Optimal encoding of triangular and quadrangular meshes of fixed topology

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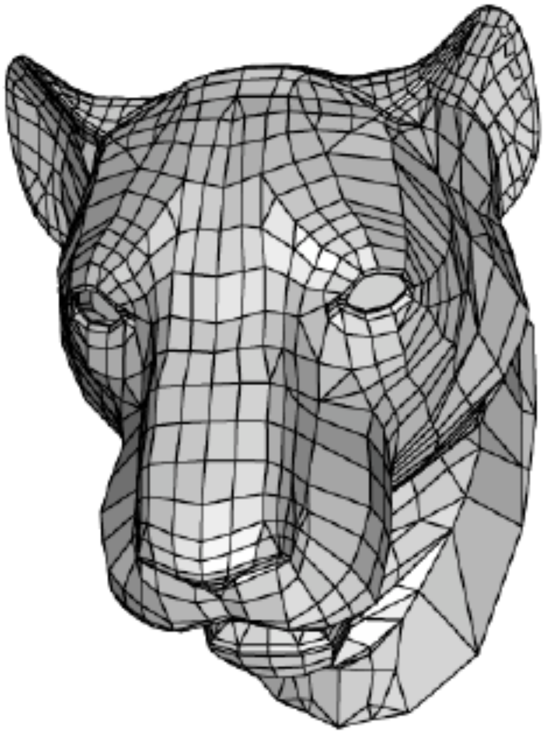
with Luca Castelli Aleardi (LIX) and Thomas Lewiner (PUC, Rio)

Part 1: Motivations and statement of results

Mesh compression



Geometry → coordinates of the vertices

Combinatorics → Incidences faces/edges/vertices

We aim at compressing efficiently the combinatorial incidences

Topology of surfaces

- An orientable surface is characterised by:
  - its genus g (number of handles)
  - its number b of boundaries

g=0
b=0

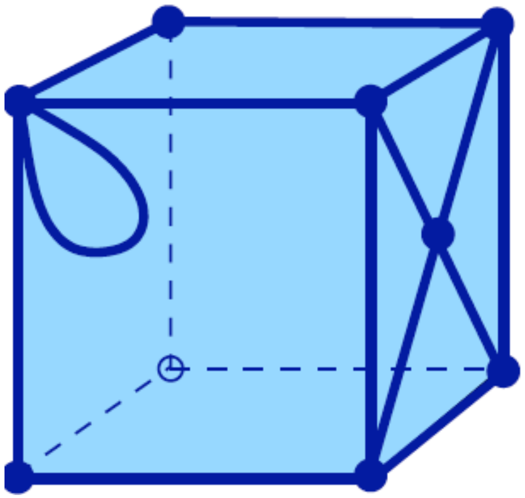
g=0
b=1

g=1
b=1

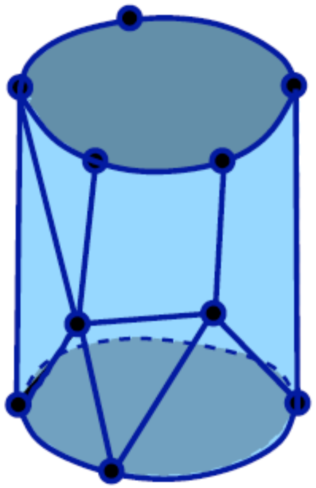
g=2
b=0

Meshes and maps

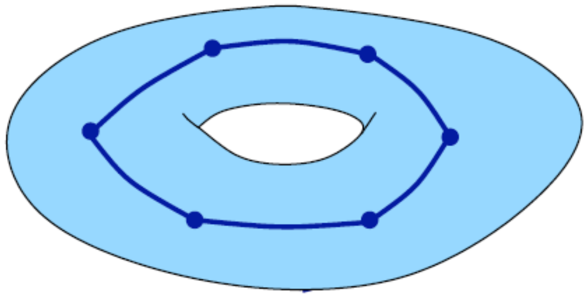
- A map is a graph embedded on a surface S such that:
  - $\partial S \subset G$
  - Components of $S \backslash G$ are topological disks



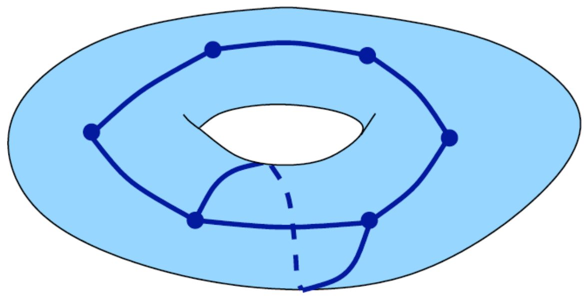
g=0 b=0



g=0 b=2



Not a map



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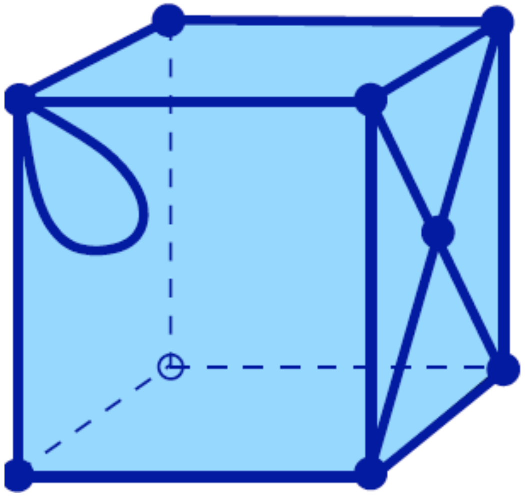
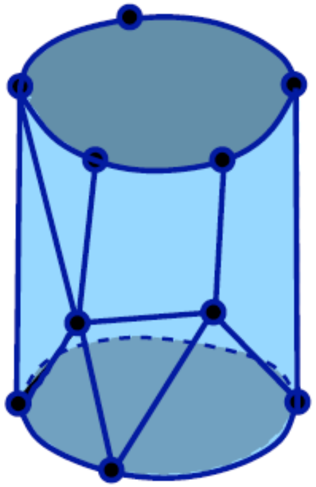


g=0 b=0



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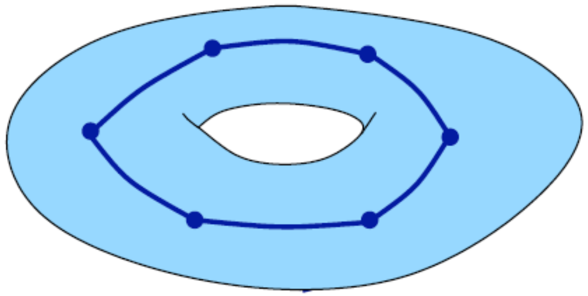

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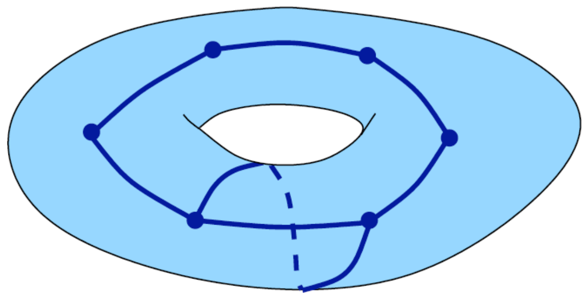


g=1 b=0

- **Remark:** The incidences face/edge/vertex of a mesh form a map

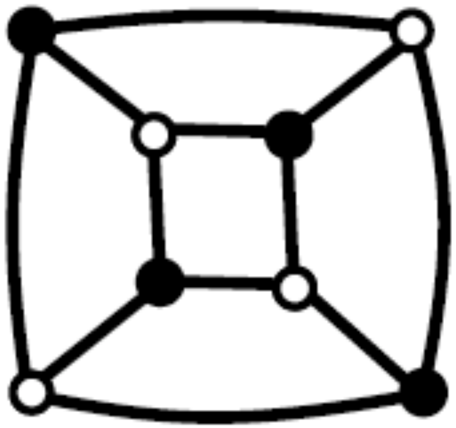
$\Longrightarrow$ encoding the incidences reduces to encoding a map

Map enumeration

Strikingly simple counting formulas (for rooted planar maps):

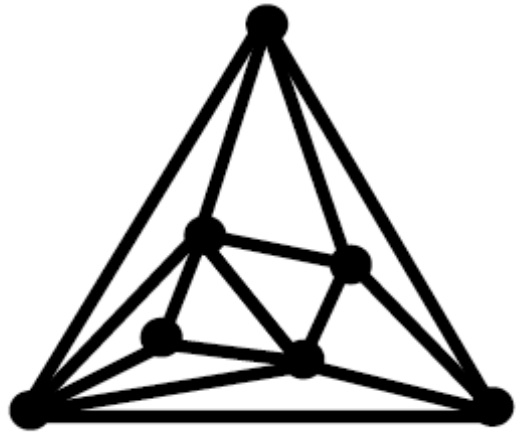
g=0
b=0

Bipartite cubic with 2n vertices



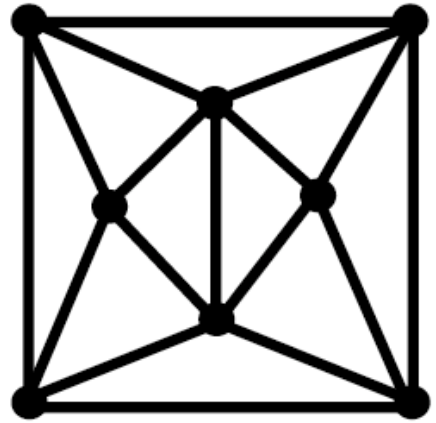
$$\frac{3\cdot 2^{n-1}(2n)!}{(n+2)!n!}$$

Triangulations with n+2 vertices



$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

Irreducible triang. with n+3 vertices



$$\frac{4(3n-3)!}{(n-1)!(2n)!}$$

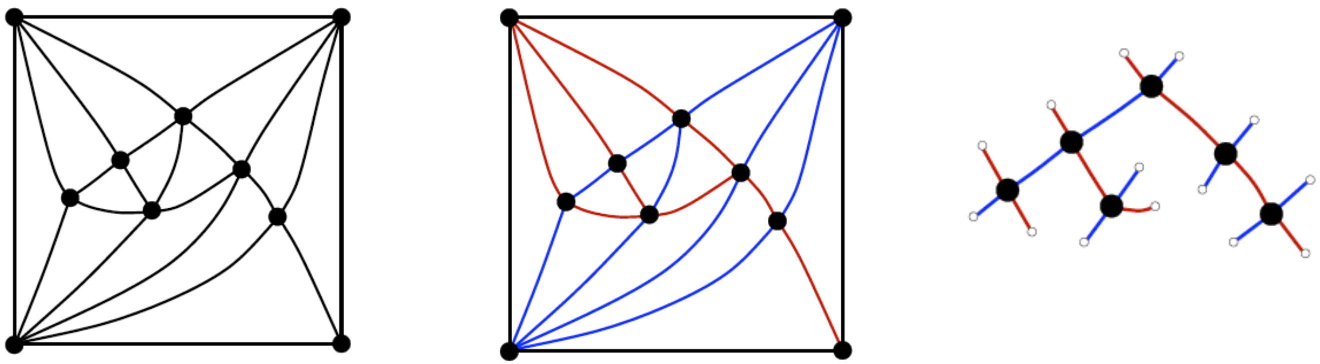
Two enumeration methods:

- Recursive + gen. functions [Tutte'62,63]
- Bijective [Cori-Vauquelin'81, Schaeffer'97]

Bijections -> encoding

The bijective method for map families yields (asymptotically) optimal encoding procedures

map ⇨ Canonical structure (orientation, coloring) ⇨ tree ⇨ contour word



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↓

$$\# = \frac{4(3n-3)!}{(n-1)!(2n)!}$$

- Applies to many planar map families (proving counting formulas)

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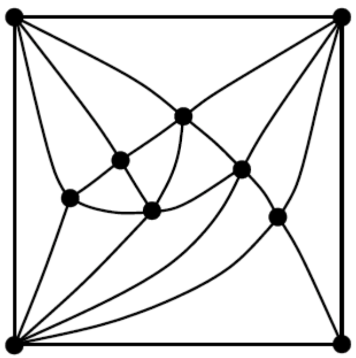

