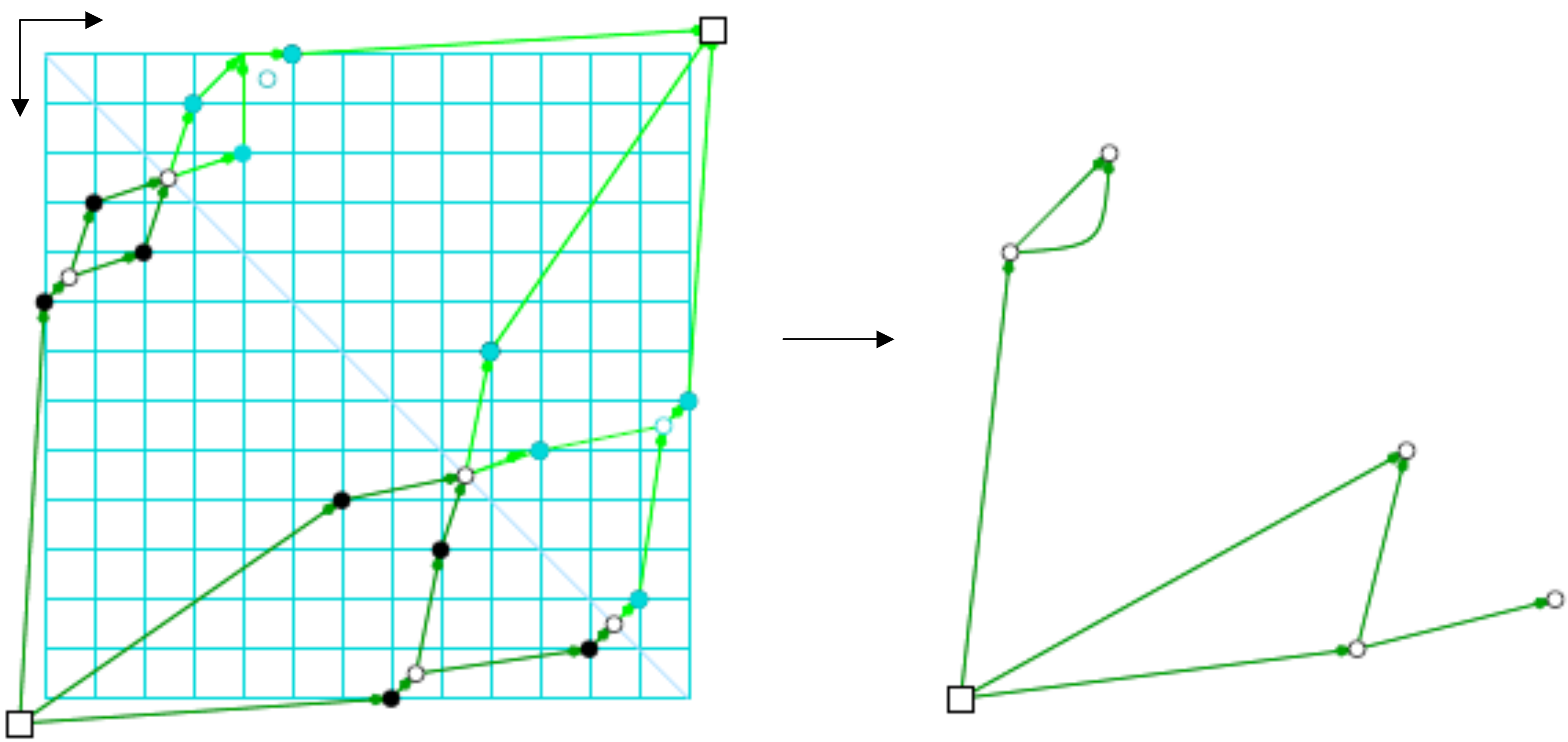
2n non-fixed points	2k descents not crossing the diagonal
p fixed points	r descents crossing the diagonals

is:
$$\frac{\binom{p+r}{r}\binom{n+p-1}{k}^2\binom{n}{t}}{nq^2(q+1)(k+1)(t+1)} \cdot \begin{vmatrix} q(q+1) & q(q-1) & s(s-1) \\ k(q+1) & (k+1)q & s(t+1) \\ k(k-1) & k(k+1) & t(t+1) \end{vmatrix}$$

