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Sujet :

Combinatoire des cartes planaires par méta-bijection

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Combinatoire des cartes planaires par méta-bijection

Résumé. Nous présentons dans ce document un cadre bijectif pour les cartes planaires (graphes plongés sur la sphère à déformation continue près). Notre méthode repose sur une 'méta-bijection' entre certaines cartes (planaires) orientées et certaines structures d'arbres appelées mobiles (bourgeonnants). Plusieurs familles de cartes planaires peuvent se caractériser par l'existence d'une certaine orientation 'canonique' et peuvent ainsi se ramener à une spécialisation de notre méta-bijection. Nous illustrons la méthode en mettant l'accent sur des familles de cartes (cartes simples biparties, quadrangulations et triangulations à bords) où les coefficients ont une forme simple factorisée, comme dans la formule des slicings de Tutte pour les cartes biparties. Nous discutons ensuite des extensions et perspectives dans le chapitre de conclusion.

Mots-clés : cartes planaires, cartes à bords, combinatoire énumérative, bijections, limite d'échelle

A master bijection method for planar maps

Abstract. In this document we present a general bijective framework for planar maps (connected graphs embedded on the sphere up to continuous deformation). Our method relies on a so-called 'master bijection' between certain oriented (planar) maps and certain tree-structures called (blossoming) mobiles; several families of maps can be shown to be characterized by certain 'canonical' orientations, making them amenable to a specialization of the master bijection. We demonstrate the method with an emphasis on some families of maps (simple bipartite maps, quadrangulations and triangulations with boundaries) where a simple factorized multivariate formula occur for the coefficients, as in Tutte's slicings formula for bipartite maps. Extensions and perspectives are then discussed in the concluding chapter.

Keywords: planar maps, maps with boundaries, enumerative combinatorics, bijections, scaling limit

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Contents

CHAPTER 1

Introduction

1.1. Definitions of maps and some properties

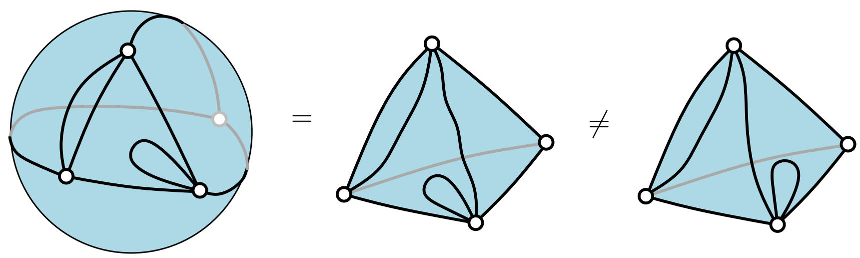


FIGURE 1.1. Left: a map. Middle: the same map, under a different (homeomorphic) representation. Right: A different map (with the same underlying graph), a loop has been moved to a different face, which has changed the combinatorial incidences.

1.1.1. Definitions. This whole document will be devoted to the combinatorial study (via bijective constructions) of objects called planar maps, which we will shortly call maps [1]. A *map* is defined as a crossing-free drawing of a connected graph G (possibly with loops and multiple edges) on the oriented sphere Σ, considered up to continuous deformation (equivalently, up to orientation-preserving homeomorphism), see Figure 1.1. A map has vertices and edges (those of the underlying graph), but also faces, which are the connected components of $\Sigma\backslash G$ (since G is planar, all faces are homeomorphic to open topological disks). A *corner* of a map is the angular sector between two consecutive half-edges around a vertex. The *degree* of a vertex v (resp. a face f) is the number of corners incident to v (resp. to f). Even if the definition of maps is given in geometric terms, maps are completely specified by the combinatorial incidences, for instance by listing for each vertex v and for each face f the cyclic list of incident corners in clockwise order around v (resp. around f).

We will consider in this document maps that are marked in several ways. A *plane map* is a map with a distinguished face. Equivalently (seeing the map as projected onto the plane from the distinguished face) a plane map is a crossing-free drawing (considered up to continuous deformation) of a graph in the plane, see Figure 1.2. The distinguished face is called the *outer face*, its degree is called the *outer degree*, and the other faces are called *inner faces*; vertices and edges are called *outer* or *inner* whether they are incident to the outer face or not.

[1] Maps can be defined on surfaces of higher genus, orientable or not. We refer to the book [**71**] for a detailed foundation, to [**68**, Chap.1] for a rigorous and accessible introduction, to [**35**] for a detailed combinatorial study via bijective constructions, and to [**44**, Chap.4] for a proof of the classification theorem for surfaces using maps.

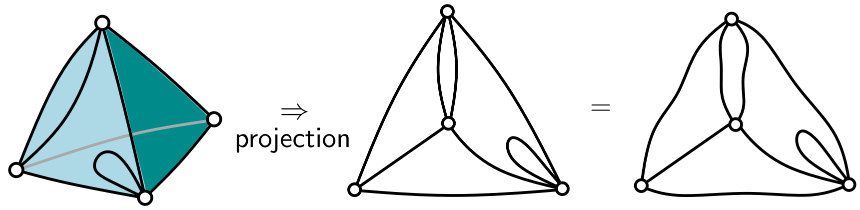


FIGURE 1.2. Left: a plane map, which is obtained from the map M of Figure 1.1 by marking the face f shown darker. Middle: the plane embedding, obtained by projecting M onto the plane from a point inside f. Right: an equivalent plane representation of the plane map.

A *vertex-pointed map* is a map with a distinguished vertex (often called the *pointed vertex*); and a *rooted map* is a map with a distinguished corner (following the convention of Olivier Bernardi [**11**]). The vertex incident to the root is called the *root vertex*, the face incident to the root is called the *root face* (to be taken as the outer face in plane embeddings), and the edge just after the root in clockwise order around the root vertex is called the *root edge*.

1.1.2. Duality. We recall here the notion of duality for maps (the notion works for any surface, orientable or nor, we will use it only in genus 0). For M a map, the *dual map* M^* is the map obtained from M as follows (see Figure 1.3):

(1) insert a vertex v_f in each face f of M,
(2) for each edge e of M, with f_1, f_2 the incident face on each side of e (possibly $f_1 = f_2$), draw a new *dual edge* e^* connecting v_{f_1} and v_{f_2} and crossing e in-between,
(3) erase the vertices and edges of M.

It is easy to check that duality is involutive, the corners of M correspond to the corners of M^*, the vertices of M correspond to the faces of M^* and vice versa, and the edges of M correspond to the edges of M^*. Note also that the dual M^* of a rooted map M is naturally a rooted map (the marked corner of M^* corresponds to the marked corner of M), and the dual of a plane map is naturally a vertex-pointed map and vice versa.

We will also use duality in the context of oriented map. An *oriented map* is a map M where every edge is assigned a direction. In the dual map M^*, the *dual orientation* is the orientation where for each edge $e \in M$ the dual edge e^* of M^* is oriented so as to cross e from its right side to its left side, see Figure 1.3 for an example [2].

1.1.3. The Euler relation. The Euler relation is a classical invariant relation for maps. It states that for any (planar) map M, the respective numbers v, e, f of vertices, edges, and faces satisfy the relation

$$v - e + f = 2. \tag{1}$$

A nice way to visualize this relation is via duality. If $M = (V, E, F)$ is a map with p vertices and q faces, let M^* be the dual map. Endow M with a spanning tree $T = (V, E')$, so that $|E'| = p - 1$. Then, as illustrated in Figure 1.4, the set of edges dual to edges in $E \backslash E'$ forms a spanning tree of M^* (it is acyclic by connectivity of T, and it is connected by

[2]Note that duality for oriented maps is not involutive: the dual of M^* equals M with the orientation reversed; as noted in [**14**] duality for oriented maps can be made involutive by taking the mirror of M^* (we will not use this convention here).

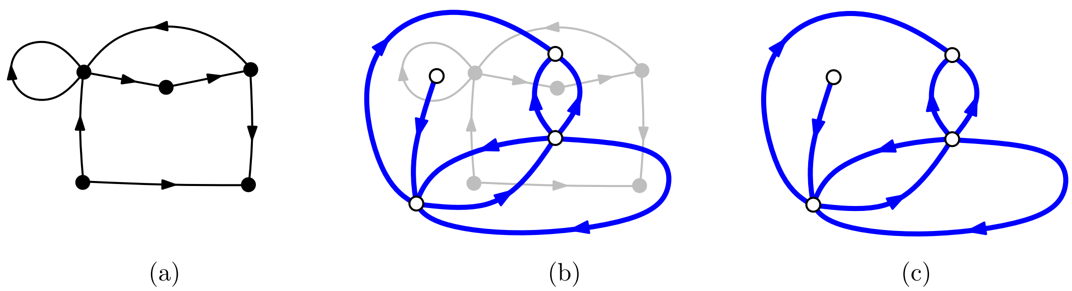


FIGURE 1.3. (a) A map M, endowed with an orientation. (b) Construction of the dual (oriented) map. (c) The dual map M^*, endowed with the dual orientation.

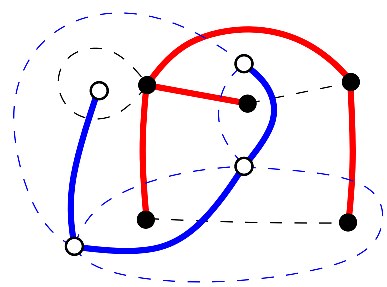


FIGURE 1.4. The map of Figure 1.3 envowed with a spanning tree, superimposed with the dual map endowed with the dual spanning tree (edges are bold if they are in the spanning tree of the map they belong to, and dashed otherwise).

acyclicity of T), hence $|E\backslash E'| = q-1$. We conclude that $n = (p-1)+(q-1) = p+q-2$, which gives (1) for $g=0$.

An application of the Euler relation is to provide a bound for the edge-density of planar graphs in terms of the number of vertices and the girth (the *girth* of a graph is defined as the length of a shortest cycle within the graph; by convention a forest has infinite girth). Let $d \geq 3$, and let G be a connected planar graph of girth d, with V the set of vertices and E the set of edges in G. Consider a planar embedding of G, with F the set of faces. It is easy to see that one can extract a cycle from the contour of each face of G, and since the girth is d, all faces have degree at least d, hence $2|E| \geq d|F|$. Together with $|V|-|E|+|F| = 2$, this gives

$$|E| \leq \frac{d}{d-2}(|V|-2). \tag{2}$$

This bound gives a necessary condition for a graph to be planar and thus provides a tool (when it applies) to show that a graph is non-planar. For instance the complete graph K_5 on 5 vertices has girth 3, and $\binom{5}{2} = 10$ edges, whereas the bound (2) gives at most 9 edges; hence K_5 is not planar. Another example is $K_{3,3}$ the complete bipartite graph on $3+3$ vertices. It has 6 vertices, girth 4 (by bipartiteness), and 9 edges, whereas the bound (2) gives at most 8 edges; hence $K_{3,3}$ is not planar.

1.2. Planar map enumeration

1.2.1. Methods for counting planar maps. The enumeration and combinatorial study of planar maps has been initiated by Tutte in the 60's. In a series of articles [**87, 88, 89**], Tutte proved strikingly simple enumeration formulas for several families of (planar) rooted maps, where the root is useful both to avoid symmetry issues and to give a starting point for a recursive decomposition.

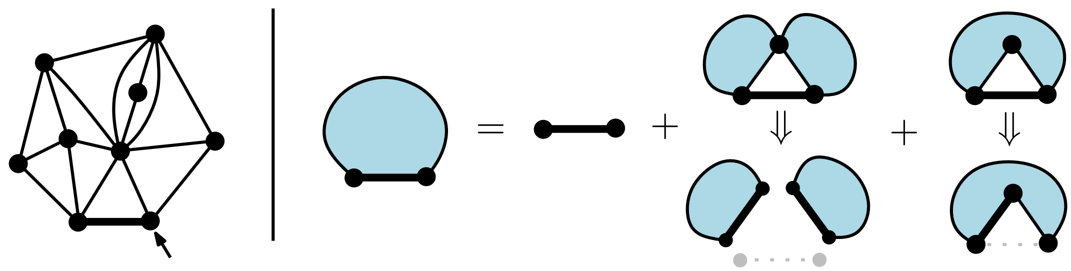


FIGURE 1.5. Left: a rooted loopless triangulation with a boundary of length 6. Right: decomposition of a (generic) rooted loopless triangulation with a boundary (the root edge is shown bolder).