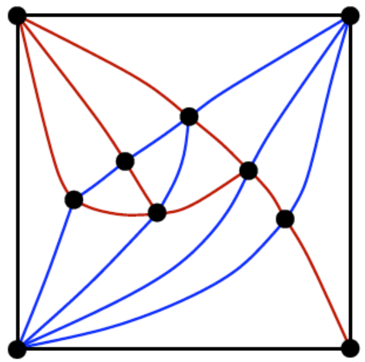

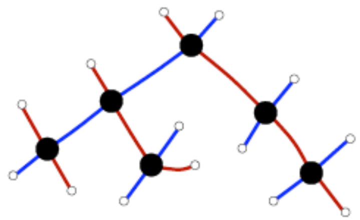

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↓

$$\# = \frac{4(3n-3)!}{(n-1)!(2n)!}$$

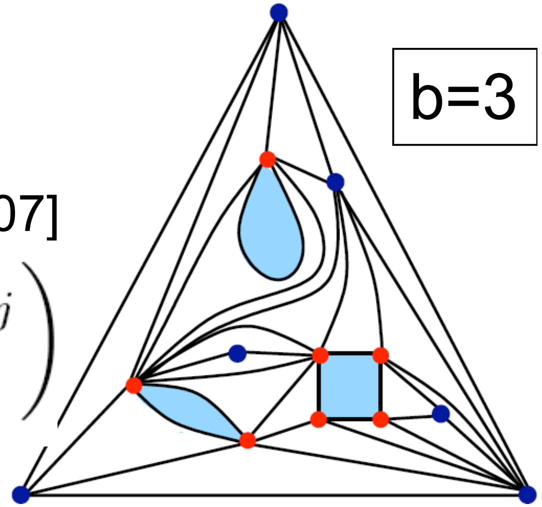
- Applies to many planar map families (proving counting formulas)

- The method can be unified in terms of orientations

$\Rightarrow$ method in [Poulalhon, Schaeffer'03] generalized in [Bernardi'06]

Planar triangulations with boundaries

- b boundaries of sizes $k_1,\ldots,k_b$, $k:=k_1+\ldots+k_b$
- n vertices not on the boundary



- With loops & multi-edges, nice formula [Krikun'07]

$$a_n^{(k_1,\ldots,k_b)} = \frac{4^{n-1}(2k+3n-5)!!}{(n-b+1)!(2k+n-1)!!}\prod_{j=1}^{b} k_j \binom{2k_j}{k_j}$$

(no bijective proof)

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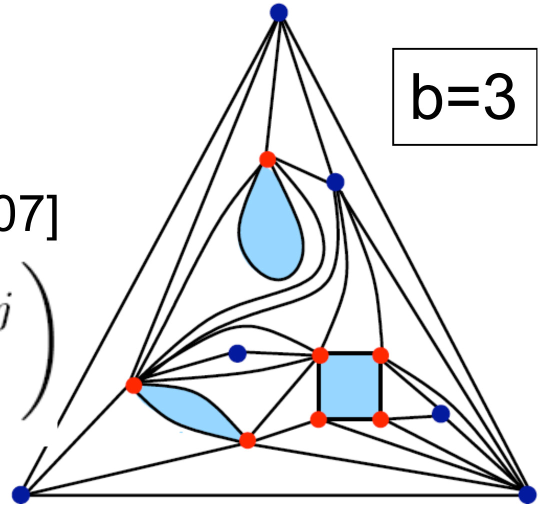

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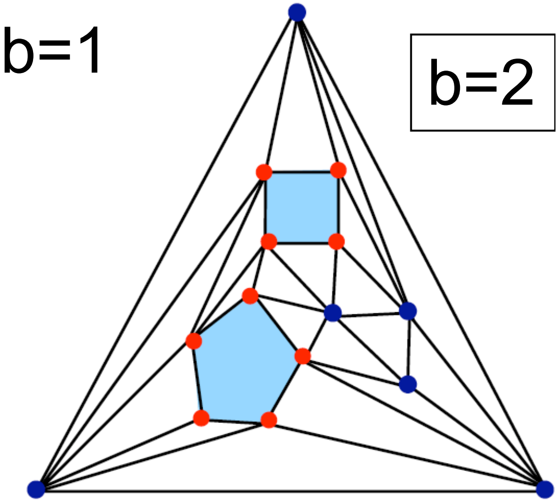
• Without loops or multi-edges, formula only for b=1

- Using the recursive method [Brown'64]
- bijective proof in [Bernardi, F'10]



$$t_n^{(k)} = \frac{2(2k-3)!}{(k-1)!(k-3)!}\,\frac{(4n+2k-5)!}{n!(3n+2k-3)!}$$

Our main result

- **The topology** $\tau = (g, b)$ **is fixed**
- S^τ := **the surface of topology** $\tau = (g, b)$
- $\mathcal{T}_{n,k}^{(\tau)}$:= **the set of triangulations on** S^τ **with**
  - k **vertices on the boundary**
  - n **vertices not on the boundary**

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Then we have a (quasi-linear) encoder such that the length $\ell_{n,k}$ **of the coding word satisfies, as** $n + k \to \infty$**:**

$$\ell_{n,k} \sim \log_2(|\mathcal{T}_{n,k}^{(\tau)}|) \sim 2k + \log_2 \binom{4n+2k}{n}$$

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- When b=0 (no boundary, k=0):

$$\log_2(T_n^{(g)}) \sim \log_2(2^8/3^3) \cdot n \approx 3.245 \cdot n$$

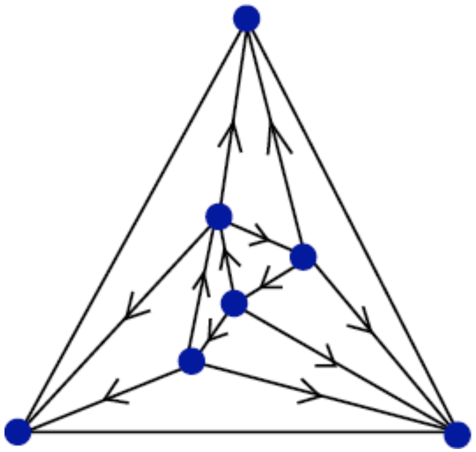
To be compared with ''Edgebreaker'': 4 bits/vertex in worst case

Part 2: Bijective encoding of maps using orientations

Orientations for map families

- Many map families are characterised by the existence of an orientation with prescribed outdegrees [Schnyder'89, Propp, Ossona de Mendez-de Fraysseix'01, Felsner'03,…]:

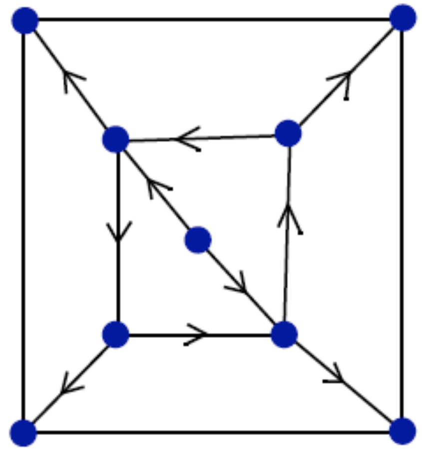
Triangulations



Outdegree = 3

$$|\mathcal{T}_n| = \frac{2(4n-3)!}{n!(3n-2)!}$$

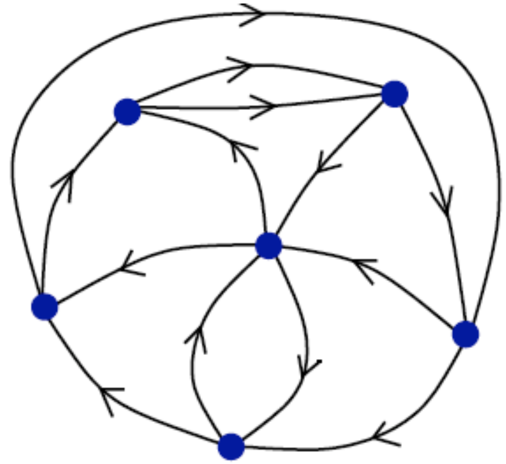
Quadrangulations



Outdegree = 2

$$|\mathcal{Q}_n| = \frac{2(3n-3)!}{n!(2n-2)!}$$

Eulerian



Out(v)=deg(v)/2

$$|\mathcal{E}_n| = \frac{2\cdot 3^n (2n)!}{n!(n+2)!}$$

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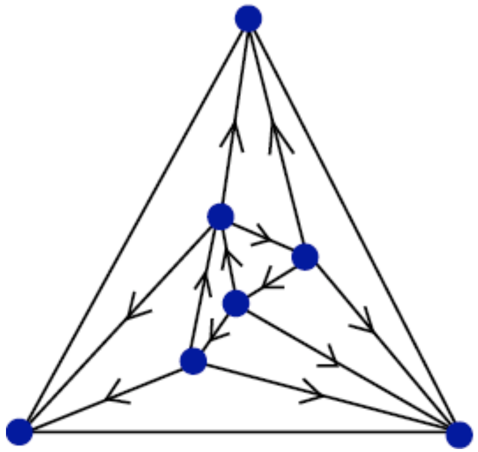
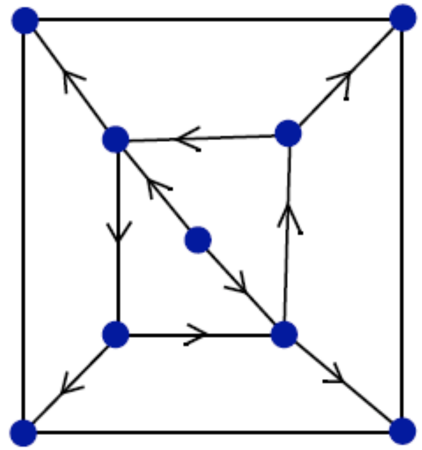
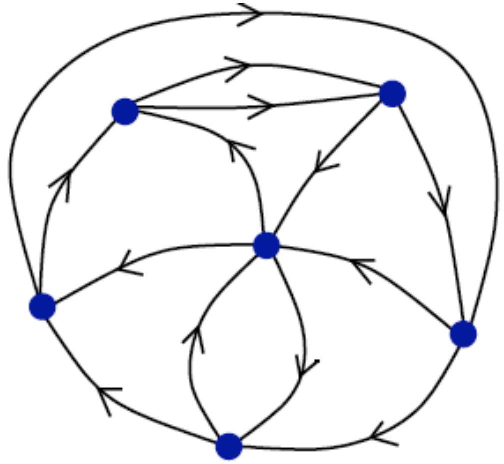
Triangulations	Quadrangulations	Eulerian
		
Outdegree = 3	Outdegree = 2	Out(v)=deg(v)/2
$\lvert\mathcal{T}_n\rvert = \frac{2(4n-3)!}{n!(3n-2)!}$	$\lvert\mathcal{Q}_n\rvert = \frac{2(3n-3)!}{n!(2n-2)!}$	$\lvert\mathcal{E}_n\rvert = \frac{2\cdot 3^n(2n)!}{n!(n+2)!}$

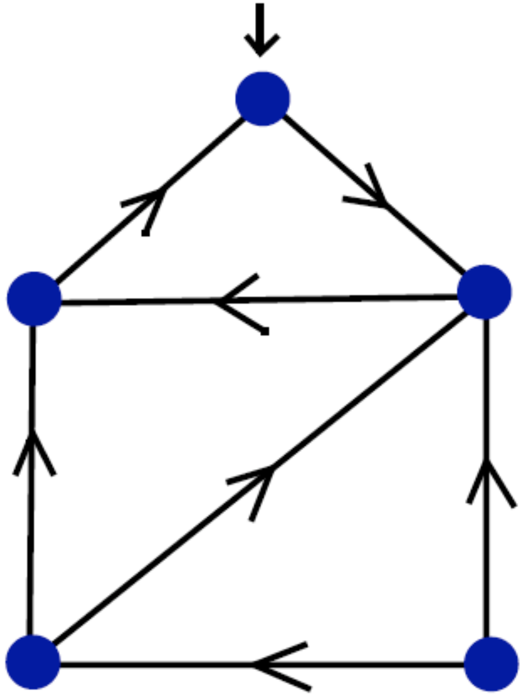
The bijective method [Poulalhon-Schaeffer'03, Bernardi'06] :

- Each map has a unique such orientation with no ccw circuit
- This orientation yields a ''canonical'' spanning tree
- The spanning tree (+decorations) is in a specific countable family

Orientation -> canonical spanning tree

[Poulalhon-Schaeffer'03, Bernardi'06]:

- Let O be an orientation with a marked corner (the root) s.t:
  - every vertex has a path to the root (accessibility)
  - there is no counterclockwise circuit (minimality)