$$where\ q := n+p-k,\ s := n-k-r,\ t := k+r$$

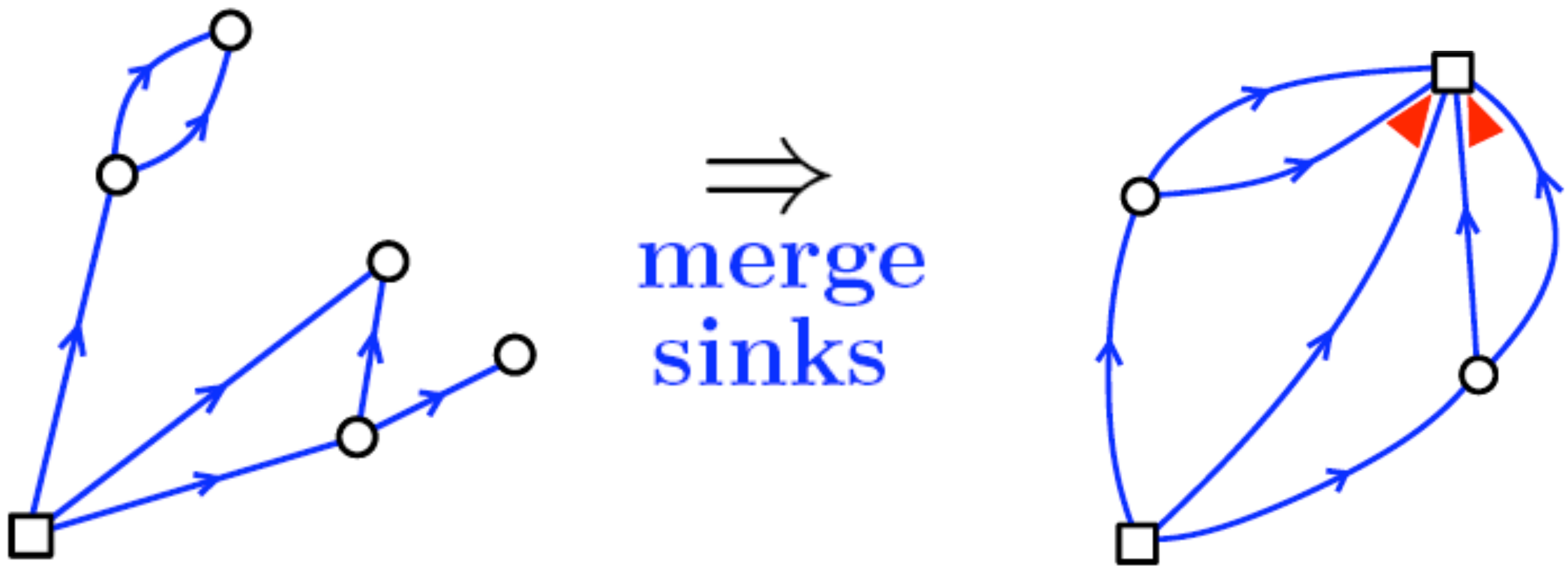
Baxter invol. -> monosource ori.

Keep the part of the picture below the axis x=y



This yields a monosource orientation
(acyclic, single source, possibly many sinks all in the outer face)