For maps, finding a decomposition often requires to enlarge the family in order for the decomposition pieces to belong to the family. For instance, if one wants to decompose a rooted loopless triangulation, a natural strategy is to delete the root-edge but the obtained object is not a triangulation anymore (it has outer degree 4). One can consider the larger family of rooted loopless triangulations with a (non self-intersecting) boundary of length $k \geq 2$ (with the root incident to the boundary face). Then two cases might occur (see Figure 1.5): either deleting the root-edge yields a pinch point in the boundary, in which case the decomposition yields two (rooted) components, or the boundary is still non self-intersecting (of length $k+1$) after the root edge deletion. Writing $a_{n,k}$ for the number of rooted loopless triangulations of boundary-length $k+2$ and with $n+k$ vertices, and $F(z,u) = 1 + \sum_{n,k} a_{n,k} z^n u^k$ for the corresponding generating function (where the term 1 accounts for the map reduced to a non-loop edge), the decomposition gives [**73**]:

$$F(z,u) = 1 + uF(z,u)^2 + z\frac{F(z,u) - F(z,0)}{u}$$

Note that, in this equation, $f(z) := F(z,0)$ is the series we are looking for (indeed, a rooted loopless triangulation with a boundary of length 2 can be identified with a rooted loopless triangulation, upon squeezing the boundary into an edge), and the additional variable u acts as an auxiliary 'catalytic' variable that helps to write a functional equation (reflecting the fact that we had to enlarge the family and keep track of a secondary parameter, the boundary length). In the beginnings of map enumeration, such equations were solved by a guess-and-check strategy. For instance, in the case of loopless triangulations with a boundary, the equation above makes it possible to compute the coefficients and conjecture the simple formula

$$a_{n,k} = \frac{(2k+1)!}{k!^2} \frac{2^{n+1}(2k+3n)!}{n!(2k+2n+2)!}, \tag{3}$$

which lifts to an expression of $F(z,u)$ as a 'Lagrangean' rational parametrization in terms of two independent auxiliary variables. The conjectured expression of $F(z,u)$ can then be checked to satisfy the functional equation, which (by uniqueness of the solution) gives a proof that the guessed expression was correct.

Since then, systematic methods have been introduced to solve such equations with a catalytic variable, starting with Tutte's quadratic method (see for instance [**55**, 2.9]). The method has been recently generalized [**19**] to arbitrary (polynomial) equations with a catalytic variable, see also the book [**49**] for a more analytic approach applied to the enumeration of maps of arbitrary genus.

Another well-known approach for map enumeration, introduced in [**32**], is by computing certain Gaussian matrix integrals in arbitrary dimension N, which (by the Wick formula) essentially gives the generating function of maps with prescribed face-degrees, arbitrary genus, and weight N for each vertex (which gives an indirect control on the genus g, using the Euler formula in arbitrary genus: $v - e + f = 2 - 2g$). It can easily be seen that the terms of dominating contribution in N are given by planar maps, which gives a way to extract expressions for the generating functions, and recover formulas found using Tutte's method. This method has proved very flexible, making it also possible to count maps equipped with statistical physics models [**59, 17, 23**], see also [**45**] for a detailed review.

Both methods mentioned above made it possible to find surprisingly simple counting formulas (such as (3)) for several families of maps, however via technical calculations that do not give a transparent proof of why such simple formulas occur. More direct bijective proofs have been later achieved, starting in the 80's with constructions by Cori and Vauquelin [**42**] and Arquès [**7, 8**], and subsequently further developed in the PhD thesis of Gilles Schaeffer [**82**]. Typically, a bijective correspondence is established between a certain family of maps (such as loopless triangulations, simple quadrangulations,...) and a certain family of decorated trees, which can be classically enumerated from a contour word encoding, or via the Lagrange inversion formula applied to the associated generating function.

Besides giving a more direct proof of the counting formulas, finding a bijective proof has also the advantage of providing very efficient random generation algorithms for maps [**83**], and more recently it has proved to be a crucial ingredient in order to study the distance properties of random maps [**38, 25, 30**] (in particular the fact that the typical distance between two random vertices is of order $n^{1/4}$), culminating with the recent results [**63, 67**] that the scaling limit of the random quadrangulation with n faces (and other families of maps [**16, 1**]) is the so-called Brownian map, introduced in [**53**].

The bijective method has been successfully applied to several families of maps [**81, 26, 77, 78**],[**T1,T2,A6,A7**], some of which making it possible to count factorizations in the symmetric group [**20, 46**] and to solve statistical physics models on random lattices [**21, 27**], see also the thesis of Jérémie Bouttier [**22**]. However, there was also often a 'guessing' component (to find out the correct family of decorated trees) guided by already known enumeration formulas. It has appeared, starting from the Poulalhon-Schaeffer bijection for simple triangulations [**78**], that the decorated tree is often naturally derived from the map once it is equipped with a suitable 'canonical' orientation, suggesting that finding a bijective construction comes down to finding the right canonical orientations that characterize the maps in the family considered.

In this document we will develop a general bijective method for planar maps, based on a so-called *master bijection*, which itself builds on a bijection (which we will review in Section 1.3.2) for tree-rooted maps due to Olivier Bernardi [**11**] and its subsequent reformulation with Guillaume Chapuy [**14**]. The master bijection itself is a correspondence between a certain family $\mathcal{O}$ of oriented maps (or the extension to the weighted bi-oriented setting, to be defined in the next chapter) and certain tree-structures called *mobiles* (or more generally, *weighted bi-mobiles*) that have been first introduced by Bouttier, Di Francesco and Guitter under a labelled formulation [**26**]. The main point is that, as mentioned above, for several families of planar maps (specified for instance by a restriction on the face, or by a condition of connectivity or girth), the maps from the family can be characterized by the existence of a certain *canonical* orientation (or weighted bi-orientation), and thus the map family can be identified with a certain subfamily of $\mathcal{O}$ on which the master bijection can be specialized.

We choose to demonstrate the method with an emphasis on providing bijective proofs of 'nice' factorized (i.e., summation-free) counting formulas for families of planar maps; we list 4 such formulas in the next section, to be proved in the following chapters.

1.2.2. The case of factorized multivariate formulas. We start recalling Tutte's slicing formula [**88**]. For $\ell_1, \ldots, \ell_r$ positive integers, denote by $\mathcal{A}[\ell_1, \ldots, \ell_r]$ the set of maps whose r faces are numbered as $f_1, \ldots, f_r$ such that $\deg(f_i) = \ell_i$ for $1 \leq i \leq r$, and where each face has a marked corner, see Figure 1.6(a) for an example; note that such maps are bipartite iff all the ℓ_i are even. Note also that r is the number of faces, $e = \sum_i \ell_i/2$ is the number of edges, and $v = e - r + 2$ is the number of vertices. Then Tutte's slicing formula can be formulated as follows:

THEOREM 1 (Tutte [**88**]). *For $a_1, \ldots, a_r$ positive integers,*

$$|\mathcal{A}[2a_1, \ldots, 2a_r]| = \frac{(e-1)!}{v!} \prod_{i=1}^{r} 2a_i \binom{2a_i - 1}{a_i - 1}, \tag{4}$$

with $e = \sum_i a_i$ the number of edges and $v = e - r + 2$ the number of vertices.

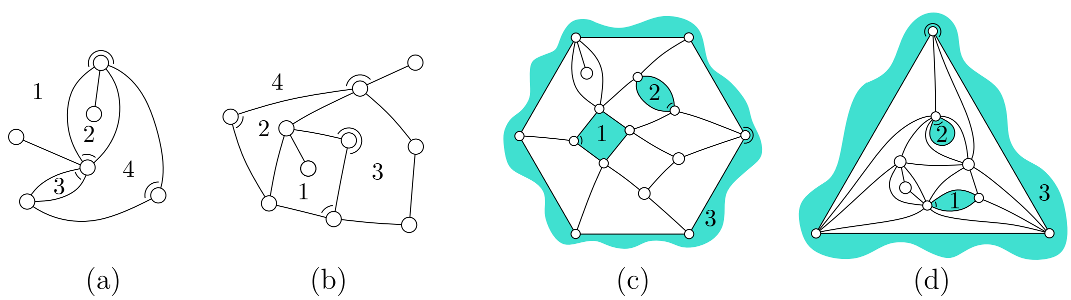


FIGURE 1.6. (a) A bipartite map in $\mathcal{A}[6, 4, 2, 4]$, (b) a simple bipartite map in $\mathcal{S}[6, 4, 6, 8]$, (c) a quadrangulation (with 3 boundaries) in $\mathcal{Q}[3; 4, 2, 6]$, (d) a triangulation (with 3 boundaries) in $\mathcal{T}[3; 2, 1, 3]$.

Tutte's original proof involved a technical guess-and-check solution of a recurrence satisfied by these coefficients. We will recall in Chapter 2 how this formula can alternatively (and more directly) be proved from a bijection of Bouttier, Di Francesco and Guitter [**26**], which itself can be obtained as a specialization of our master bijection, as we will see in Chapter 2 (see also [**14**, Sec.8]). Then, in Chapter 3 we are going to show a similarly looking formula, this time for *simple* bipartite maps (no multiple edges). For $\ell_1, \ldots, \ell_r$ positive integers, denote by $\mathcal{S}[\ell_1, \ldots, \ell_r]$ the subset of maps in $\mathcal{A}[\ell_1, \ldots, \ell_r]$ that are simple (no multiple edges), see Figure 1.6(b) for an example.

THEOREM 2. *For $a_1, \dots, a_r$ positive integers,*

$$|\mathcal{S}[2a_1, \dots, 2a_r]| = \frac{(e+r-3)!}{e!} \prod_{i=1}^{r} 2a_i \binom{2a_i - 1}{a_i - 2}, \tag{5}$$

with $e = \sum_i a_i$ the number of edges.

Then, in Chapter 4 we will explain how to prove two nice factorized formulas for bipartite quadrangulations and for triangulations with boundaries of prescribed lengths (the formula for triangulations has been discovered by Krikun [**60**], the one for bipartite quadrangulations is new to our knowledge). Define a *quadrangulation with boundaries* (resp. a *triangulation with boundaries*) as a map with all faces of degree 4 (resp. of degree 3) except for a set of $r \geq 1$ marked faces called *boundary-faces*, such that the contours of boundary-faces are cycles that are not self-intersecting and pairwise vertex-disjoint. Vertices not incident to a boundary-face are called *internal*. For $m \geq 0$ and $\ell_1, \dots, \ell_r$ positive integers, define $\mathcal{Q}(m; \ell_1, \dots, \ell_r)$ (resp. $\mathcal{T}(m; \ell_1, \dots, \ell_r)$) as the set of quadrangulations (resp. triangulations) with m internal vertices, and where the r boundary-faces are numbered as $f_1, \dots, f_r$, such that for $i \in [1..r]$ (with the notation $[a..b]$ for the set of integers from a to b, including a and b), $\deg(f_i) = \ell_i$ and f_i has a distinguished corner; see Figure 1.6(c)-(d) for examples. Krikun [**60**] has found the following beautiful formula (similarly as Tutte's proof of the slicing formula, his proof relies on a guessing/checking approach, with the checking part requiring technical calculations), where we recall the notation $n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (n - 2i)$:

THEOREM 3 (Krikun [**60**]). *For $m \geq 0$ and $a_1, \dots, a_r$ positive integers,*

$$|\mathcal{T}[m; a_1, \dots, a_r]| = \frac{4^k (e-2)!!}{m!(2b+k)!!} \prod_{i=1}^{r} a_i \binom{2a_i}{a_i}, \tag{6}$$

where $b := \sum_{i=1}^{r} a_i$ is the total boundary length, $k := r + m - 2$, and $e = 2b + 3k$ is the number of edges.