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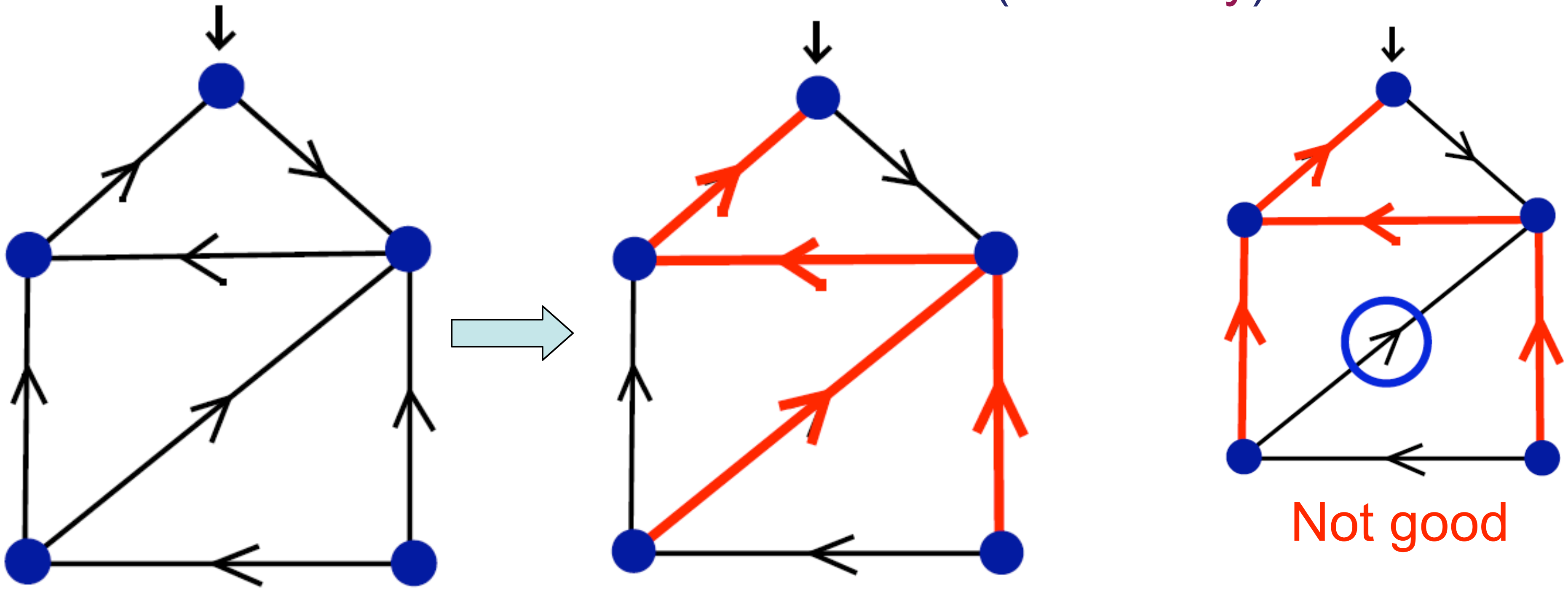


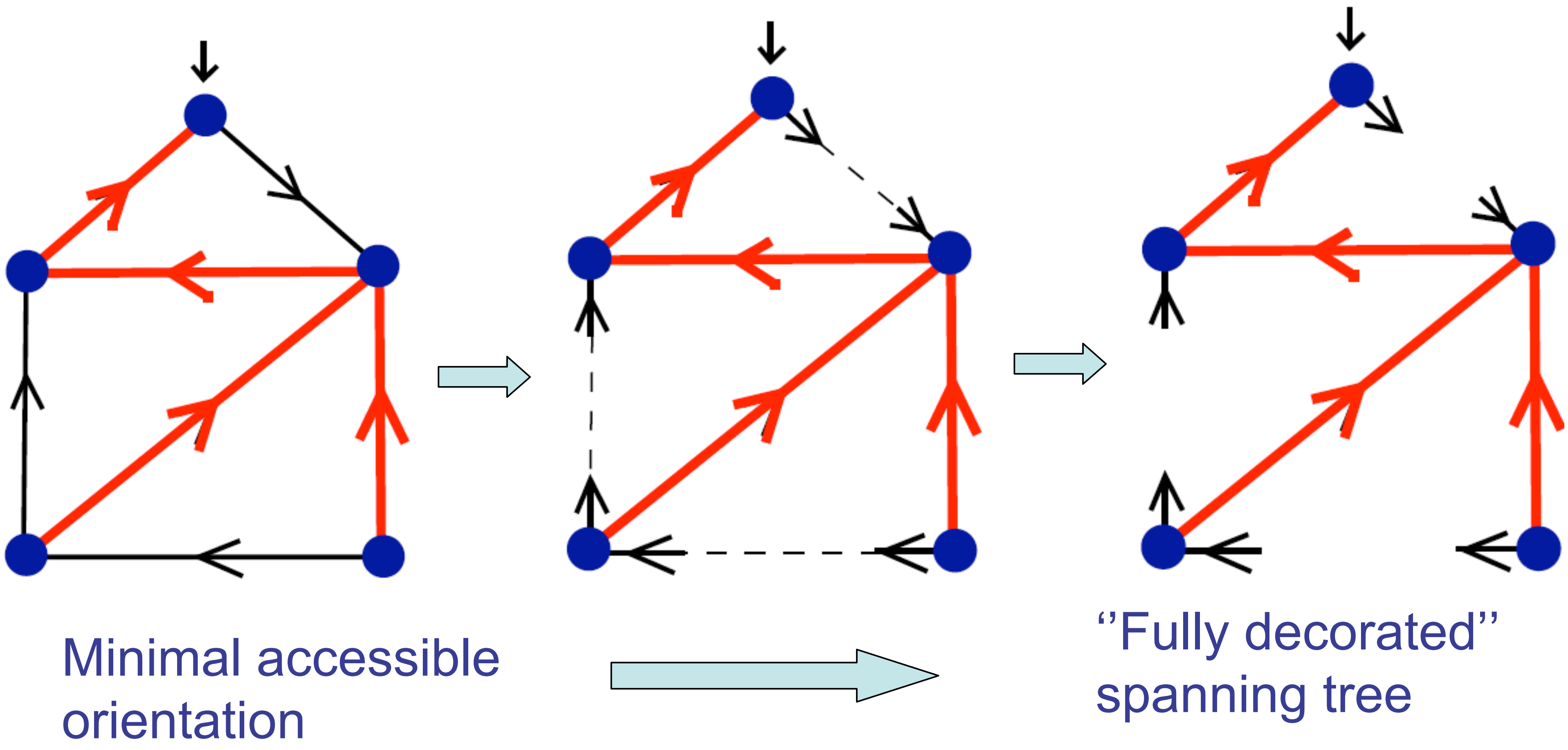
Then O has a unique spanning tree T such that:

- T is oriented to the root
- Every edge e of O\T is clockwise on the unique cycle of T+e

This ''canonical'' spanning tree is computed by a d.f.s. traversal

Encoding using the canonical spanning tree

Crucial obervation: no loss of information when cutting at their middle the edges that are not in the canonical spanning tree



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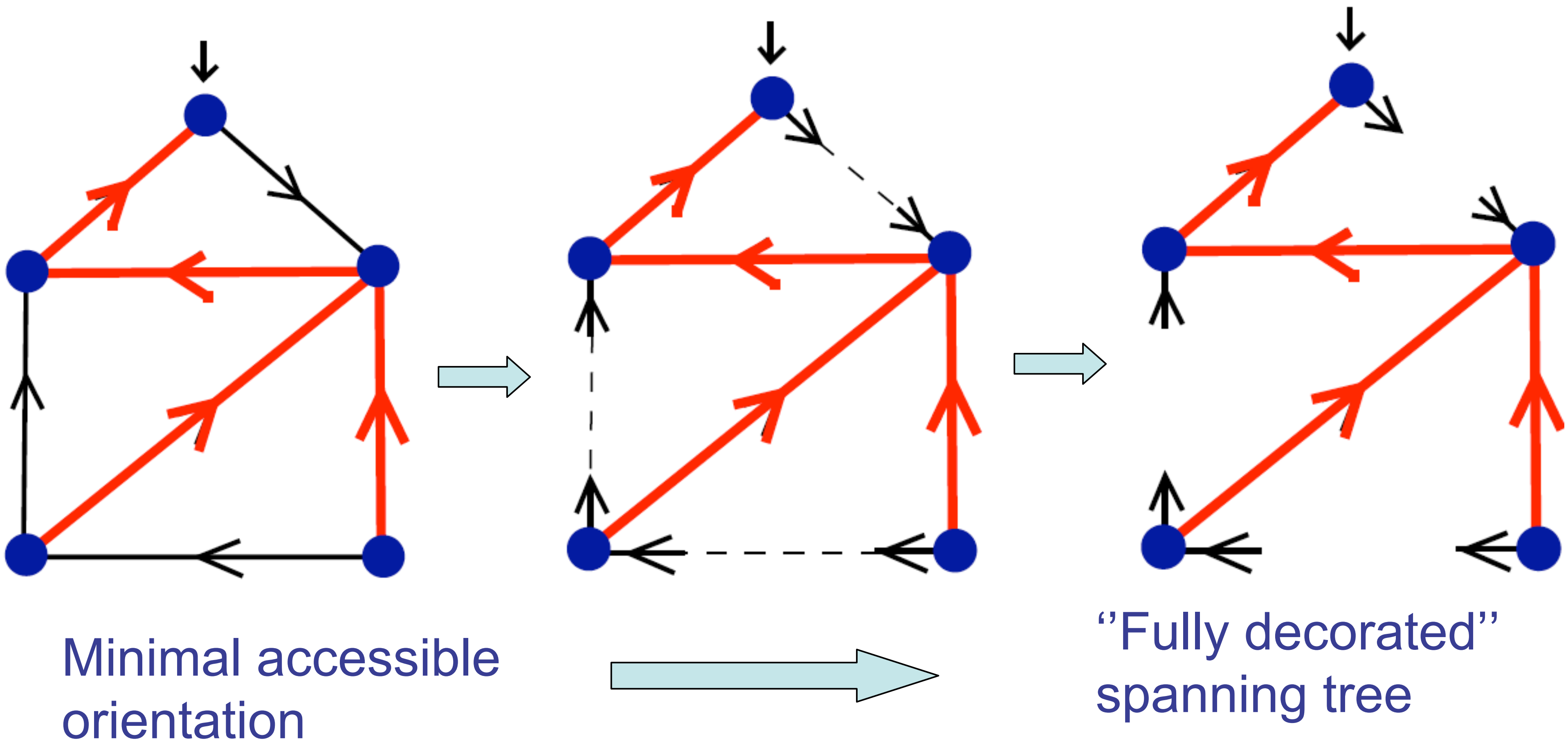
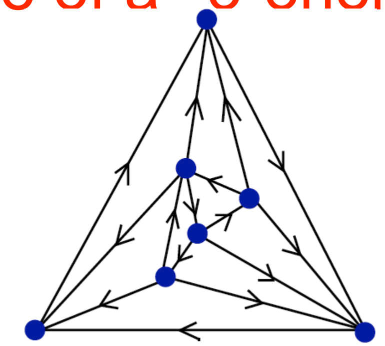


The fully decorated spanning tree encodes the planar orientation

Illustration on triangulations (g=0, b=0)

- [De Fraysseix-Pollack-Pach, Schnyder'90]: simple triangulations are characterised by the existence of a ''3-orientation'', that is,
  - outer vertices have outdegree 1
  - inner vertices have outdegree 3



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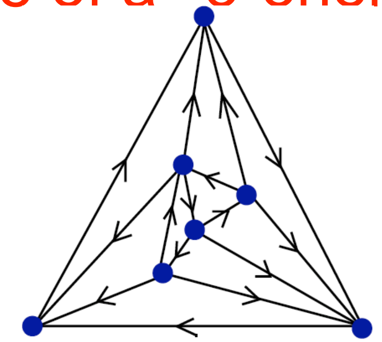
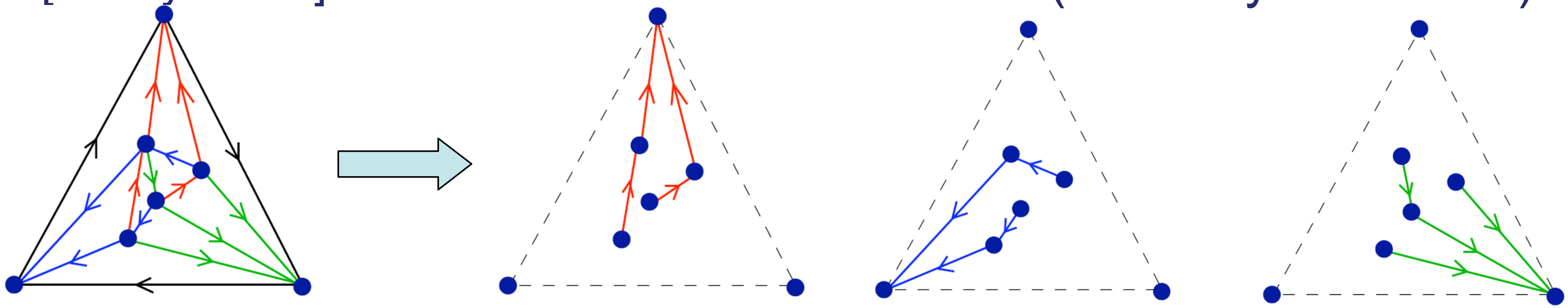