Encoding monosource orientations



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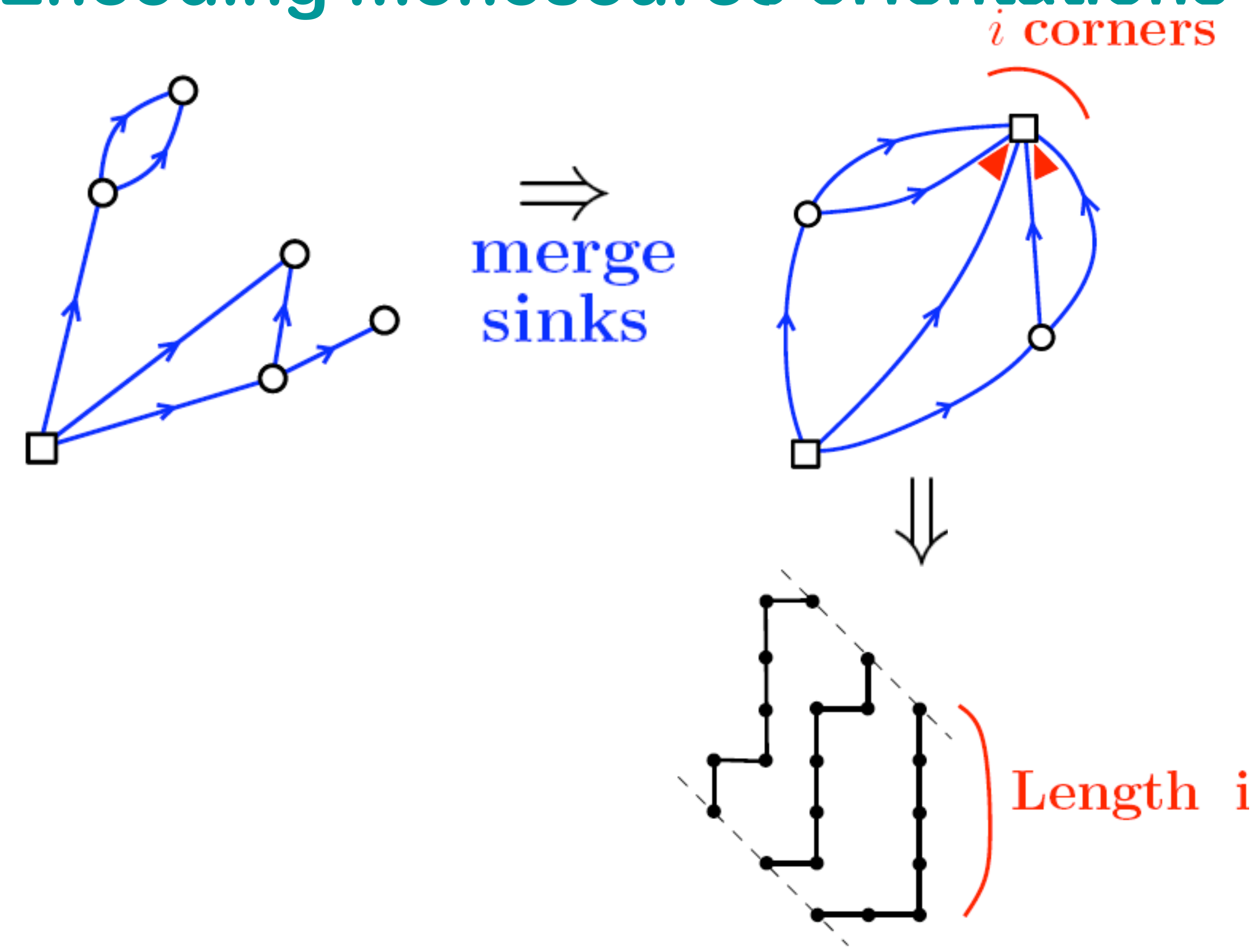
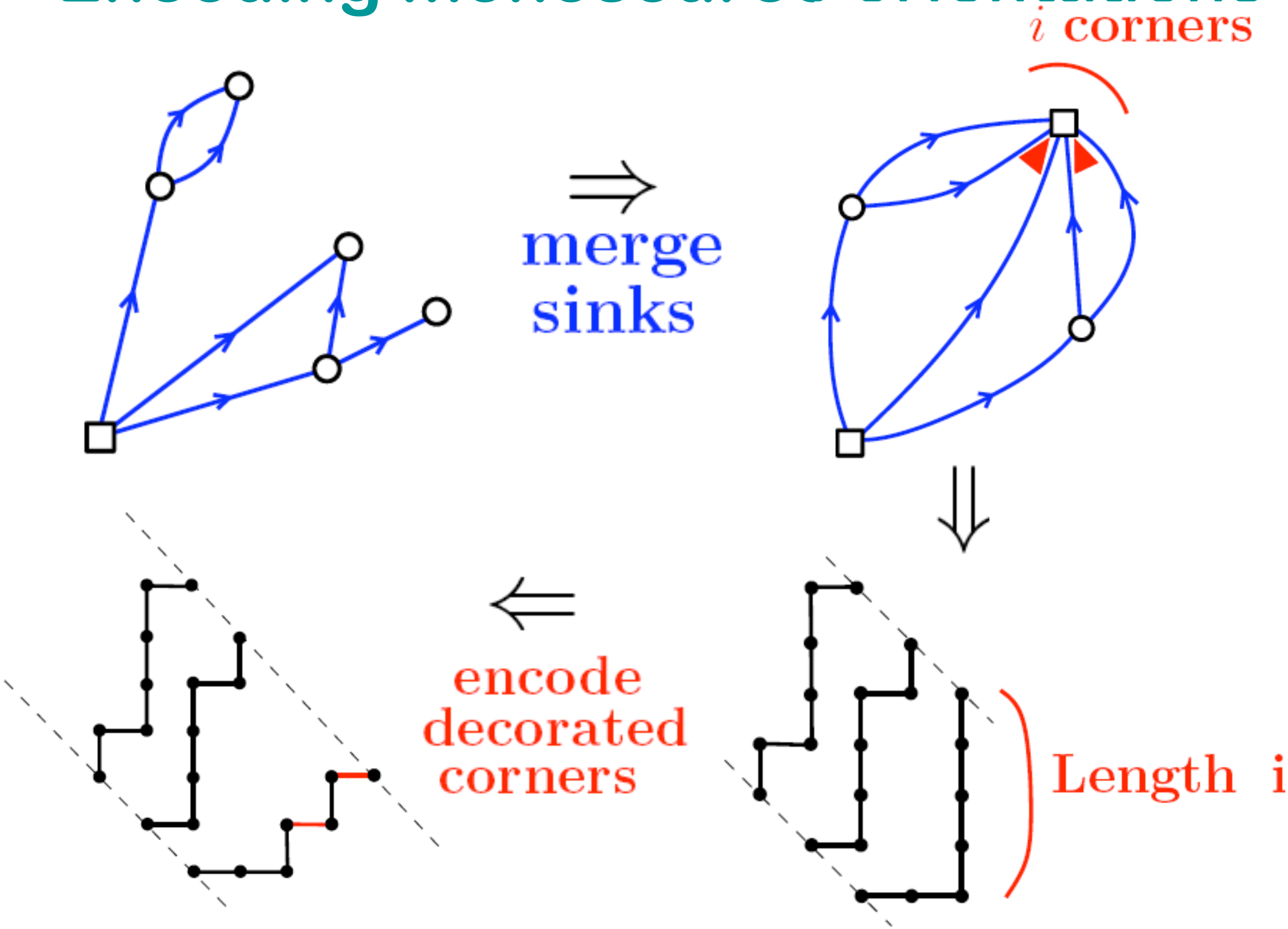
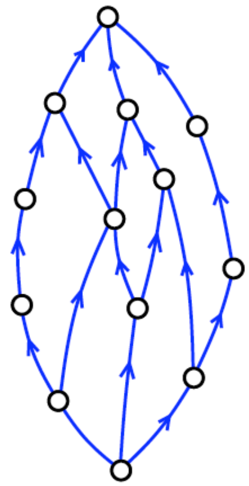