We will give a bijective proof of this formula in Chapter 4, using the master bijection strategy adapted to the context of maps with boundaries. We will actually apply first the strategy to bipartite quadrangulations with boundaries (being bipartite is equivalent to the fact that $\ell_1, \dots, \ell_r$ are all even), for which we will prove a similarly looking factorized formula (which is new to our knowledge):

THEOREM 4. *For $m \geq 0$ and $a_1, \dots, a_r$ positive integers,*

$$|\mathcal{Q}[m; 2a_1, \dots, 2a_r]| = \frac{3^k (e-1)!}{m!(3b+k)!} \prod_{i=1}^{r} 2a_i \binom{3a_i}{a_i}, \tag{7}$$

where $b := \sum_{i=1}^{r} a_i$ is the half-total boundary length, $k := r + m - 2$, and $e = 3b + 2k$ is the number of edges.

To prove these counting formulas ((4) to (7)) from our bijective constructions, we will decompose the corresponding tree-structures (typically a decomposition at the root), and translate the decomposition into an expression for the associated generating function $F(t)$, typically $F(t)$ will be given as the power of a certain generating function $g(t)$, itself specified by an equation of the form $g(t) = \phi(g(t))$. We will then extract an expression for the coefficient, using the Lagrange inversion formula, which we recall here (we use the notation $[t^n]A(t)$ to denote the nth coefficient of a generating function $A(t)$):

Theorem 5 (Lagrange inversion formula [**85**] Theo.5.4.2). *Let R be a ring* [3] *containing $\mathbb{Q}$, and let $\phi(y)=\sum_{n\geq 0} a_n y^n$ be a generating function with coefficients in R. Let $g(t)$ be the generating function (with coefficients in R) specified by the equation*

$$g(t)=t\,\phi(g(t)).$$

Then, for any $k\geq 1$ and $n\geq 1$,

$$[t^n]g(t)^k=\frac{k}{n}[y^{n-k}]\phi(y)^n. \tag{8}$$

1.3. Encoding tree-rooted maps and application to map encoding

1.3.1. Encoding by a shuffle of two parenthesis words. A *tree-rooted map* is a pair (M,T), where M is a rooted map and T is a spanning tree of M (i.e., a subgraph of M that forms a tree and covers all the vertices of M). It turns out that tree-rooted maps are easier to count than rooted maps (the spanning tree helps to design an encoding procedure).

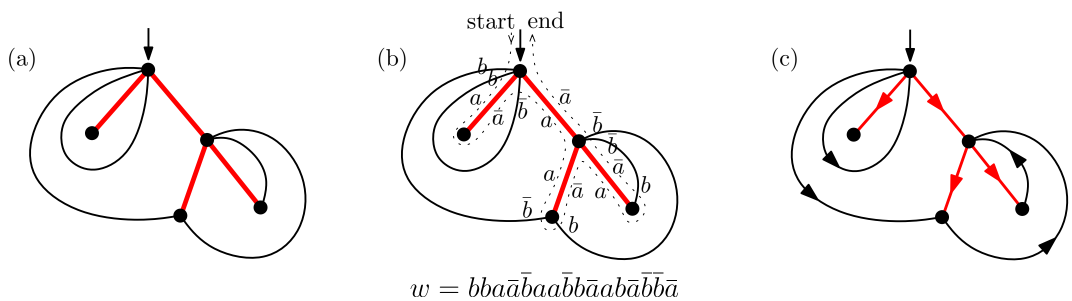


Figure 1.7. (a) A tree-rooted map. (b) Encoding by a shuffle of two parenthesis words. (c) The associated canonical orientation.

We review here a well known bijection [**72, 90**] with shuffles of two parenthesis words. For $p,q\geq 0$, let (M,T) be a tree-rooted map where M has $p+1$ vertices and $q+1$ faces, equivalently T has p edges, called *internal edges*, and $M\backslash T$ has q edges, called *external edges*. We encode (M,T) by a word on the alphabet $\{a,\bar{a},b,\bar{b}\}$ as follows: walking in counterclockwise order around T (i.e., with T on our left), we write out a letter a (resp. $\bar{a}$) when traversing an internal edge for the first (resp. second) time, and we write out a letter b (resp. $\bar{b}$) when crossing an external edge for the first (resp. second) time, see Figure 1.7(b) for an example. The obtained word w is easily checked to be a shuffle of a parenthesis word $w_a\in\mathfrak{S}(a^p\bar{a}^p)$ and a parenthesis word $w_b\in\mathfrak{S}(b^q\bar{b}^q)$; indeed by planarity of T the matched pairs $(a,\bar{a})$ of w_a correspond to the p internal edges, and by planarity of M the matched pairs $(b,\bar{b})$ in w_b correspond to the q external edges (w_a is the contour word for T and w_b is the contour word for the dual spanning tree, as shown in Figure 1.4). Moreover, the mapping is clearly invertible. Denoting by $t_{p,q}$ the number of tree-rooted maps with $p+1$ vertices and $q+1$ faces (hence $n=p+q$ edges, by the Euler relation), the above encoding gives, with $C_k=\frac{(2k)!}{k!(k+1)!}$ the kth Catalan number,

$$t_{p,q}=\binom{2p+2q}{2p}C_pC_q=\frac{(2p+2q)!}{p!(p+1)!q!(q+1)!}. \tag{9}$$

[3]Typically R can be $\mathbb{Q}$ or $\mathbb{C}$, but we will in most cases use the formula with R the set of multivariate power series over a field, such as $R=\mathbb{Q}[x_0,\dots,x_h]$.

Hence, if we denote by $t_n = \sum_{p=0}^{n} t_{p,n-p}$ the number of tree-rooted maps with n edges, we obtain, using the Vandermonde identity:

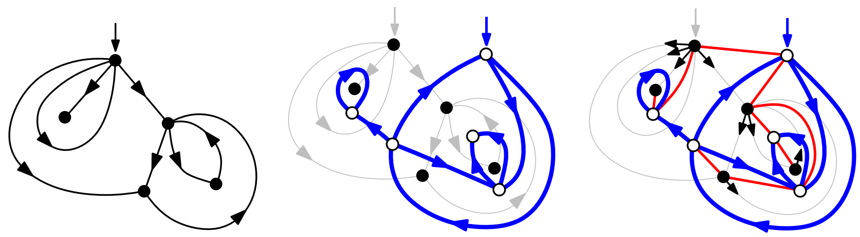
$$t_n = \sum_{p=0}^{n} \frac{(2n)!}{p!(p+1)!(n-p)!(n-p+1)!} = \frac{(2n)!}{(n+1)!^2} \sum_{p=0}^{n} \binom{n+1}{p}\binom{n+1}{n-p} = \frac{(2n)!}{(n+1)!^2}\binom{2n+2}{n},$$

hence

$$t_n = C_n C_{n+1}. \tag{10}$$

1.3.2. Encoding by a pair of rooted plane trees. The above formula (10) for t_n suggests the possibility of encoding tree-rooted maps with n edges by a pair of rooted plane trees that have respectively n edges and $n+1$ edges (which does not clearly appear from the encoding presented in the previous section). This has been recently achieved by Olivier Bernardi, who designed a beautiful alternative encoding procedure for tree-rooted maps [**11**]. As we will see, this encoding, which crucially relies on certain orientations, also provides powerful tools for the encoding of (non tree-rooted) maps.

Let $\mathcal{T}_n$ be the set of tree-rooted maps with n edges. The first step in [**11**] is a bijection between $\mathcal{T}_n$ and certain oriented rooted maps. Let M be a map endowed with an orientation X (i.e., the assignment of a direction to every edge). For f a face of M, X is said to be *minimal with respect to* f if there is no counterclockwise (shortly ccw) cycle in the plane embedding of M with f as the outer (infinite) face; similarly X is called *maximal with respect to* f if there is no clockwise (shortly cw) cycle in the plane embedding with f as the outer face (the terminology of “minimal” and “maximal” will be justified later on when stating Proposition 8). If M is a plane map, minimality or maximality will always be considered with respect to the outer face). And for v a vertex of M, X is said to be *accessible from* v if for every vertex $v' \in M$ there is a directed path from v to v'. Denote by $\mathcal{R}_n^{\circlearrowleft}$ (resp. $\mathcal{R}_n^{\circlearrowright}$) the set of rooted maps with n edges and endowed with an orientation that is maximal (resp. minimal) with respect to the root face and accessible from the root vertex; and define $\mathcal{R}^{\circlearrowleft} := \bigcup_n \mathcal{R}_n^{\circlearrowleft}$ and $\mathcal{R}^{\circlearrowright} := \bigcup_n \mathcal{R}_n^{\circlearrowright}$. A nice property shown in [**14**] and illustrated in Figure 1.8(a)-(b) is that duality maps (bijectively) $\mathcal{R}_n^{\circlearrowleft}$ to $\mathcal{R}_n^{\circlearrowright}$.



(a) (b) (c)

FIGURE 1.8. (a) An oriented rooted map $M \in \mathcal{R}_n^{\circlearrowleft}$ ($n=8$). (b) The map M superimposed with the dual oriented rooted map $M^* \in \mathcal{R}_n^{\circlearrowright}$. (c) Both maps M, M^* superimposed with the associated mobile T_2 (adding the outgoing half-edges of M, shown in black, T_2 forms a blossoming mobile of excess 1).

For $(M,T) \in \mathcal{T}_n$, the *canonical orientation* for (M,T) is the orientation of M such that:

- the edges of T are oriented away from the root (i.e., from the extremity of smaller depth to the extremity of larger depth in T),

- for every edge $e \in M \backslash T$, let C_e be the cycle formed by e and the (unique) path of edges of T connecting the two extremities of e; then e is directed so as to have the interior of C_e on its left.

It is easy to check that M endowed with the canonical orientation is in $\mathcal{R}_n^{\circlearrowleft}$. In addition the mapping is invertible: for M a rooted oriented map in $\mathcal{R}_n^{\circlearrowleft}$ there is a certain traversal procedure (originally introduced in [**78**] for the case of 3-orientations of simple triangulations, and generalized in [**11**]) that finds a spanning tree T oriented away from the root and with the property that T is the unique spanning tree of M that has the considered orientation as canonical orientation. The above mapping is thus a bijection between $\mathcal{T}_n$ and $\mathcal{R}_n^{\circlearrowleft}$.

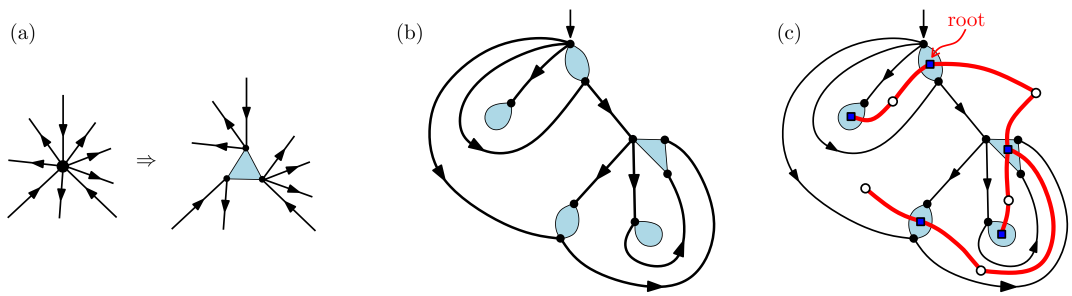


FIGURE 1.9. (a) The explosion rule to be performed at every vertex of a rooted oriented map in $\mathcal{R}_n^{\circlearrowleft}$. (b) Application to the oriented rooted map of Figure 1.8(c). (c) The associated mobile is obtained by drawing the dual of the edges that are incident to the light faces.