- [Schnyder'90]: 3-orientations are accessible (cf Schnyder woods)



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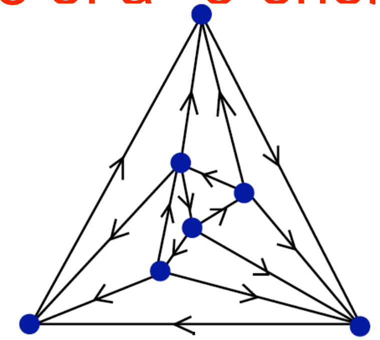

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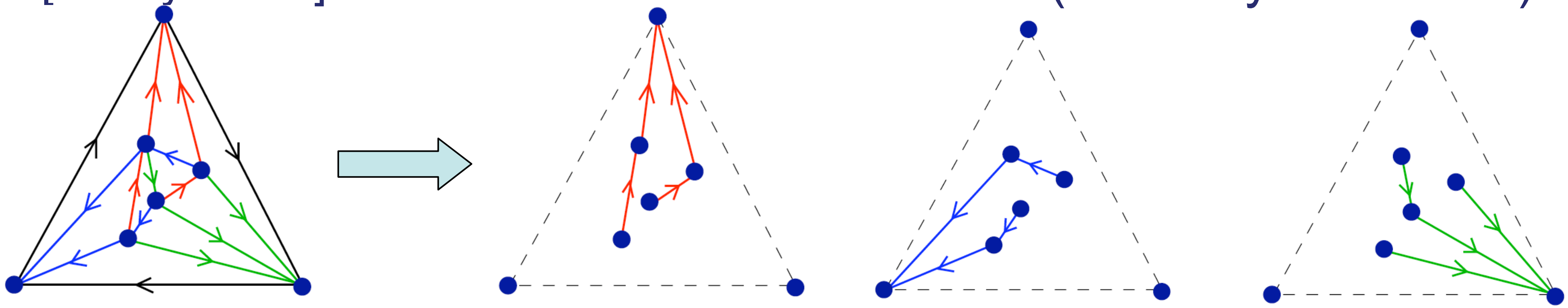
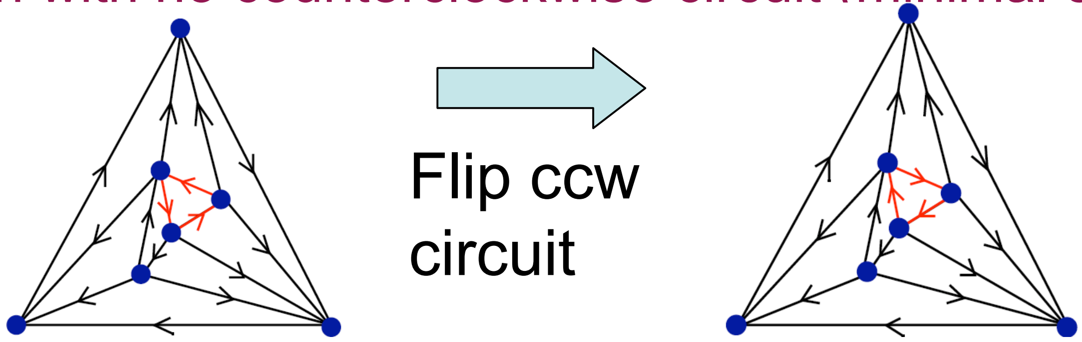


- [Ossona de Mendez'94, Propp, Felsner]: any triangulation has a unique 3-orientation with no counterclockwise circuit (minimal orientation)



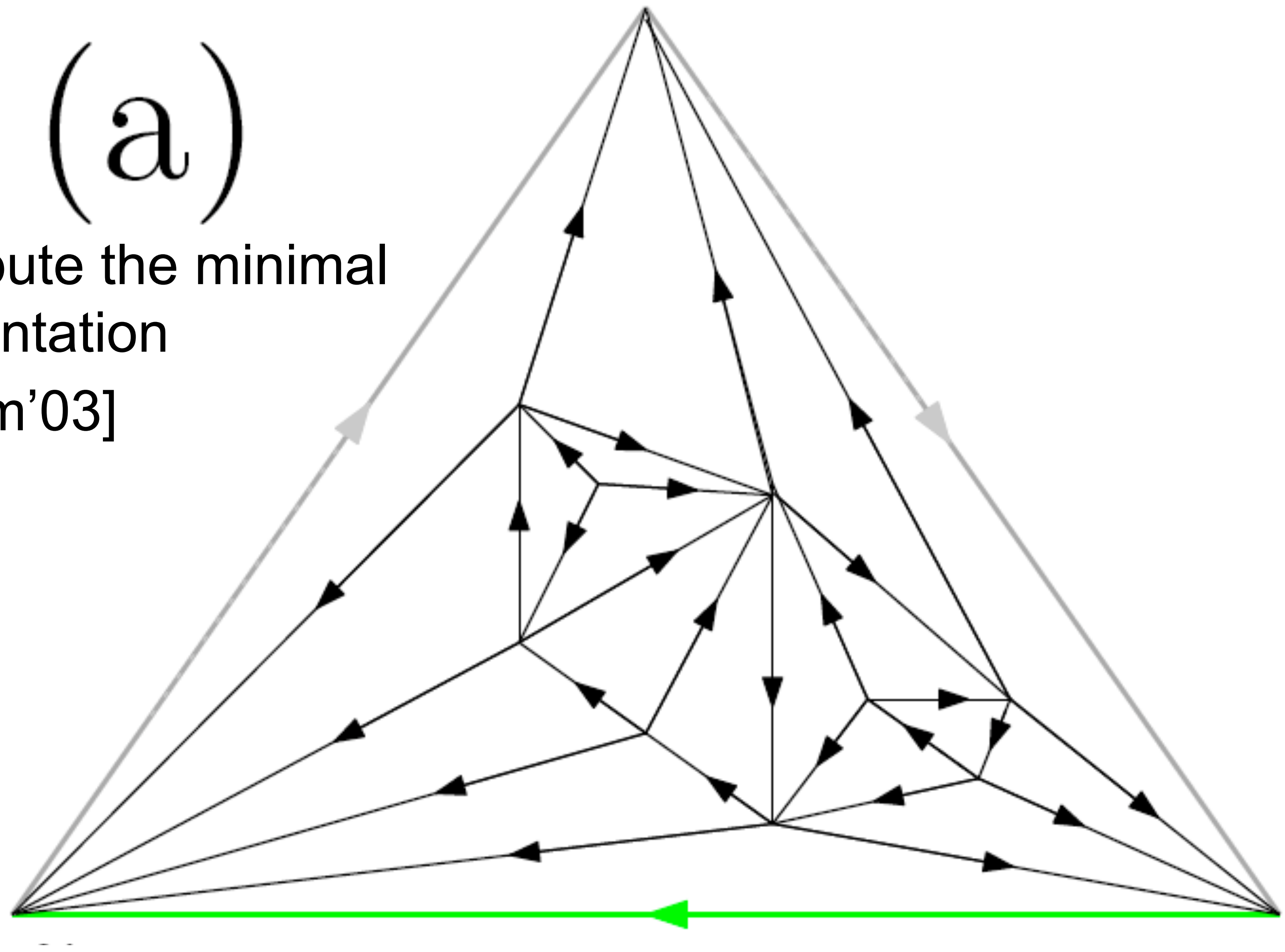
Encoding a triangulation (g=0, b=0)

[Poulalhon-Schaeffer’03] :

(a)

Compute the minimal 3-orientation

[Brehm’03]

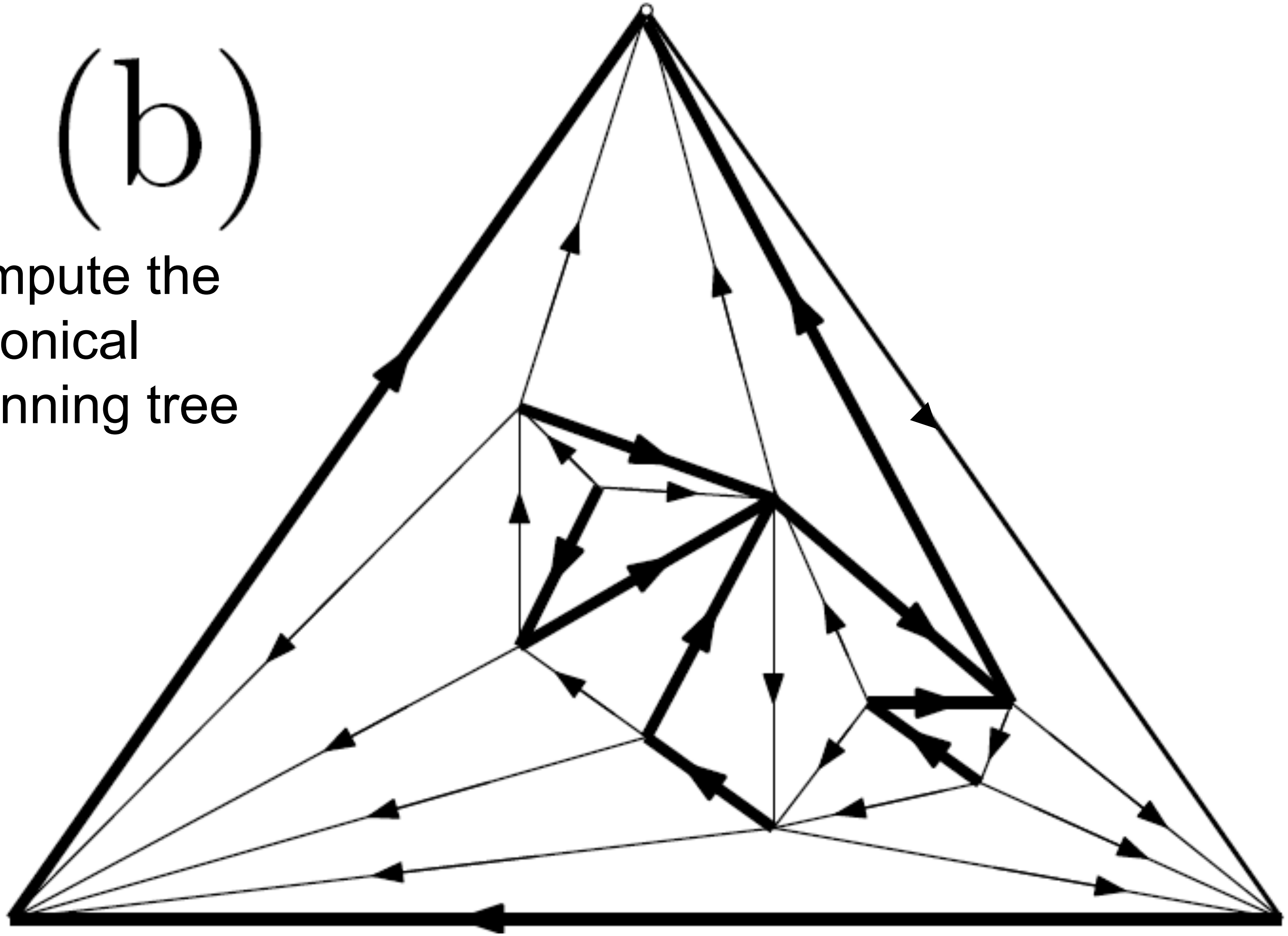


Encoding a triangulation (g=0, b=0)

[Poulalhon-Schaeffer'03] :

(b)