Encoding monosource orientations

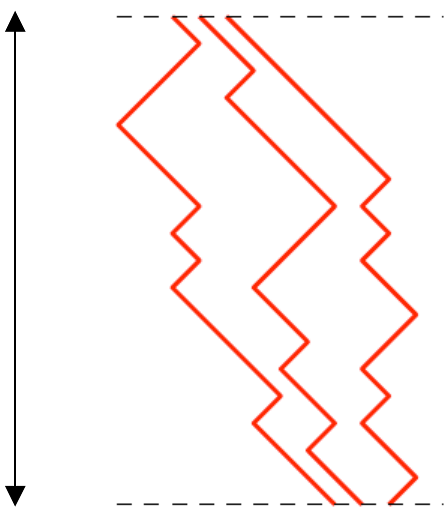


Generic picture

Baxter
permutations
n elements

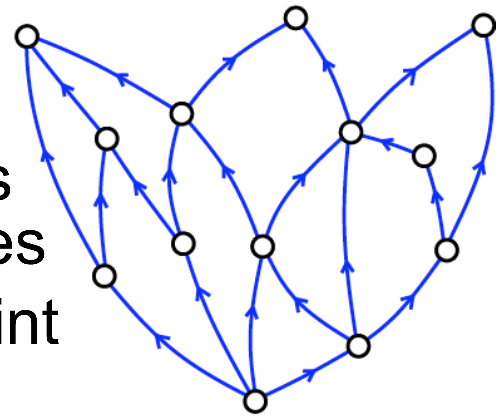


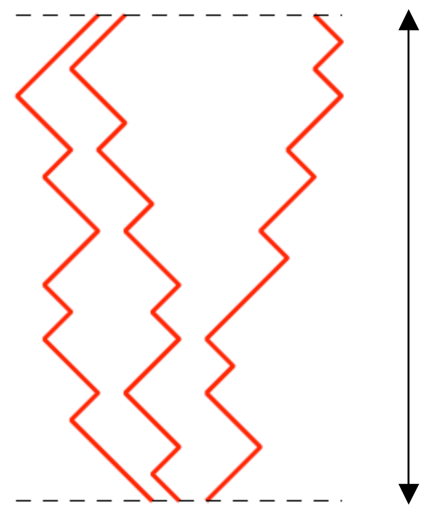
n-1
steps



$$\frac{1}{\binom{n+1}{1}\binom{n+1}{2}}\sum_{r=0}^{n-1}\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}$$

Baxter
involutions
n two-cycles
no fixed point



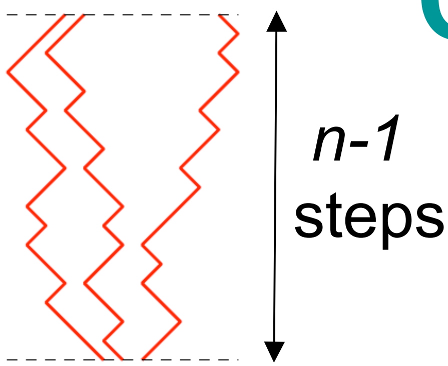


n-1
steps

$$\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$$

Counting

Want to count:



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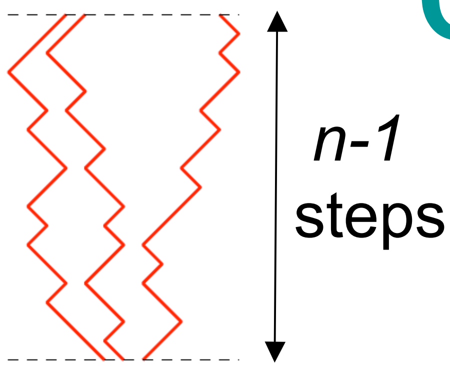
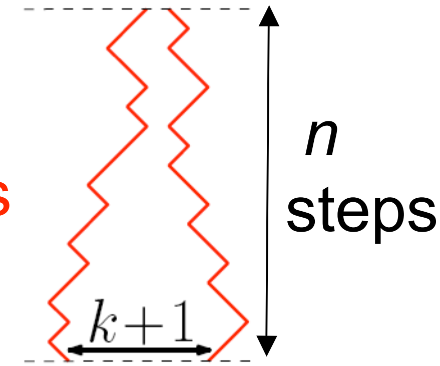


Useful lemma:
2 non-cross. paths
[J. Levine'59]



$$\boxed{\mathbf{a_{n,k} = (2k+1)\frac{(2n+2k)!}{n!(n+2k+1)!}}}$$

Counting

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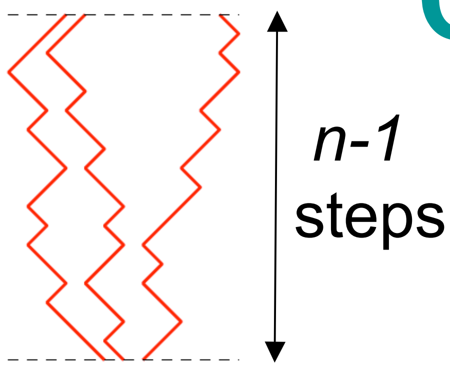