The second step in [**11**] is a procedure to encode any oriented rooted map $M \in \mathcal{R}_n^{\circlearrowleft}$ by a pair of rooted plane trees with n edges and $n+1$ edges, respectively (here the root is represented as an ingoing half-edge, which matters in the construction). Perform at each vertex of M the operation shown in Figure 1.9(a), i.e., a vertex v of indegree k, with $e_1, \ldots, e_k$ the ingoing edges (one of which is the root half-edge if v is the root vertex) in counterclockwise order around v, is blown into a k-gon so that each vertex v_i (for $i \in [1..k]$), of the k-gon receives the ingoing edge e_i together with the interval of outgoing edges between e_i and e_{i+1}, see Figure 1.9(b) for an example. The new k-gonal face is called a *light face*. After performing this operation at each vertex of M, the graph T_1 formed by the n oriented edges, rooted at the root half-edge, can be shown to be a tree [**11**] (what is easy to show is that there are $n+1$ vertices after performing the blowing operations, since each vertex has indegree 1, and the contribution to the indegrees is given by the n edges and by the root half-edge, hence a total contribution of $n+1$).

The encoding is completed by a second rooted plane tree (which indicates how the vertices of T_1 have to be identified in order to recover M) that has $n+2$ vertices. Insert a vertex in each face of M, considered as a round white vertex, and insert a vertex in each light face (after the blowing operations), called a light vertex. Then for each edge of a light face, draw a dual edge connecting the light vertex in the incident light face and the white vertex in the incident face of M. Let T_2 be the embedded subgraph formed by the light vertices, the white vertices, and the new dual edges. Then it can be shown that T_2 is a tree (which is naturally rooted at the corner 'at the root', see Figure 1.9) called the *mobile* of M (again what is easy to see is that T_2 has $n+1$ edges and, if M has $p+1$ vertices and $q+1$ faces, then T_2 has $p+1$ light vertices and $q+1$ white vertices, hence has $n+2$ vertices by the

Euler relation). The mapping from M to (T_1, T_2) can be shown to be a bijection between $\mathcal{R}_n^{\circlearrowleft}$ and $\mathcal{C}_n \times \mathcal{C}_{n+1}$; the oriented map is recovered by "winding" the tree T_1 in counterclockwise order around T_2. This encoding thus gives a direct bijective proof of (10). It also gives an alternative proof of (9); indeed the encoding ensures that $t_{p,q}$ counts pairs (T_1, T_2) of rooted plane trees such that T_1 has $n = p + q$ edges, and T_2 has $p + 1$ vertices at even depth and $q + 1$ vertices at odd depth. It is well known that the number of possibilities for T_2 is given by the Narayana number $N_{p,q} = \frac{1}{n+1}\binom{n+1}{p}\binom{n+1}{q}$, so that we recover

$$t_{p,q} = C_n N_{p,q} = \frac{(2n)!}{p!(p+1)!q!(q+1)!}, \quad \text{with } n = p + q.$$

In the original formulation [**11**] of the bijection, the second Catalan structure (describing how the vertices of T_1 have to be identified) was given as a noncrossing partition, the reformulation with T_2 as a mobile has been given in [**14**], where the procedure is extended to higher genus. We will focus in this document only on the planar case, and show how some variations on the encoding procedure in [**11, 14**] make it possible to develop a general bijective strategy for planar maps.

1.3.3. Application to map encoding. In order to apply the procedure of the previous section to map encoding, we will use the second step, namely the bijection between $\mathcal{R}_n^{\circlearrowleft}$ (or $\mathcal{R}_n^{\circlearrowright}$ if we use duality) and pairs of rooted plane trees. The main idea is that, given a certain family $\mathcal{F} = \bigcup_n \mathcal{F}_n$ of rooted maps, it is often possible to endow maps from $\mathcal{F}$ with a certain "canonical orientation" in $\mathcal{R}_n^{\circlearrowleft}$ and thereby establish a bijection between $\mathcal{F}$ and a certain subfamily of $\mathcal{R}^{\circlearrowleft}$; then hopefully the bijective encoding by pairs of trees specializes nicely onto the subfamily.

Before demonstrating the methodology on an example (we will discuss the case of Eulerian maps), let us slightly reformulate the encoding of a rooted oriented map $M \in \mathcal{R}_n^{\circlearrowleft}$, turning the corresponding pair (T_1, T_2) of rooted plane trees into a single tree-structure called a blossoming mobile. Precisely, a *blossoming mobile* is defined as an unrooted bipartite plane tree (bipartite means that there are black vertices and white vertices and each edge connects a black vertex to a white vertex) where the black vertices might carry additional dangling half-edges called *buds*, see Figure 1.10(c) for an example. The *excess* of a blossoming mobile is the number of edges minus the number of buds.

Let M be a rooted oriented map in $\mathcal{R}_n^{\circlearrowleft}$. Note that the corresponding mobile T_2 can be directly computed (see Figure 1.10(b)) from M using the following local rule (we consider the ingoing half-edge at the root as a complete directed edge when applying this rule): "insert a white vertex w_f in each face $f \in M$ and then, for each edge $e \in M$ with f the face on its left, draw a new edge from w_f to the end-vertex of e".

Next, the fact that T_1 is "winding" in counterclockwise order around T_2 ensures that there is no loss of information if we just record the arity of each vertex of T_1, which amounts to recording the number of outgoing edges in each corner of T_2. A way of doing this is to cut each edge of M in its middle (which disconnects the outgoing part from the ingoing part) and then delete all ingoing half-edges (including the root), see Figure 1.10(c) for an example. This yields a blossoming mobile T with $n + 1$ edges (one edge for each edge of M, plus one edge for the root of M) and n buds (one bud for each edge of M), hence a blossoming mobile of excess 1. Conversely, one can recover M from T as follows (see Figure 1.10(d)): (i) calling *down-corner* a corner of T just after an edge (not a bud) in counterclockwise order around a black vertex, we insert an ingoing half-edge called a *bid* in each of the $n + 1$ down-corners of T, (ii) we match the buds with the bids according to a counterclockwise walk around T where buds (resp. bids) are considered as opening (resp. closing) parentheses, (iii) we

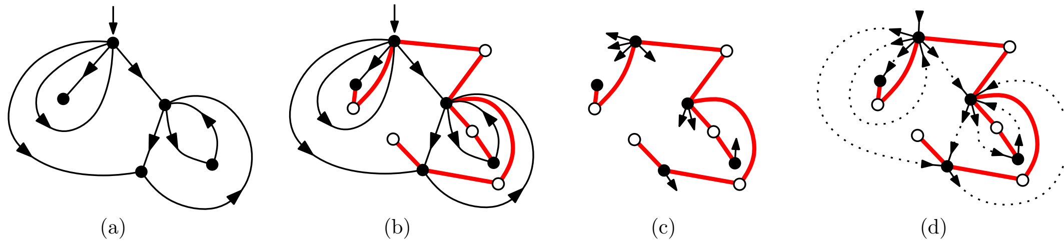


FIGURE 1.10. (a) The oriented map $M \in \mathcal{R}_n^{\circlearrowleft}$ ($n = 8$) obtained in Figure 1.7. (b) The mobile T_2 for M, computed directly on M. (c) Deleting the ingoing half-edges of M yields a blossoming mobile T of excess 1. (d) The map M is recovered from T as follows: insert an ingoing half-edge (called a bid) in each corner just after an edge in ccw order around a black vertex, and match the buds with the bids according to a ccw walk around T where buds (resp. bids) are considered as opening (resp. closing) parentheses; the unique unmatched bid is taken as the root.

create the n directed edges of M out of the n matched pairs bud/bid, and the unique bid left unmatched is declared as the root of M. To summarize we obtain:

CLAIM 6 (from [**14**]). *The above procedure gives a bijection between $\mathcal{R}_n^{\circlearrowleft}$ and blossoming mobiles of excess* 1 *with* $n+1$ *edges.*

REMARK 7. *Let $M \in \mathcal{R}_n^{\circlearrowleft}$, T the associated blossoming mobile of excess* 1*, and $M^* \in \mathcal{R}_n^{\circlearrowleft}$ the dual of M. Then, as illustrated in Figure 1.8(c), T is obtained from M^* (whose vertices are white) by following the local rule: "insert a black vertex b_f in each face f of M^*, and then for each edge $e = (u, v) \in M^*$, with f_r the face on the right and f_ℓ the face on the left, insert a new edge connecting b_{f_r} to v (including if e is the root half-edge) and insert a bud in b_{f_ℓ} pointing toward u (excluding e being the root half-edge). The master bijection we will present in Chapter 2 can be seen as an adaptation of this mapping to a more general setting. Let us also mention that the simple local rules to obtain the edges of T from the orientation are reminiscent of the local rules given in* [**26**] *in order to associate a (labelled) mobile to a vertex-pointed bipartite map endowed with its geodesic labelling (we will see this more precisely when reviewing the BDG bijection in Section 2.3).*

Let us now discuss how this bijection makes it possible to encode Eulerian maps (i.e., maps with all vertices of even degree), and in doing this we also introduce the general terminology of so-called α-orientations [**50**]. Define the *indegree* of a vertex v in an oriented map M as the number of edges of M whose end-vertex is v. For $M = (V, E)$ a map and $\alpha : V \to \mathbb{N}$, an *$\alpha$-orientation* of M is an orientation of M such that every vertex $v \in V$ has indegree $\alpha(v)$ (if M is a rooted map, the root, which is depicted as an ingoing half-edge, is neither counted in the indegree nor in the degree of the root vertex).

It is known that several families of maps are characterized by the existence of certain α-orientations, for instance [**50**]:

- any Eulerian map (map with all vertices of even degree) admits an Eulerian orientation, i.e., an α-orientation for $\alpha(v) = \deg(v)/2$,
- a plane triangulation (plane map with all faces of degree 3) is simple (no loops nor multiple edges) iff it admits an α-orientation such that $\alpha(v) = 3$ for all inner vertices and $\alpha(v) = 1$ for all three outer vertices,

- a plane quadrangulation (plane map with all faces of degree 4) is simple iff it admits an α-orientation such that $\alpha(v) = 2$ for all inner vertices and $\alpha(v) = 1$ for all four outer vertices.

The following general result for α-orientations is well-known:

LEMMA 8 ([**50, 80, 75**]). *If M is a plane (or rooted) map and has an α-orientation, then M has a unique minimal α-orientation and a unique maximal α-orientation* [4].

To prove that any Eulerian map has an Eulerian orientation as mentioned above, one can for instance derive the existence from the existence of an Eulerian tour, which also ensures that the Eulerian orientation is accessible from any vertex (i.e., is strongly connected). An alternative method of proof is to use the following general existence lemma (which can be proved by induction on the number of edges [**50**, page 3]):

LEMMA 9. *Let $M = (V, E)$ be a map and let $\alpha : V \to \mathbb{N}$. For any $S \subseteq V$, define $\alpha(S) := \sum_{v \in S} \alpha(v)$ and let E_S denote the set of edges with both ends in S. Then M admits an α-orientation iff for all $S \subseteq V$, $\alpha(S) \geq |E_S|$, with equality for $S = V$. In addition, for $v_0 \in V$, either none or all α-orientations are accessible from v_0, and the later occurs iff for all $S \subseteq V \backslash \{v_0\}$, $\alpha(S) > |E_S|$.*

This makes it easy to prove the existence of an Eulerian orientation for any Eulerian map, and accessibility from any vertex. Indeed with the notations of Lemma 9, for Eulerian orientations we have $\alpha(S) = \sum_{v \in S} \deg(v)/2$, whereas $|E_S| = \sum_{v \in S} \deg_S(v)/2$, where $\deg_S(v)$ denotes the degree of v when taking only the edges of E_S into account; hence $\alpha(S) \geq |E_S|$, with equality iff $S = V$.