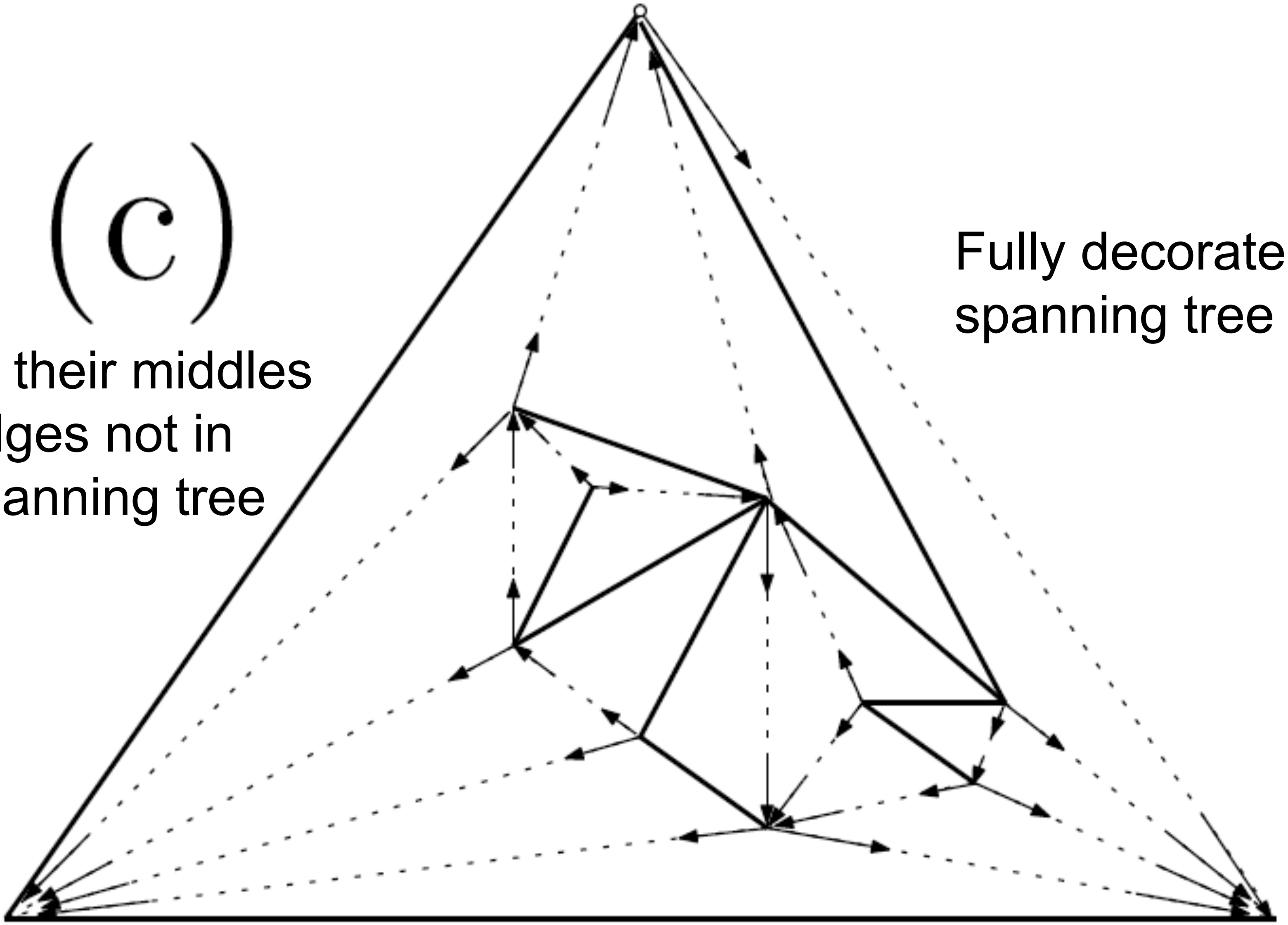
Compute the canonical spanning tree



Encoding a triangulation (g=0, b=0)

[Poulalhon-Schaeffer'03] :

Cut at their middles the edges not in the spanning tree



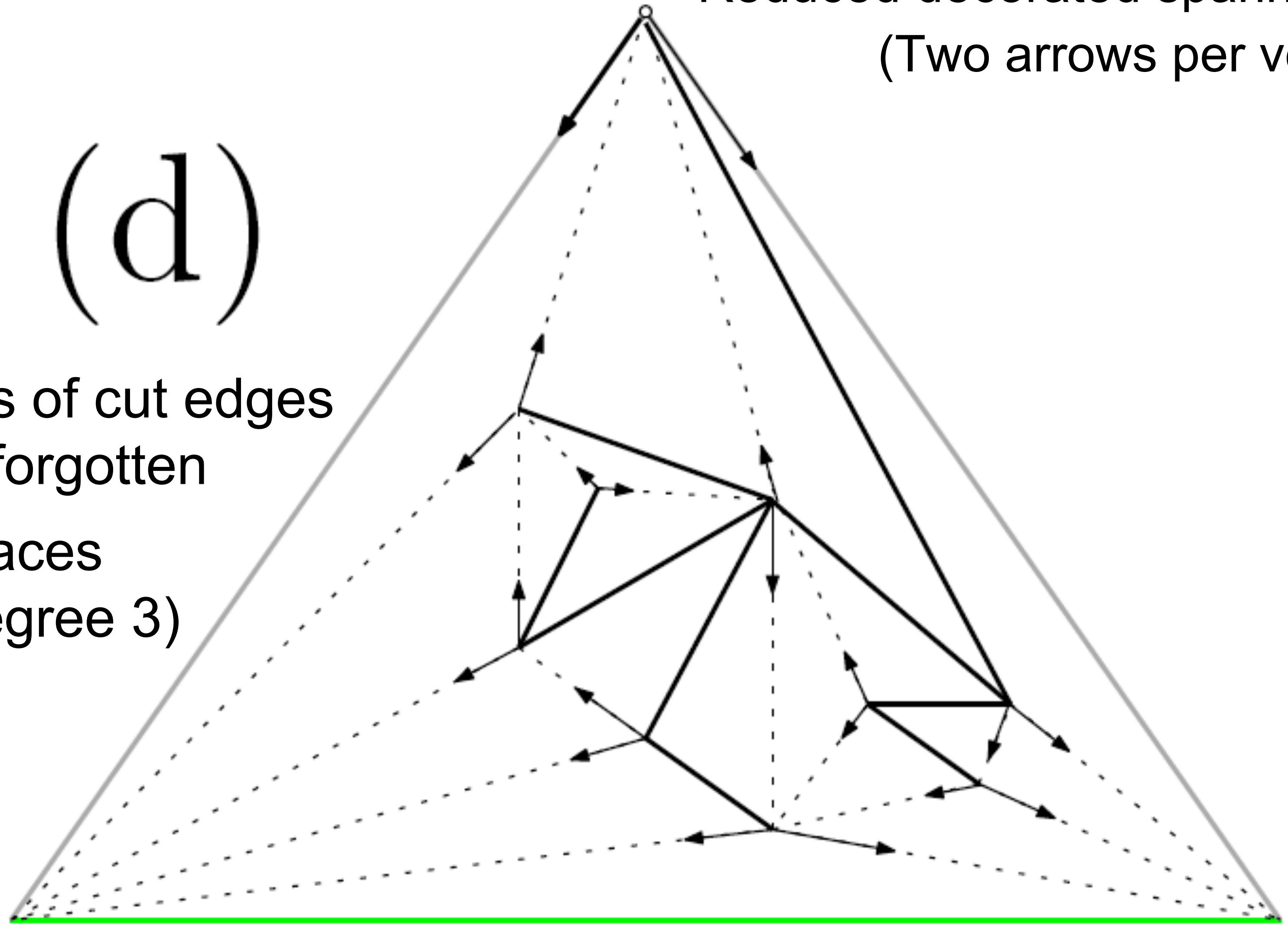
Fully decorated spanning tree

Encoding a triangulation (g=0, b=0)

[Poulalhon-Schaeffer'03] :

Reduced decorated spanning tree

(Two arrows per vertex)



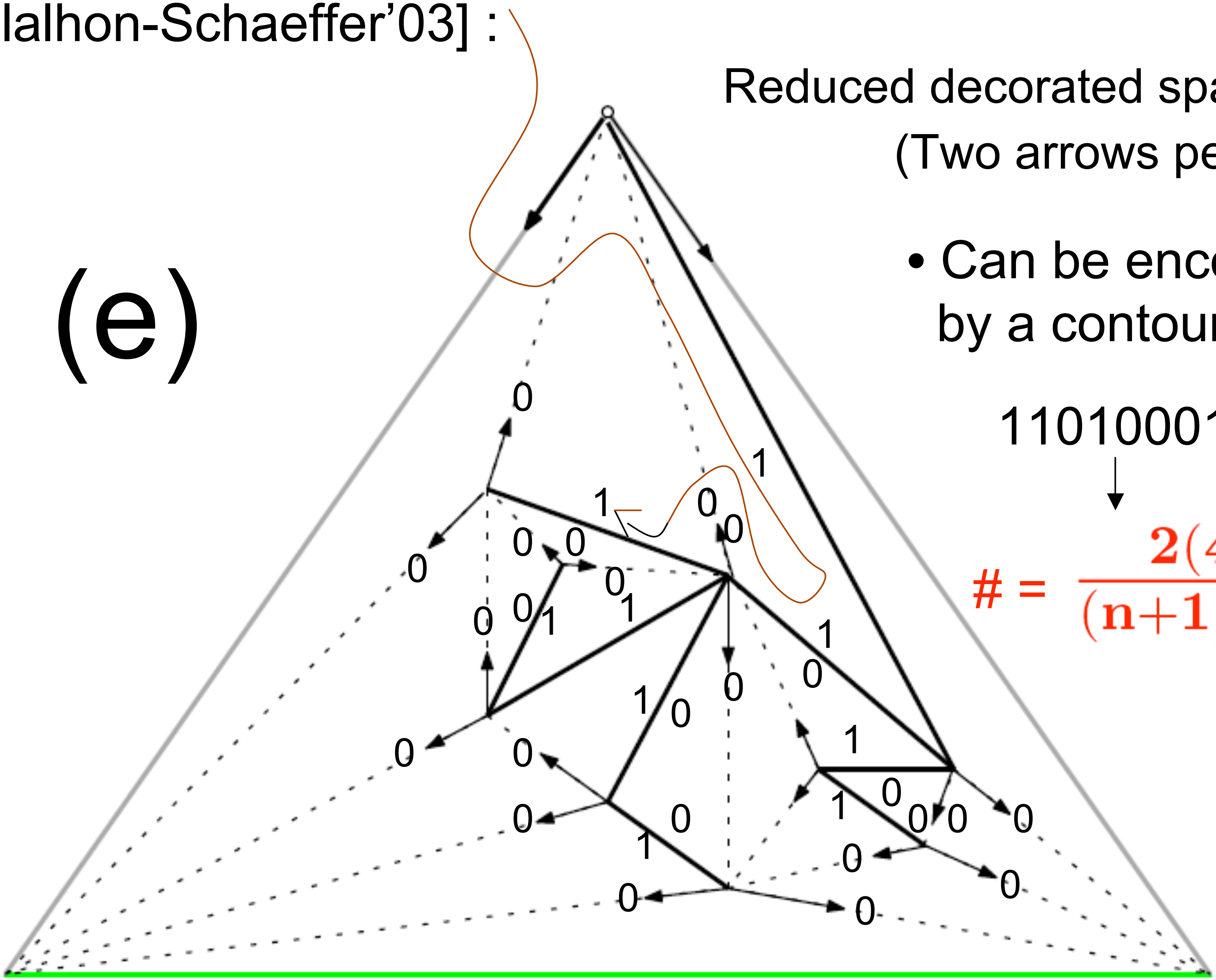
The tails of cut edges can be forgotten

(since faces have degree 3)

Encoding a triangulation (g=0, b=0)

[Poulalhon-Schaeffer'03] :

(e)



Reduced decorated spanning tree

(Two arrows per vertex)