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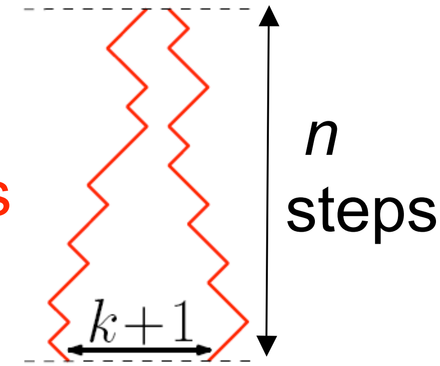
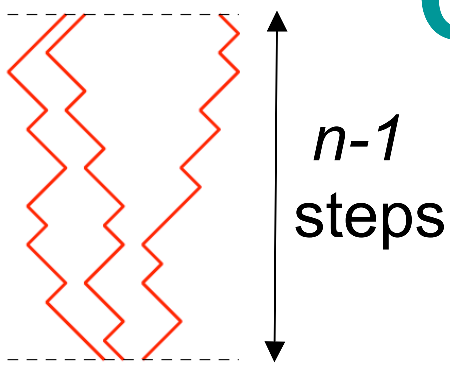


$$\mathbf{a_{n,k}} = (\mathbf{2k+1})\frac{(\mathbf{2n+2k})!}{\mathbf{n}!(\mathbf{n+2k+1})!}$$

Counting

Want to count:



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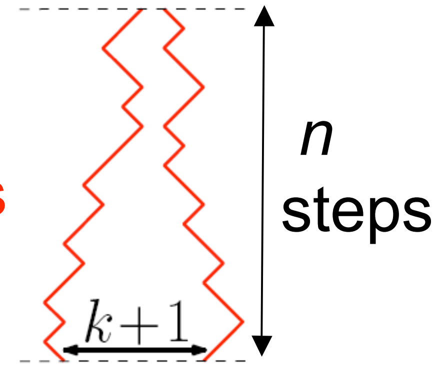

$$\boxed{\mathbf{a_{n,k} = (2k+1)\frac{(2n+2k)!}{n!(n+2k+1)!}}}$$

$$\Downarrow$$

$$\mathbf{a_{n-1,0} \cdot 2^{n-1}}$$

Counting

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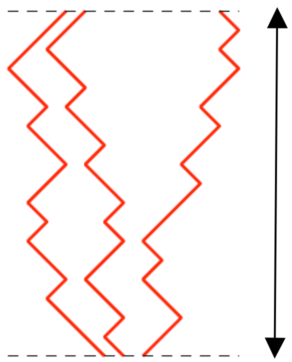

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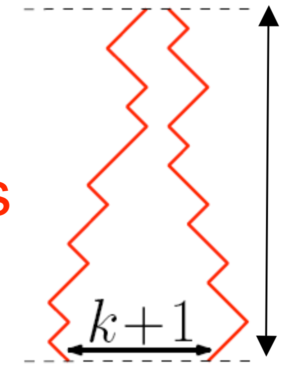