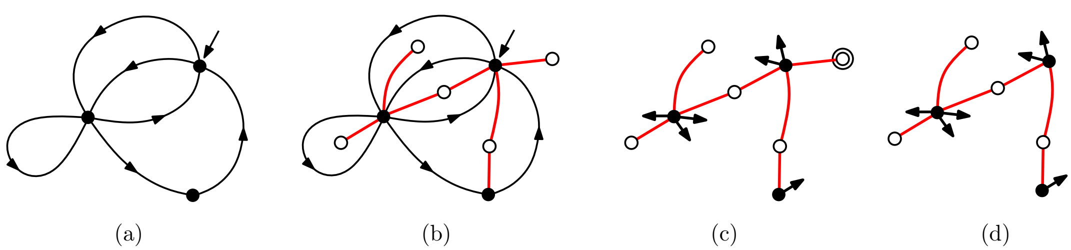


FIGURE 1.11. (a) A rooted Eulerian map M endowed with its unique Eulerian orientation in $\mathcal{R}^{\circlearrowleft}$. (b) M superimposed with its mobile. (c) The associated blossoming mobile, which is admissible quasi-balanced. (d) The induced balanced blossoming mobile.

Hence the family $\mathcal{E}$ of rooted Eulerian maps can be identified with the subfamily $\widetilde{\mathcal{E}}$ of rooted oriented maps in $\mathcal{R}^{\circlearrowleft}$ such that every vertex has same oudegree as indegree; and thus (by Claim 6), $\mathcal{E}$ is in bijection with the corresponding subfamily of blossoming mobiles of excess 1, which is to be characterized. A blossoming mobile is called *balanced* if every black vertex has as many buds as white neighbours; hence such a mobile clearly has excess 0. A blossoming mobile T of excess 1 is called *quasi-balanced* if it is obtained from a balanced mobile T' by adding a distinguished *pending edge* at a corner of a black vertex, connected

[4] More precisely the set of α-orientations of M forms a so-called distributive lattice where an orientation O is smaller than O' if O' can be obtained from O by successively reversing cw cycles into ccw cycles; the smallest (resp. largest) element of the lattice is the unique element with no ccw cycle (resp. no cw cycle).

to an additional white vertex of degree 1; T' is called the *reduction* of T. And a quasi-balanced blossoming mobile T is called *admissible* if, in the corresponding rooted oriented map $M \in \mathcal{R}^{\circlearrowleft}$, the root ingoing half-edge corresponds to the pending edge of the mobile. It can be proved (based on the strong connectivity property) that in a rooted oriented map in $\widetilde{\mathcal{E}}$ every outer edge is counterclockwise (i.e., has the outer face on its right), which implies quite easily that $\widetilde{\mathcal{E}}$ is mapped to the family of *admissible* quasi-balanced blossoming mobiles, see Figure 1.11 for an example.

Balanced mobiles (and quasi-balanced mobiles) are characterized only by local conditions, hence are easy to manage combinatorially, which is not the case when adding the admissibility condition, which is non-local. A way to bypass this difficulty is to observe that if two rooted Eulerian maps differ from each other by changing the marked corner in the outer face, then the two corresponding blossoming mobiles have the same reduction, so that, by taking the 'quotient' of the above correspondence, one obtains [5]:

Claim 10. *Plane Eulerian maps are in bijection with balanced blossoming mobiles.*

Formulated in this way, the bijection deals with simpler tree-structures, where the conditions (to be balanced) are the same at all vertices. Our master bijection Φ to be presented in the next chapter essentially adapts (and extends to the more general setting of weighted bi-orientations) the bijection of Claim 6 (in the dual setting, as in Remark 7) so as to formulate correspondences for plane (or vertex-pointed) oriented maps (rather than for rooted oriented maps, where the associated tree-structure can be more delicate to characterize, as we have seen for Eulerian maps). We will see that the bijection of Claim 10 (in the dual setting) can be recovered as a specialization of Φ and is equivalent to the Bouttier-Di Francesco-Guitter for vertex-pointed bipartite maps [**26**] (balanced blossoming mobiles correspond to so-called labelled mobiles in [**26**] and admissible quasi-balanced blossoming mobiles correspond to labelled mobiles rooted at a vertex of smallest label, a non-local condition known to be delicate to handle combinatorially, already for labelled mobiles corresponding to quadrangulations [**42**]).

1.4. Related work and overview of the document

It appears that (to our knowledge) every bijection between a map family and a family of "decorated" tree-structures introduced in the literature, with the breakthrough in Gilles Schaeffer's PhD thesis [**82**], somehow relies on endowing the map with a certain canonical orientation to get the encoding tree-structure from the maps, in particular the first bijections [**81, 26**] (for unconstrained bipartite maps) rely on the geodesic labelling/orientation (on the map or its dual), the bijections in [**82**] for loopless triangulations and non-separable maps rely on the minimal 2-orientations for simple quadrangulations, and the bijection in [**78**] for simple triangulations relies on the minimal 3-orientations (that for triangulations characterize the property of not having loops nor multiple edges).

This document aims at demonstrating that many of the bijections known so far (and new bijections) can be put under one roof, as specializations of a "master bijection" that is closely related to the second step of the bijection in [**11, 14**]. Note that, instead of using the second step of the bijection, one can alternatively use the first step, i.e., encode an orientation in $\mathcal{R}^{\circlearrowleft}$ using the associated canonical spanning tree (instead of using the associated mobile). The canonical spanning tree is computed from the orientation using a traversal procedure that has first been introduced in [**78**] for simple triangulations (endowed with a so-called 3-orientation

[5]Note also that the correspondence keeps track of the degree-distribution of the vertices, which corresponds to the degree-distribution of black vertices in the associated balanced blossoming mobiles.

in $\mathcal{R}^{\circlearrowleft}$), and generalized in [**11**] to any orientation in $\mathcal{R}^{\circlearrowleft}$. Similarly as with mobiles, an encoding tree-structure called a *blossoming tree* is typically obtained by deleting the ingoing half-edge of each edge not belonging to the canonical spanning tree. Characterizing the associated blossoming trees might take some work depending on the map family considered. Some examples (loopless triangulations, simple quadrangulations,...) are discussed in [**84**] and in [**T2**], and the recent article [**3**] makes this method more systematic, in particular it makes it possible to treat in a nice way the case of maps with a boundary (such as d-angulations of girth d with a boundary), by generalizing the canonical spanning tree approach to orientations that are minimal with respect to a fixed face and accessible with respect to a fixed vertex (not necessarily incident to the fixed face, as for rooted maps), which is convenient to characterize the encoding blossoming tree by local conditions only. However, there are still several examples of map families, in particular those of Chapter 3 (except the case of simple quadrangulations) and of Chapter 4 in this document, where the bijective approach based on mobiles succeeds whereas it is not known if the bijective enumeration can be performed using the canonical spanning tree approach (we believe that an advantage of mobiles is that they are deduced from the orientation in a very simple local way, whereas computing the canonical spanning tree necessitates a traversal procedure).

Another combinatorial method for map enumeration, based on so-called *slices* (certain portions of maps) and their decomposition (relying on the geodesic labelling), has been recently introduced [**31**] and proves very powerful: it makes it possible to count in a unified way maps with a control on the face-degree distribution and on the girth. With this method one can obtain the results of Chapter 3 (and their generalizations mentioned in Section 5.1 and Section 5.2.1 of Chapter 5), but at the moment this approach has not been adapted to deal with maps with boundaries (results of Chapter 4 on triangulations and quadrangulations with multiple boundaries), nor extended to hypermaps (results mentioned in Section 5.1.6).

Articles from which this document is extracted. This document relies on 4 articles with Olivier Bernardi (two of which are published), listed below with the same reference as in the publication list given at the end of the document:

[A21]. "A bijection for triangulations, quadrangulations, pentagulations, etc.", J. Combin. Theory Ser. A, 119(1), pp. 218-244, 2012.

[A24]. "Unified bijections for maps with prescribed degrees and girth", J. Combin. Theory Ser. A, 119(6), pp. 1352-1387, 2012.

[S1]. "Unified bijections for planar hypermaps with general cycle-length constraints", arXiv:1403.5371, 2014.

[S5]. "Bijections for maps with boundaries: Krikun's formula for triangulations, and a quadrangulation analogue", arXiv:1510.05194, 2015.

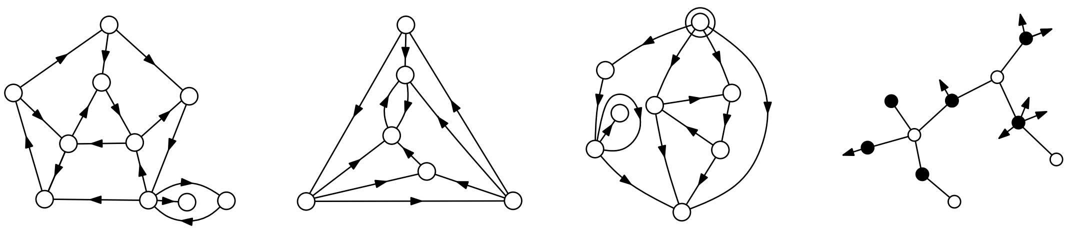
We have introduced the master bijection (Chapter 2) in [**A21**], where the proof of the bijectivity of the mapping is done by a reduction to the bijection in [**11, 14**]. We have extended the master bijection to hypermaps in [**S1**], where we have provided a self-contained proof and a new presentation of the inverse mapping via cacti graphs. In this document we only mention briefly (in Chapter 5) the extension to hypermaps, but we retain from [**S1**] the presentation of the inverse mapping using cacti-graphs, and the self-contained proof of bijectivity. Reference [**A24**] applies the master bijection strategy to count maps with control on the face-degrees and the girth parameter. Chapter 3 gives the main ingredients of [**A24**] applied to the case of simple bipartite maps (girth at least 4). The extension of these arguments to the general case is then explained (without details) in Chapter 5. Reference **[S5]** is the content of Chapter 4: it adapts the master bijection to the context of maps with boundaries, and applies the strategy to two classes: bipartite quadrangulations with boundaries, and triangulations with boundaries.

In the concluding chapter, we give extensions and other results related to the results in the preceding chapters, in particular regarding the extension to higher girth of the results of Chapter 3 for bipartite maps, non-necessarily bipartite maps (in **[A24]**), and hypermaps (in **[S1]**). Regarding maps with boundaries, we discuss the obstacles to an extension of the results of Chapter 4 to higher girth, and we indicate some cases where the strategy should work. Then we explain how a labelled reformulation of the master bijection makes it possible to estimate/bound the distances, and we sketch how this could make it possible to prove convergence to the Brownian map for families of maps amenable to the master bijection (in particular maps with conditions on the girth and face-degrees). An additional section at the end of the document (before the publication list and the bibliography) describes without details some other results obtained in previous years, regarding the 2-point and 3-point function of planar maps (with J. Bouttier and E. Guitter), counting unicellular maps in any given genus (with G. Chapuy and V. Feray), the combinatorics of Baxter families (with D. Poulalhon, G. Schaeffer, S. Felsner, M. Noy, D. Orden, N. Bonichon, M. Bousquet-Mélou, S. Burrill, J. Courtiel, S. Melczer, and M. Mishna), and the enumeration of intervals in the m-Tamari lattice (with M. Bousquet-Mélou and L.-F. Préville-Ratelle).

CHAPTER 2

The master bijection

In this chapter we present the master bijection, which for each $\delta \in \mathbb{Z}$ gives a bijective correspondence between a certain family $\mathcal{O}_\delta$ of oriented maps and the family of blossoming mobiles (shortly called mobiles from now on) of excess δ; as we will see the correspondence for rooted oriented maps given in Claim 6 corresponds to the case $\delta = 1$. We also extend the construction (for any δ) to the more general setting of weighted bi-orientations, which will prove useful in the next chapters to encode bijectively several families of maps by specializing the master bijection.