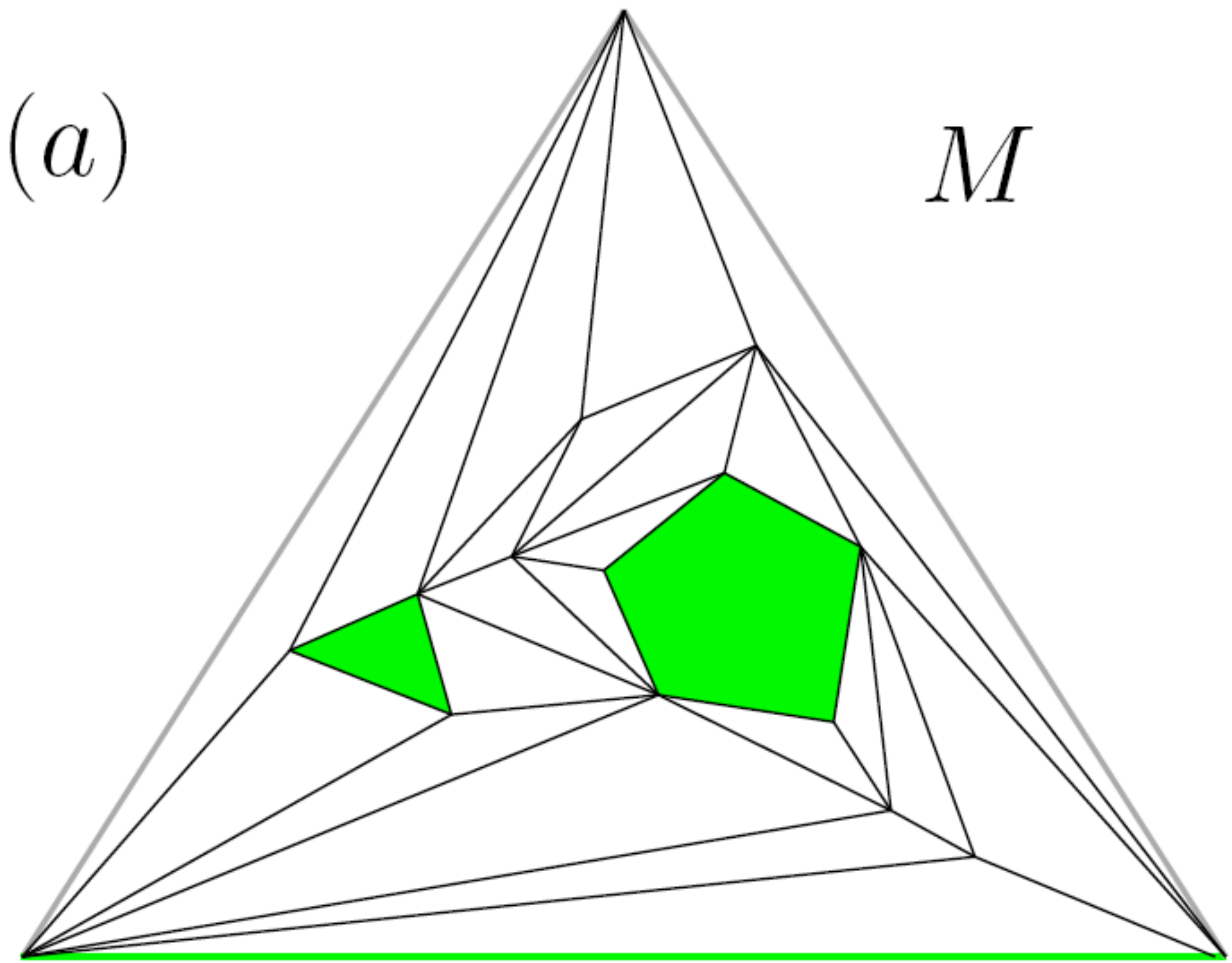
- Can be encoded by a contour word

1101000110000…

$$\# = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

Dealing with boundaries (g=0, b>0)

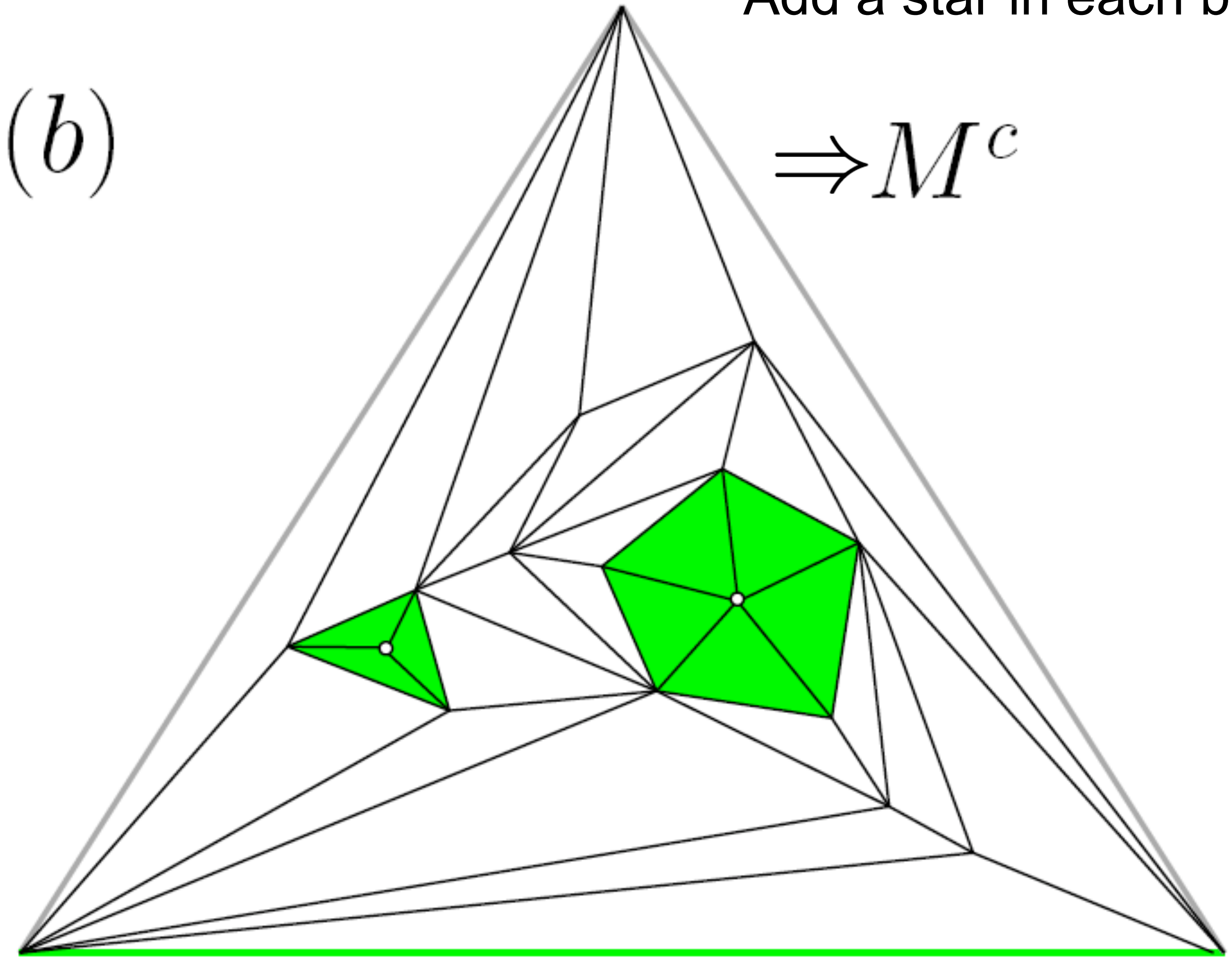
[Castelli,F,Lewiner'10]



Dealing with boundaries (g=0, b>0)

[Castelli,F,Lewiner'10]

Add a star in each boundary-face



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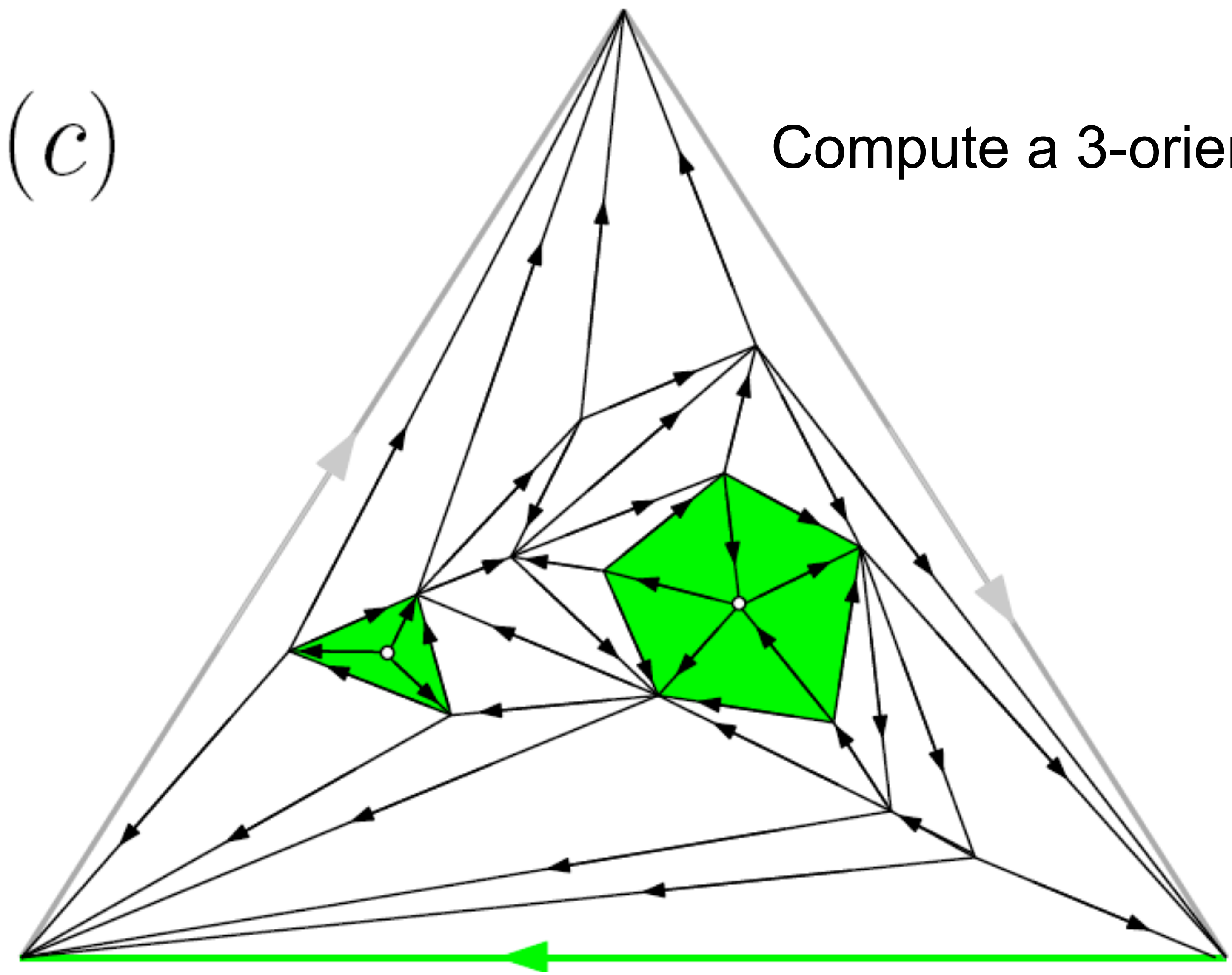
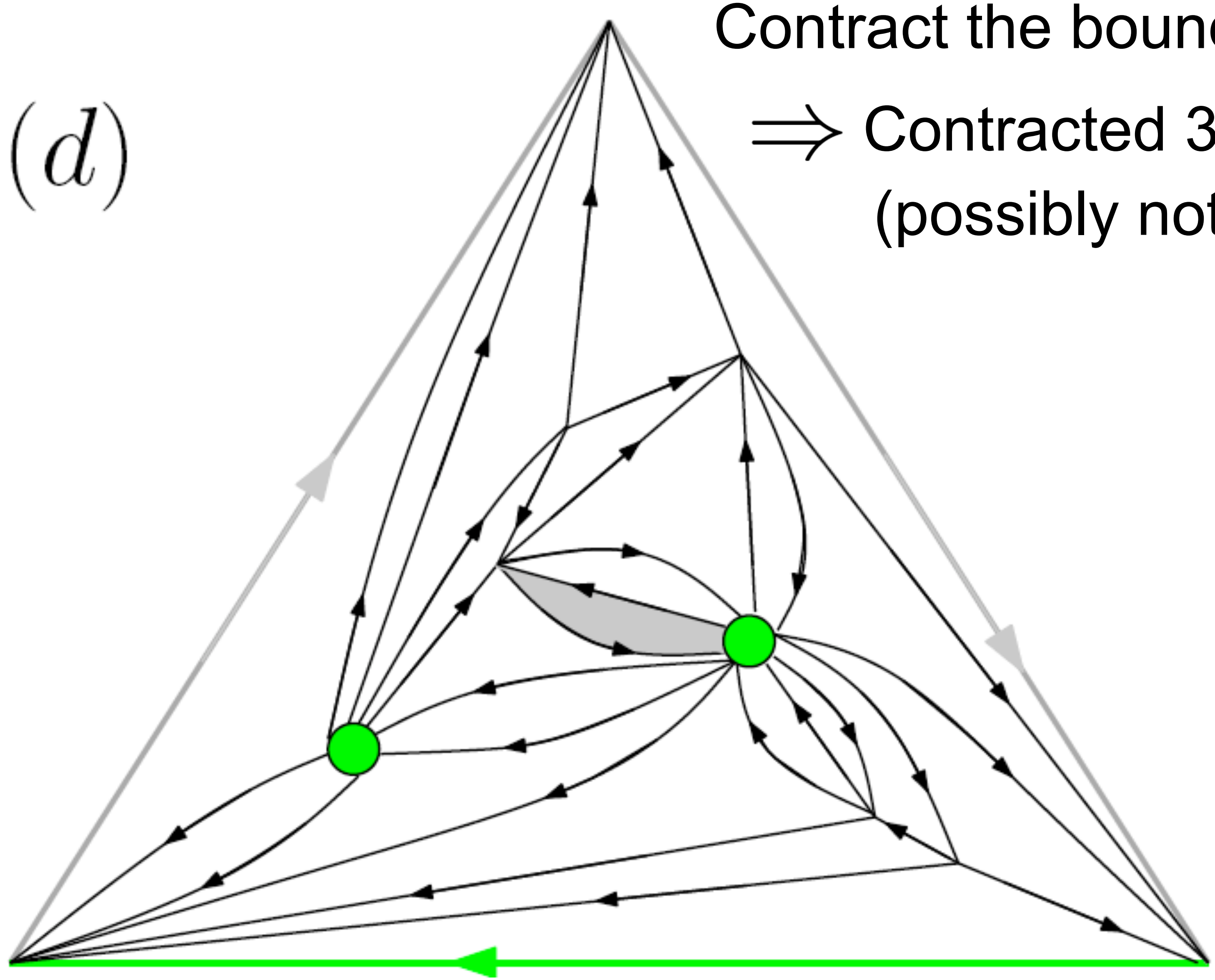