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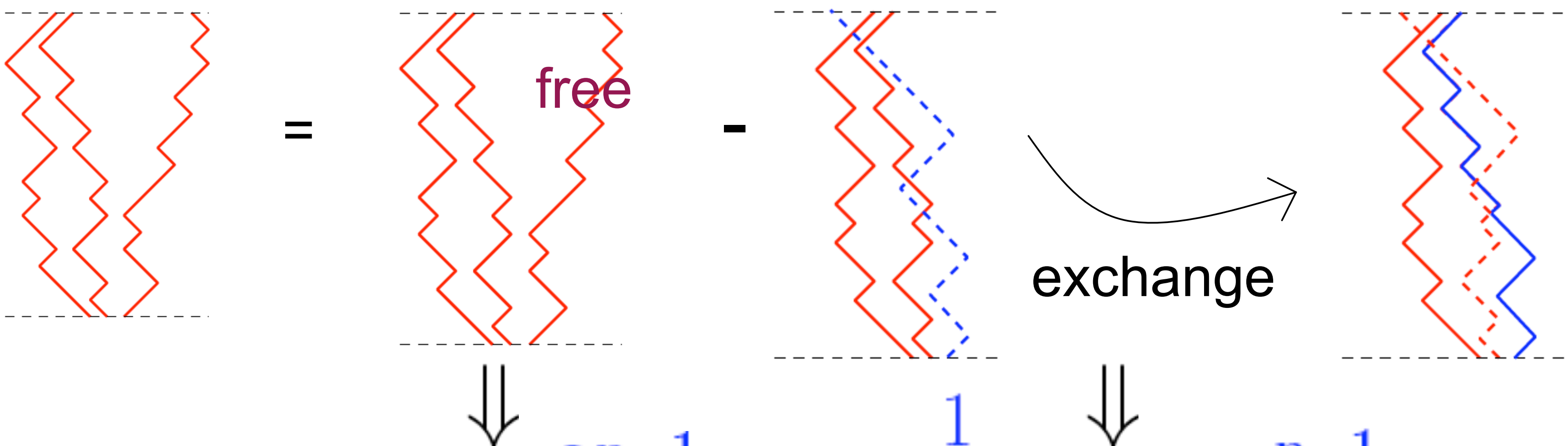
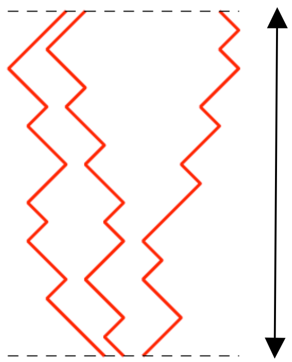


$$a_{n-1,0}\cdot 2^{n-1} \quad - \quad \frac{1}{2}a_{n-1,1}\cdot 2^{n-1}$$

Counting

Want to count:



n-1 steps

Useful lemma:
2 non-cross. paths
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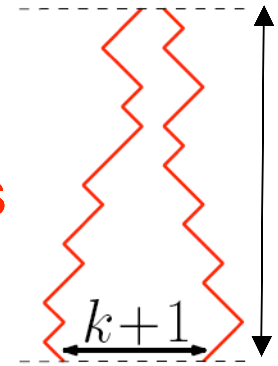

n steps

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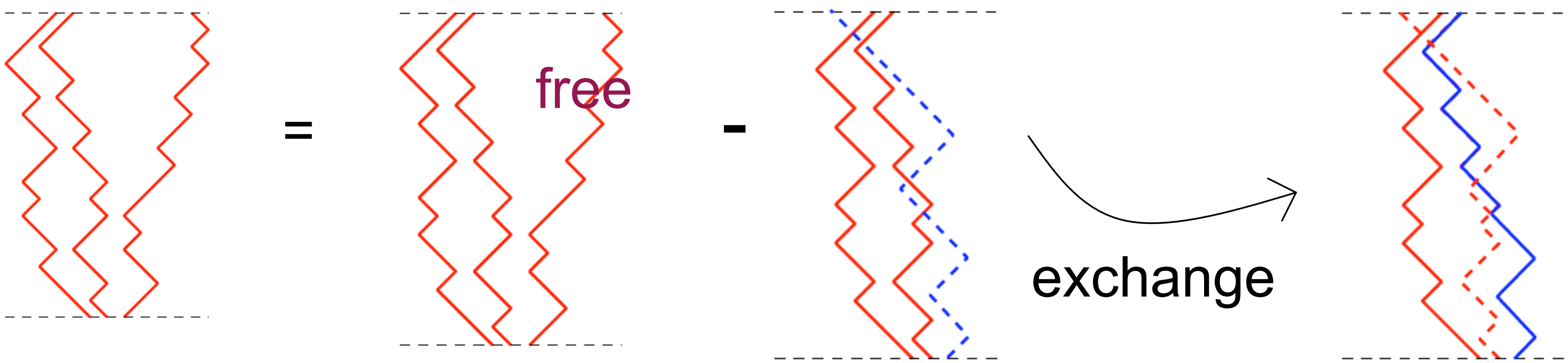

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\mathbf{a_{n-1,0} \cdot 2^{n-1}} \quad - \quad \mathbf{\frac{1}{2} a_{n-1,1} \cdot 2^{n-1}}$$

$$= \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}$$