(a) (b) (c) (d)

FIGURE 2.1. (a) An orientation in $\mathcal{O}_7$, (b) an orientation in $\mathcal{O}_{-3}$, (c) an orientation in $\mathcal{O}_0$ (to be seen as embedded on the sphere, with a marked vertex), (d) a mobile of excess 2.

2.1. From oriented maps to mobiles

Let us first define the families of oriented maps (by a slight abuse of terminology, we will often use the term "orientation" for an oriented map) that will be involved in the master bijection; there is one family $\mathcal{O}_\delta$ for each $\delta \in \mathbb{Z}$, (examples for $\delta \in \{-4, 5, 0\}$ are given in the top-row of Figure 2.3). For $\delta > 0$, let $\mathcal{O}_\delta$ be the family of plane orientations of outer degree δ, that are minimal (no counterclockwise cycle) and are accessible with respect to each of the outer vertices. Note that, for $O \in \mathcal{O}_\delta$, each outer edge $e = (u, v)$ of O has an inner face on its right (assuming not, since the orientation is accessible with respect to v, there exists an oriented path P from v to u, and the cycle formed by P and e is counterclockwise, a contradiction). Let $\mathcal{P}_\delta$ be the subfamily of $\mathcal{O}_\delta$ where all outer vertices have indegree 1, in which case the outer cycle must be a simple clockwise cycle. Note also that, for $O \in \mathcal{P}_\delta$, any incidence of an inner edge e with an outer vertex v is such that e is going out of v. Define $\mathcal{O}_{-\delta}$ as the family of plane orientations that are obtained from orientations in $\mathcal{P}_\delta$ by returning the outer cycle (from cw to ccw). Finally, define $\mathcal{O}_0$ as the family of vertex-pointed orientations that are accessible with respect to the pointed vertex v_0, such that v_0 is a source, and there is no directed cycle C such that the part of the sphere on the right of C contains

v_0 (equivalently, the orientation is minimal with respect to any of the faces incident to the pointed vertex). Denote by $\mathcal{O}$ the union of the families $\mathcal{O}_\delta$ over $\delta \in \mathbb{Z}$.

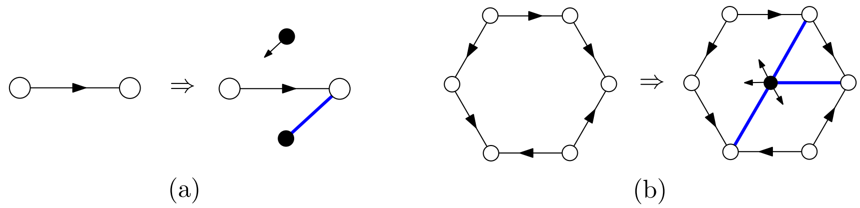


FIGURE 2.2. (a) The local rule performed at each edge in the master bijection. (b) The effect of the local rule on a face.

We now define a mapping Φ from $\mathcal{O}$ to mobiles based on simple local operations (herafter we will shortly talk about mobiles for blossoming mobiles). For $\delta \in \mathbb{Z}$ and $O \in \mathcal{O}_\delta$, where the vertices of O are considered as white, let $\Phi(O)$ be the embedded graph with buds obtained as follows, see Figure 2.3 for examples (where respectively $\delta < 0$, $\delta > 0$ and $\delta = 0$):

(1) Insert a black vertex v_f in each face f of O.
(2) For each edge $e = (u, v)$ of O, with f_ℓ (resp. f_r) the face on the left (resp. right) of e, insert a new edge from v_{f_r} to v and insert a bud at v_{f_ℓ} pointing toward (but not reaching) u, see Figure 2.2(a).
(3) Delete all the edges of O (for any $\delta \in \mathbb{Z}$). Then, if $\delta > 0$, the black vertex in the outer face of O carries only buds: delete this black vertex and its buds. If $\delta < 0$ the black vertex b in the outer face of O has no bud and is the unique neighbour of each of the outer vertices v_i for $i \in [1..|\delta|]$: delete $b, v_1, \ldots, v_{|\delta|}$ and the $|\delta|$ edges $\{b, v_i\}$ for $i \in [1..|\delta|]$. If $\delta = 0$ delete the pointed vertex of O.

Note that the effect of the local rules at faces is as follows: turning in cw order around each face f of O, for each clockwise edge (u, v) (i.e., with f on its right), insert a new edge from v_f to v, and for each counterclockwise edge (u, v) (i.e., with f on its left), insert a bud at v_f directed toward u (but not reaching to u), see Figure 2.2(b).

THEOREM 11. *For $\delta \in \mathbb{Z}$, the mapping Φ is a bijection from $\mathcal{O}_\delta$ to mobiles of excess δ.*

We will prove Theorem 11 in Section 2.5. Let us for now justify the correspondence for the excess parameter. Note that each edge e of $O \in \mathcal{O}_\delta$ gives rise to an edge of $T = \Phi(O)$, in the face f_r on the right of e, except if $\delta < 0$ and e is an outer edge; and e gives rise to a bud of T in the face f_ℓ on the left of e, except if $\delta > 0$ and e is an outer edge. Hence, the excess of T is δ.

REMARK 12. *Note that a rooted oriented map $M \in \mathcal{R}_n^{\circlearrowright}$ can be identified with an oriented map in $\mathcal{O}_1$, by extending the root half-edge into a (clockwise) loop starting from the root-vertex and forming the outer contour. Under this identification, the bijection of Claim 6, reformulated in a dual setting in Remark 7, is equivalent to the master bijection Φ in the case $\delta = 1$.*

The local rules to obtain the mobile from the orientation are very simple, so that one can keep track of several paremeters of the oriented maps, making Φ amenable to specializations. For $O \in \mathcal{O}_\delta$ and $T = \Phi(O)$ the associated mobile, each inner face f (resp. each face if $\delta = 0$) of O corresponds to a black vertex b of the same degree in T; in addition the number of cw edges (resp. ccw edges) around f corresponds to the number of white neighbours (resp.

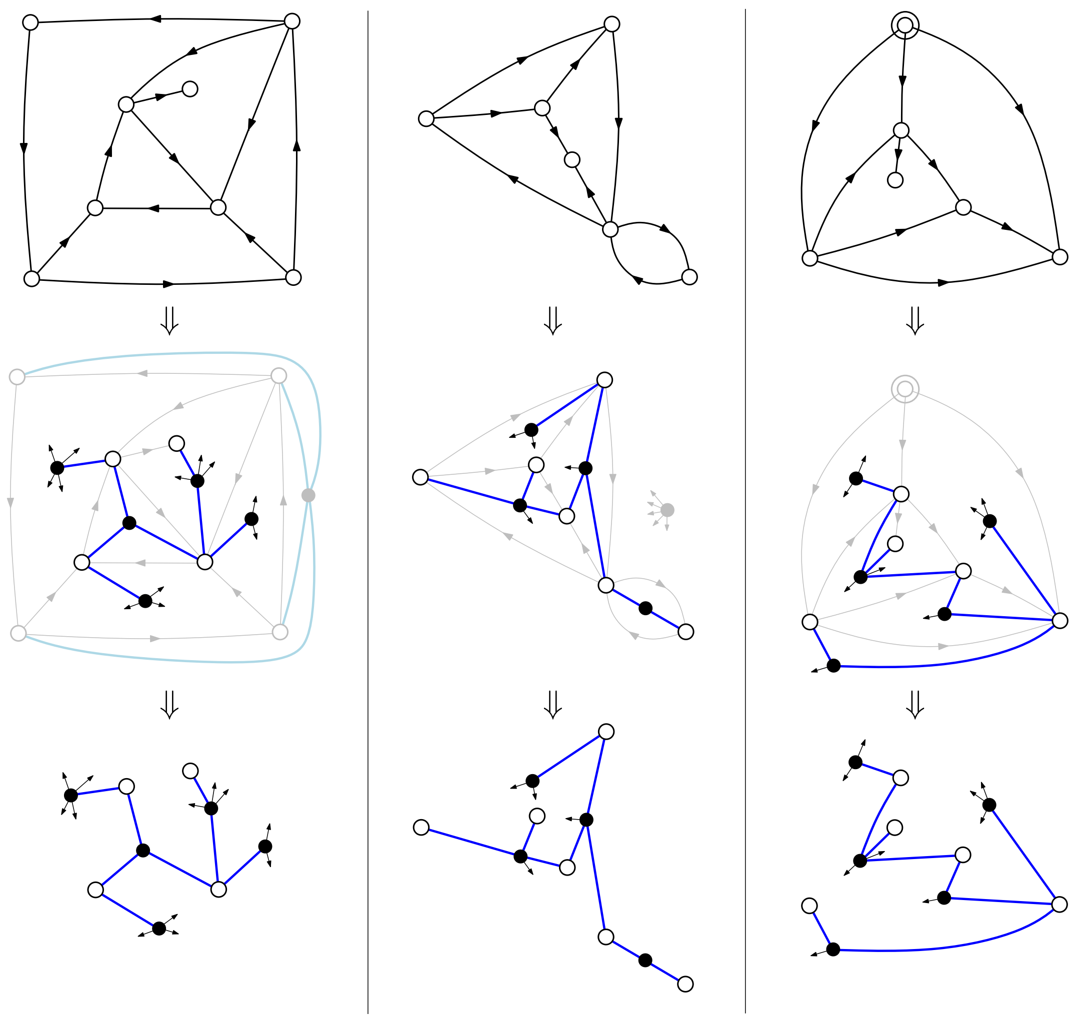


FIGURE 2.3. The master bijection on 3 examples: left with negative excess -4, middle with positive excess $+5$, right with zero excess (in zero excess the map is to be seen as embedded on the sphere, with no outer face).

incident buds) of b; in particular f has a cw contour iff b has no bud. Moreover, each vertex (resp. non-marked vertex if $\delta = 0$, inner vertex if $\delta < 0$) of O corresponds to a white vertex w in T, such that the indegree of v in O equals to degree of w in T. And each edge of O (resp. inner edge if $\delta < 0$) corresponds to an edge in T.

2.2. From mobiles to oriented maps

We now give the inverse bijection Ψ of Φ in two equivalent formulations: one using closure operations on the mobile (as in Figure 1.10(d)) and then duality, and one where the mobile is first turned into a so-called cactus on which the closure operations are performed.