Compute a 3-orientation of M^c

Dealing with boundaries (g=0, b>0)

[Castelli,F,Lewiner'10]

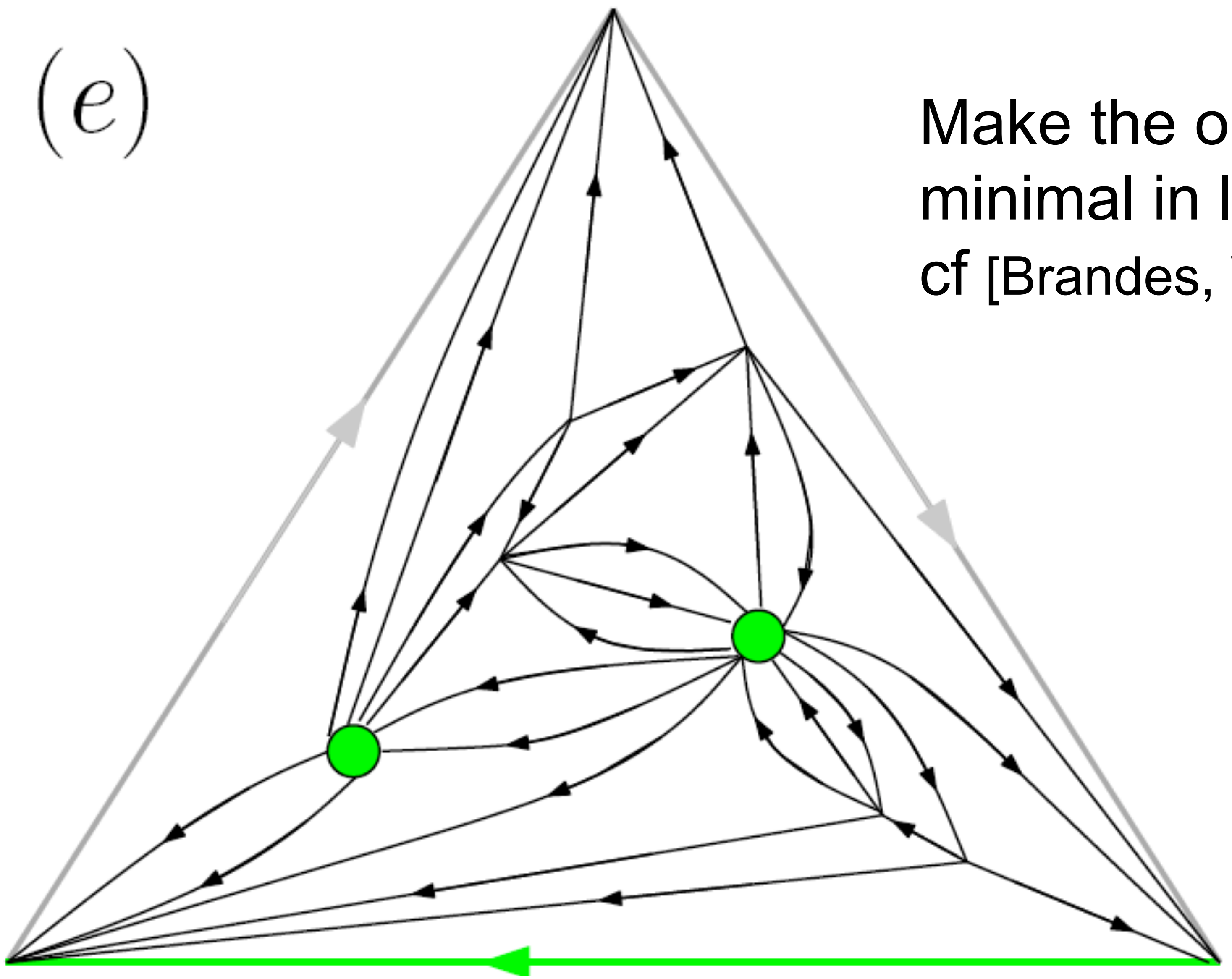


Contract the boundary-faces

$\Longrightarrow$ Contracted 3-orientation (possibly not minimal)

Dealing with boundaries (g=0, b>0)

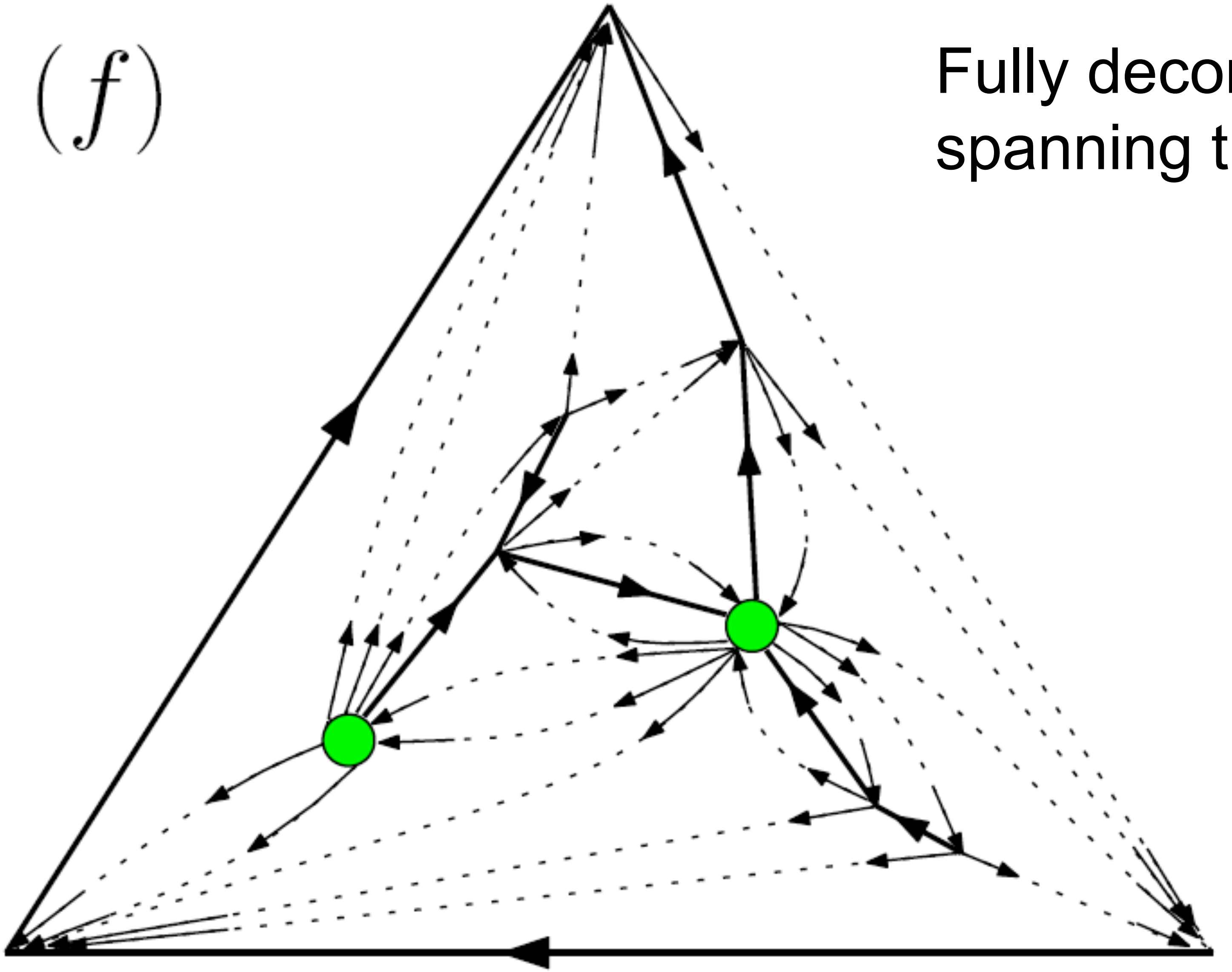
[Castelli,F,Lewiner'10]



Make the orientation minimal in linear time cf [Brandes, Wagner'00]

Dealing with boundaries (g=0, b>0)

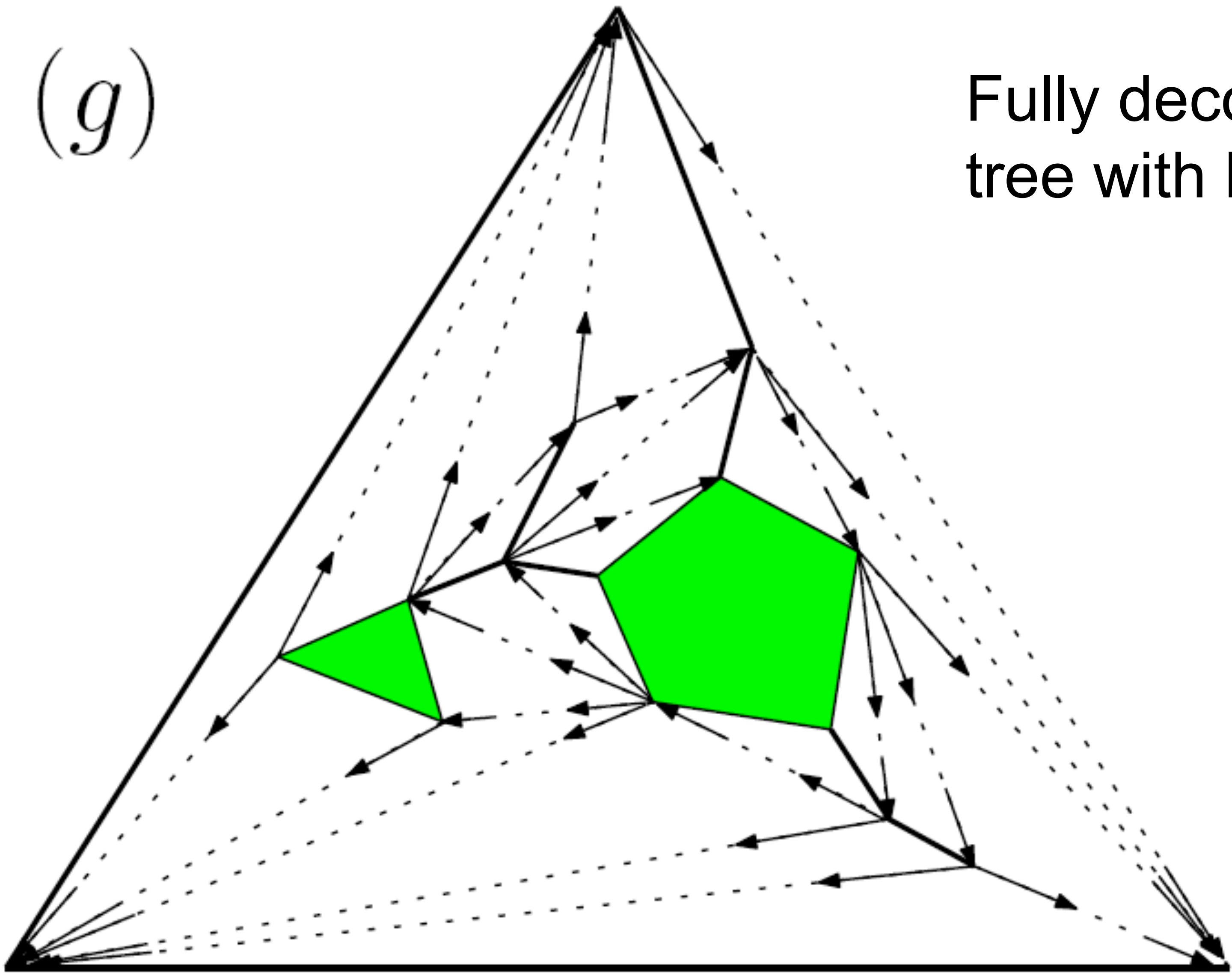
[Castelli,F,Lewiner'10]



Fully decorated spanning tree

Dealing with boundaries (g=0, b>0)