Formulation by closures and then duality. Let $\delta \in \mathbb{Z}$, and let T be a mobile of excess δ. Call *down-corner* of T a corner that follows an edge (not a bud) in counterclockwise

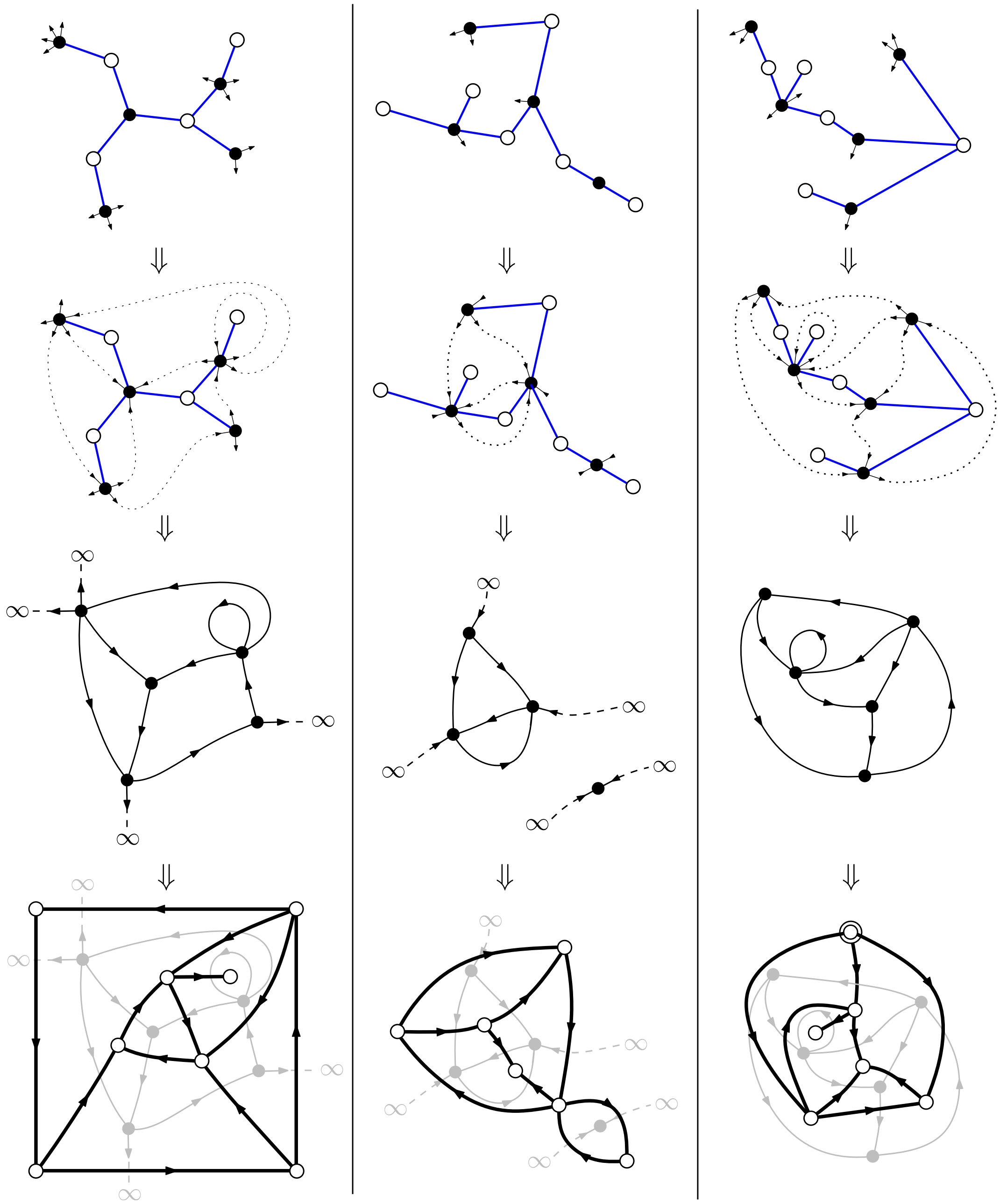


FIGURE 2.4. The inverse Ψ of the master bijection, performed on the 3 examples of Figure 2.3: formulation using duality.

order around a black vertex of T. Let T' be obtained from T by inserting a dangling ingoing half-edge, called a *bid*, in each down-corner. Let w be the cyclic word on the alphabet $\{a, \bar{a}\}$ obtained from a (cyclic) walk in counterclockwise order around T', where each encountered bud is encoded by a letter a, and each encountered bid is encoded by a letter $\bar{a}$. This cyclic word yields a (partial) matching of the a's with the $\bar{a}$'s; precisely each letter a is matched with the unique letter $\bar{a}$ (if it exists) such that the word starting just after a and ending just before $\bar{a}$ is a parenthesis word. There are exactly $|\delta|$ letters left unmatched, which are all a's for $\delta > 0$, and are all $\bar{a}$'s for $\delta < 0$. Accordingly the buds are matched with the bids and for $\delta > 0$ (resp. $\delta < 0$) there are $|\delta|$ bids (resp. buds) left unmatched, which are called *exposed*. Let O' be the oriented map (with a distinguished vertex for $\delta \neq 0$, resp. with a distinguished face for $\delta = 0$) obtained as follows:

(1) create a directed edge out of each matched pair bud/bid,
(2) for $\delta > 0$ (resp. $\delta < 0$), connect the $|\delta|$ unmatched bids (resp. buds) to a new vertex v_∞ "at infinity", taken as the pointed vertex,
(3) erase the edges and the white vertices of T.

Define $O = \Psi(T)$ as the oriented map (plane map for $\delta \neq 0$, vertex-pointed map for $\delta = 0$) that is the dual of O', see Figure 2.4 for examples.

Formulation via cacti. Alternatively, we can associate to T a *cactus-graph* $\widehat{T}$ (plane map whose inner faces are simple polygons connected together at vertices in a tree-like way) by growing around each black vertex b a polygon of same degree as b so as to satisfy the local rule of Figure 2.2(b), as in Figure 2.5 below (see also the second row of Figure 2.6 for

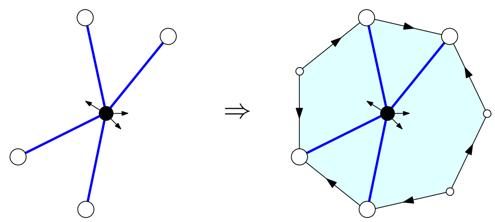


FIGURE 2.5. Growing a polygon around a black vertex of a mobile.

examples). Note that each bud of T yields a new vertex in the grown polygon (depicted as a small white vertex in the figures), called a *floating vertex*. Note also that every white vertex of degree i of T is incident to i polygons in $\widehat{T}$, while the floating vertices are incident to exactly one polygon.

Let T be a mobile and let $\widehat{T}$ be the cactus-graph for T. An edge of $\widehat{T}$ is called *cw-outer* (resp. *ccw-outer*) if it has the outer face on its left (resp. on its right). Similarly as before, a cyclic word on $\{a, \bar{a}\}$ is obtained from a counterclockwise walk around $\widehat{T}$ where each traversed ccw-edge is encoded by a letter a and each traversed cw-edge is encoded by a letter $\bar{a}$. Again we consider the matching of the a's with the $\bar{a}$'s given by the cyclic word, which leaves $|\delta|$ letters a (resp. $\bar{a}$) unmatched for $\delta > 0$ (resp. $\delta < 0$). Accordingly the ccw-outer edges are matched with the cw-outer edges. We obtain from $\widehat{T}$ an oriented map O by merging each matched pair ccw-edge/cw-edge into a directed edge. For $\delta \neq 0$, O is a plane oriented map of outer degree $|\delta|$, the outer contour being made of $|\delta|$ cw-outer (resp. ccw-outer) edges for $\delta > 0$ (resp. $\delta < 0$); for $\delta = 0$ all the outer edges of $\widehat{T}$ are glued, we distinguish the unique vertex of O that is a source (i.e., has no ingoing edge), so that O is a vertex-pointed oriented map. We define $O = \Psi(T)$.

The two formulations of Ψ are easily shown to be equivalent; indeed, creating an edge out of a matched pair bud/bid from T is equivalent via duality to creating an edge out of a matched pair ccw-edge/cw-edge from $\widehat{T}$.

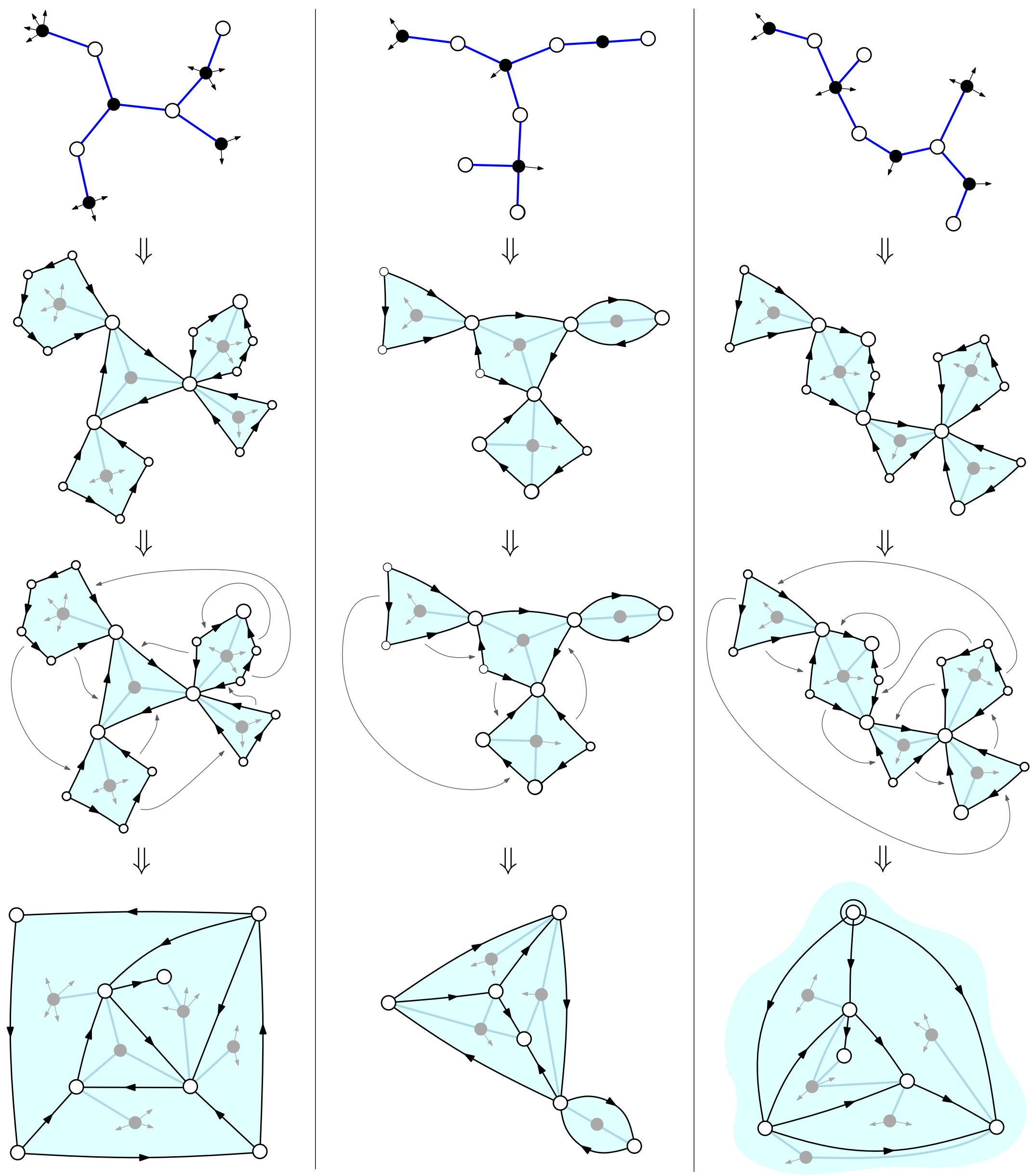


FIGURE 2.6. The inverse Ψ of the master bijection, performed on the 3 examples of Figure 2.3: formulation via cacti.

2.3. Application to bipartite maps

We now explain how to apply the master bijection strategy to vertex-pointed bipartite maps, and thereby recover the bijection from [**26**]. For M a vertex-pointed bipartite map, define a *balanced orientation* of M as an orientation of M such that around each face there are as many clockwise as counterclockwise edges. And define an *admissible labelling* of M as an assignment of labels in $\mathbb{Z}$ to every vertex v of M such that the pointed vertex has label 0 and the labels of any two adjacent vertices differ by 1 in absolute value. An admissible labelling induces a balanced orientation upon orienting each edge from the smaller to the larger label extremity, and it is easy to see that the mapping is a bijection. An example of admissible labelling is the *distance-labelling* with respect to the pointed vertex v_0, where each vertex is labelled by its distance from v_0.