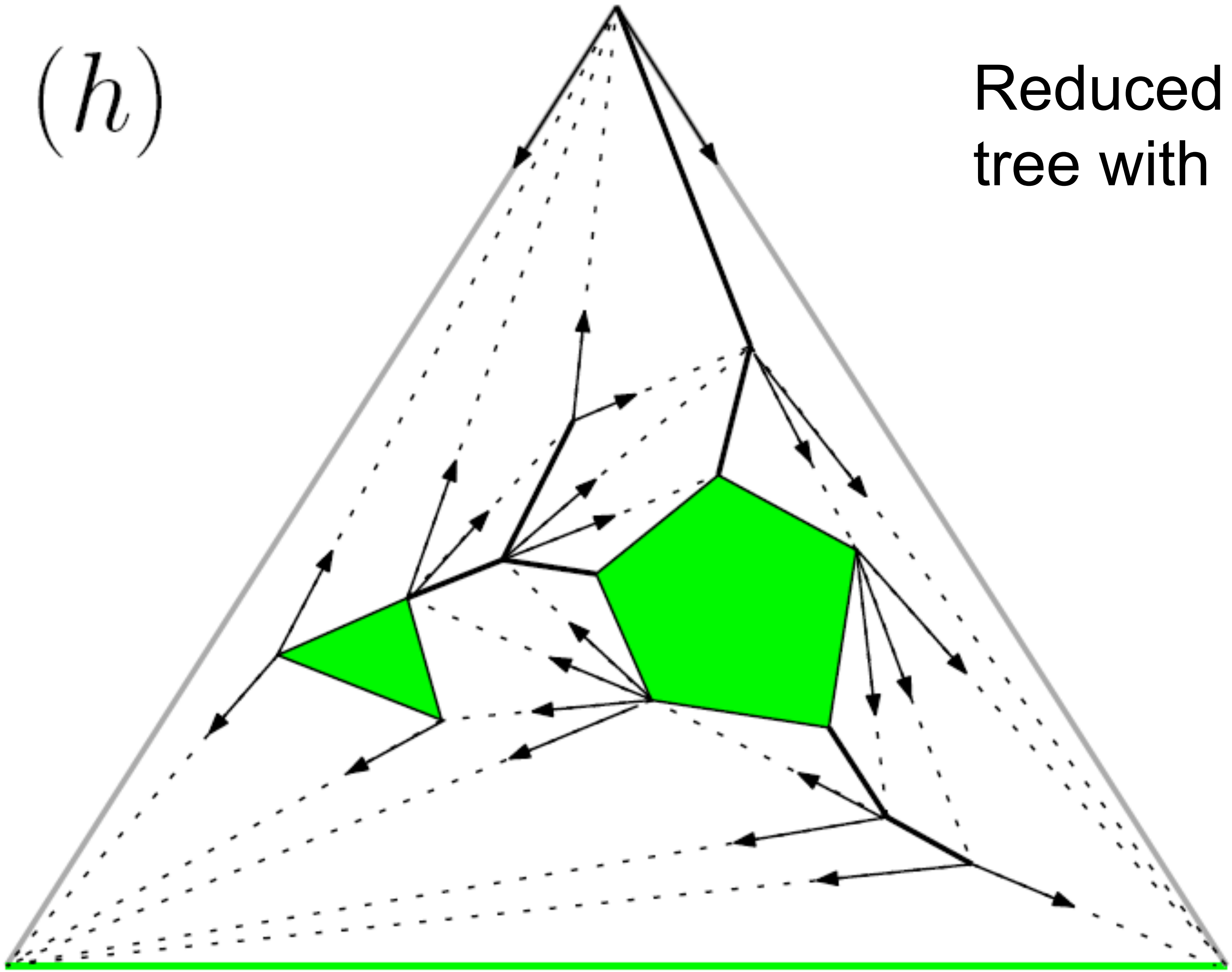
[Castelli,F,Lewiner'10]



Fully decorated
tree with boundaries

Dealing with boundaries (g=0, b>0)

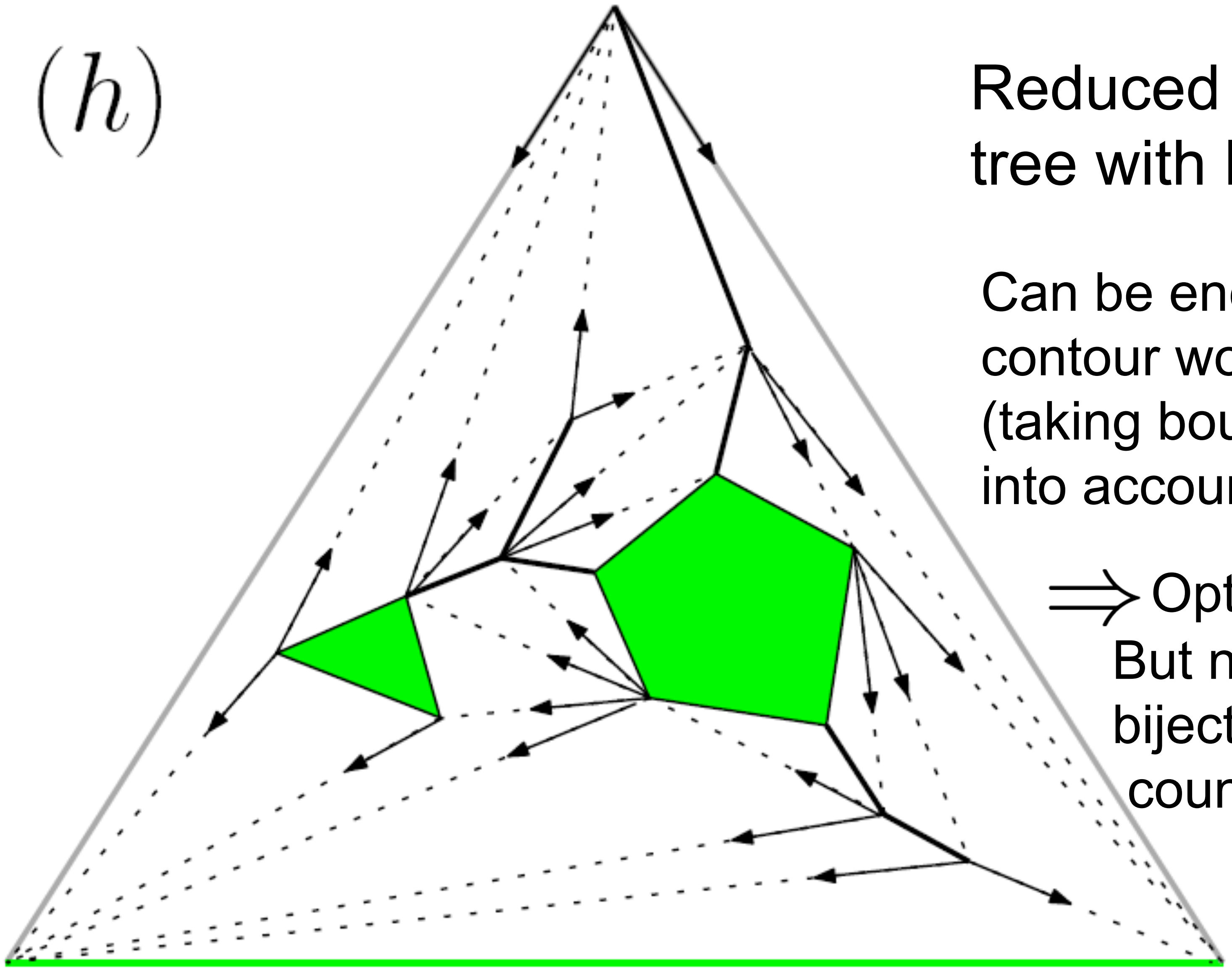
[Castelli,F,Lewiner'10]



Reduced decorated tree with boundaries

Dealing with boundaries (g=0, b>0)

[Castelli,F,Lewiner'10]



Reduced decorated tree with boundaries

Can be encoded by contour word (taking boundary faces into account)

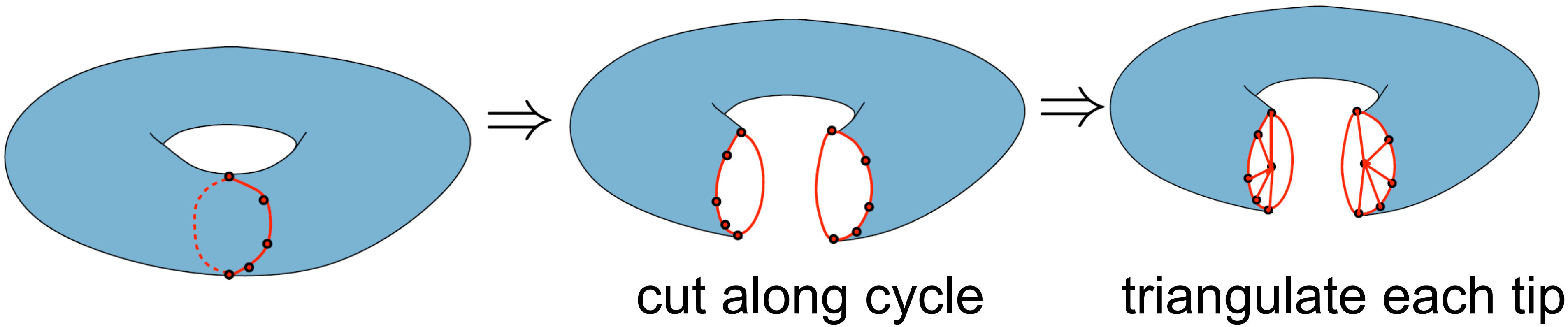
$\Longrightarrow$ Optimal encoding
But not a ''perfect'' bijection (gives no counting formula)

Dealing with higher genus (g>0)

Reduces to the planar case using results of graphs on surfaces:

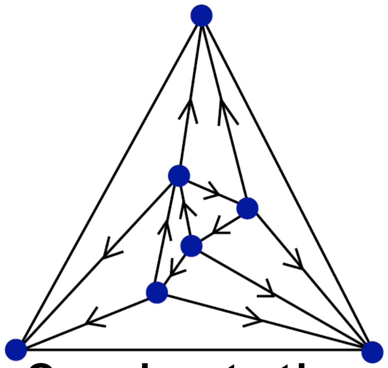
- there is a non-contractible cycle meeting the triangulation in at most $\sqrt{(2n+2k)}$ vertices, cf [Mc Diarmid'08]

- the shortest non-contractible cycle can be computed in time $O_g(n \log(n))$ [Kutz'06, Cabello-Chambers'07]

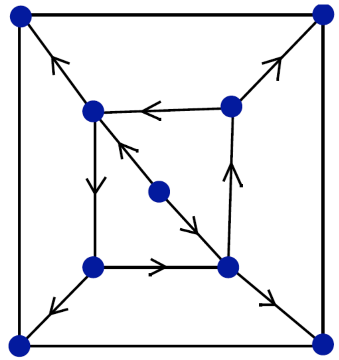


Other results, extensions

- The same approach works also for bipartite quadrangulations



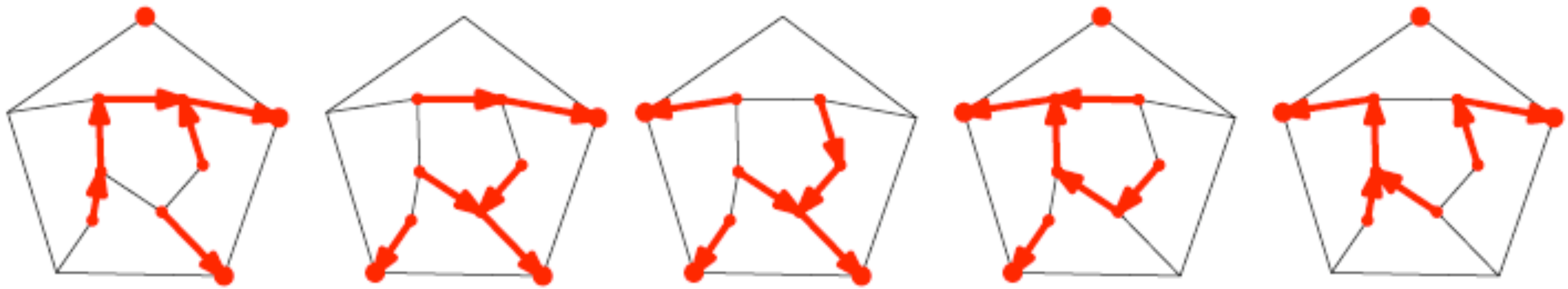
3-orientation



2-orientation

- Recent extension in [Bernardi F'10] in terms of the girth
  girth of a graph = length of shortest cycle

Bijective counting of planar d-angulations of girth d=3,4,5,…
relies on structures/orientations generalising Schnyder woods



d=5: d-angulation with d spanning trees, each edge in d-2 trees