CLAIM 13. *For M a vertex-pointed bipartite map with pointed vertex v_0, the distance-labelling is the unique admissible labelling of M such that each vertex $v \neq v_0$ has a neighbour of smaller label. Hence, the induced balanced orientation, called the* geodesic orientation *of M, is the unique balanced orientation of M that is accessible with respect to v_0. This orientation has v_0 as a source and is clearly acyclic, hence is in $\mathcal{O}_0$.*

PROOF. Let $\ell(v)$ be an admissible labelling having the stated properties, and let $d(v)$ be the distance-labelling with respect to v_0. Since each vertex $v \neq v_0$ has a neighbour of label $\ell(v)-1$, there is a label-decreasing path from v to v_0 (the label decreasing by 1 along each edge) and finishing at label 0. Hence this path has length $\ell(v)$, so that $\ell(v) \geq d(v)$. Now consider a geodesic path from v to v_0. Since the labels decrease by at most 1 along each edge of the path and the ending label is 0, we have $\ell(v) \leq d(v)$. Hence $\ell(v) = d(v)$. ☐

Define a *balanced mobile* as a mobile that has as many edges as buds at each black vertex; clearly such a mobile has excess 0. The master bijection Φ specializes to a bijection between balanced orientations in $\mathcal{O}_0$ and balanced mobiles. In addition, Claim 13 ensures that vertex-pointed bipartite maps can be identified (via the geodesic orientation) with balanced orientations in $\mathcal{O}_0$. Together with the parameter-correspondence for Φ we obtain:

PROPOSITION 14. *Vertex-pointed bipartite maps are in bijection (via the master bijection) with balanced mobiles. For M a vertex-pointed bipartite map and T the associated balanced mobile, each non-pointed vertex of M corresponds to a white vertex of T, and each face in M corresponds to a black vertex of same (even) degree in T.*

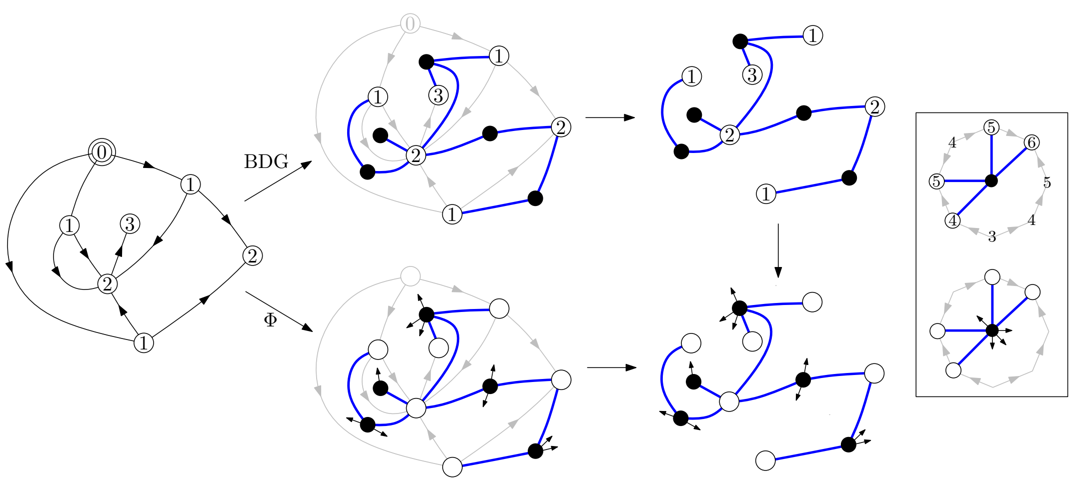


FIGURE 2.7. The bijection for vertex-pointed bipartite maps: bottom-row as a specialization of the master bijection, top-row the original formulation of Bouttier, Di Francesco and Guitter [**26**].

Remark 15. *For M a plane Eulerian map and M^* the dual vertex-pointed bipartite map, it is easy to see that the Eulerian orientations of M correspond by duality to the balanced orientations of M^*, and moreover it can be checked that the dual of the maximal Eulerian orientation of M is the geodesic orientation of M^*. This ensures that the bijection of Proposition 14 is equivalent to the bijection of Claim 10 between plane Eulerian maps and balanced (blossoming) mobiles.*

The bijection is equivalent to the one in [**26**]*, where the only difference in the formulation is that in our mobiles we erase the distance-labels on white vertices (these are implicitly recorded by the buds), whereas in* [**26**] *the mobiles have labels at white vertices (recording the distance-labels) and have no buds, see Figure 2.7 for an example.*

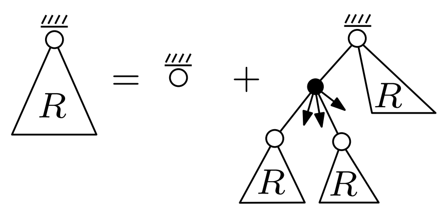


Figure 2.8. The recursive decomposition of a rooted mobile: it is either reduced to a single vertex, or else the leftmost (black) child b has degree $2i$ for some $i \geq 1$ ($i = 3$ in the example), so that there is a total of i hanging rooted submobiles (counted by R), and a factor $\binom{2i-1}{i} = \binom{2i-1}{i-1}$ to account for the number of ways to place the i buds at b.

As we review now, the bijection makes it possible to recover Tutte's slicing formula (Theorem 1). Call here *rooted mobile* a balanced mobile with a marked corner at a white vertex (for convenience we consider the tree reduced to a single white vertex as a rooted mobile). Denote by $R \equiv R(t; x_1, x_2, \ldots)$ the series of rooted mobiles where t is conjugate to the number of white vertices and x_i is conjugate to the number of black vertices of degree $2i$, for $i \geq 1$. As illustrated in Figure 2.8 (see also [**26**]), a decomposition at the root ensures that

$$R = t + R \sum_{i \geq 1} x_i \binom{2i-1}{i-1} R^{i-1},$$

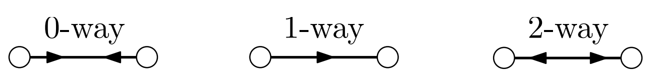
which rewrites as $R = t\phi(R)$, with $\phi(y) = (1 - \sum_{i\geq 1} x_i \binom{2i-1}{i-1} y^{i-1})^{-1}$. For nonnegative integers $v, n_1, \ldots, n_h$, the bijection of Proposition 14 ensures that $[t^{v-1} x_1^{n_1} \cdots x_h^{n_h}]R$ is the number of vertex-pointed bipartite maps with a marked edge (due to the marked corner in the mobile), v vertices, n_i faces of degree $2i$ for $i \in [1..h]$, and no face of degree larger than $2h$. By the Lagrange inversion formula, with $e = \sum_i i n_i$ the number of edges, $r = \sum_i n_i$ the number of faces, and $v = e - r + 2 = 2 + \sum_i (i-1)n_i$ the number of vertices, we have

$$
\begin{aligned}
[t^{v-1}x_1^{n_1}\dots x_h^{n_h}]R &= \frac{1}{v-1}[x_1^{n_1}\dots x_h^{n_h}][y^{v-2}]\phi(y)^{v-1}\\
&= \frac{1}{v-1}[y^{v-2}x_1^{n_1}\dots x_h^{n_h}]\Big(1-\sum_{i\geq 1} x_i\binom{2i-1}{i-1}y^{i-1}\Big)^{-v+1}\\
&= \frac{1}{v-1}[x_1^{n_1}\dots x_h^{n_h}]\Big(1-\sum_{i\geq 1} x_i\binom{2i-1}{i-1}\Big)^{-v+1}\\
&= \frac{1}{v-1}\binom{v-2+n_1+\cdots+n_h}{v-2, n_1, \dots, n_h}\prod_{i=1}^{h}\binom{2i-1}{i-1}^{n_i}\\
&= \frac{e!}{(v-1)!}\prod_{i=1}^{h}\frac{1}{n_i!}\binom{2i-1}{i-1}^{n_i},
\end{aligned}
$$

which gives (4), upon multiplying by $\frac{1}{v\cdot e}\prod_{i=1}^{h} n_i!(2i)^{n_i}$ to account for unmarking the marked vertex and marked edge, numbering the faces, and marking a corner in each face.

2.4. Extension to weighted bi-orientations

We explain here how the master bijection Φ can easily be extended to the more general setting of weighted bi-orientations. A *bi-orientation* of a map M is the choice of a direction for each half-edge of M. For $i = 0, 1, 2$ we call *i-way* an edge with i ingoing half-edges, as shown in the figure below:



The *indegree* of a vertex v is the number of ingoing half-edges at v, and the clockwise-degree of a face f is the number of outgoing half-edges that have f on their right. This extends the concept of orientation, since an orientation of a map M can be seen as a special kind of bi-orientation, in which each edge is 1-way. Moreover, a bi-oriented map B gives rise to an oriented map $O = \mu(B)$, by blowing each 2-way edge into a clockwise face of degree 2, and inserting a vertex of degree 2 in the middle of each 0-way edge (such a vertex is thus a sink); to distinguish these new faces and vertices of degree 2 we consider them as marked. We can now extend the definition of the families $\mathcal{O}_\delta$ to bi-orientations; by definition a bi-orientation O is in $\mathcal{O}_\delta$ if the oriented map $\mu(O)$ is in the already defined family $\mathcal{O}_\delta$ of orientations. Define now a *bi-mobile* as a plane tree with two kinds of vertices, black or white, where buds might be attached at each black vertex (thus the only difference with mobiles is that edges are allowed to connect two vertices of the same type). The excess of a bi-mobile T is defined as $e_{\bullet-\circ} + 2e_{\circ-\circ} - b$, with $e_{\bullet-\circ}$ the number of black-white edges, $e_{\circ-\circ}$ the number of white-white edges, and b the number of buds.

Let us now explain how the master bijection Φ can be extended to bi-oriented maps. For O a bi-oriented map in $\mathcal{O}_\delta$, consider the mobile $T = \Phi(\mu(O))$. This mobile has marked black (resp. white) vertices of degree 2 associated to the marked faces (resp. marked sinks) of degree 2 in $\mu(O)$. The mobile T can then be simplified into a bi-mobile $\lambda(T)$ by erasing the marked black and white vertices of degree 2, and furthermore erasing the buds associated to the edges incident to marked sinks. The bi-mobile $\lambda(T)$ can easily be seen to have same excess as T. Now we simply extend the master bijection Φ by setting $\Phi(O) = \lambda(T)$, see Figure 2.9. The extended master bijection Φ is now a bijection, for each $\delta \in \mathbb{Z}$, between

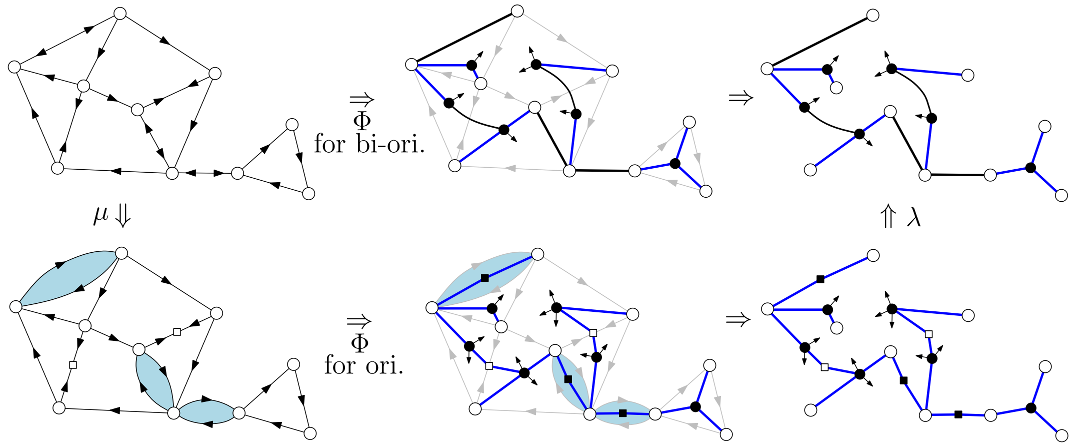


FIGURE 2.9. The master bijection Φ for bi-orientations ($\delta = 10$ here) can be obtained by a reduction to the oriented case (bottow-row) and simplification of the obtained mobile. As shown in Figure 2.10 it can also be obtained (top-row) by applying the local rules of Figure 2.11 to each edge.

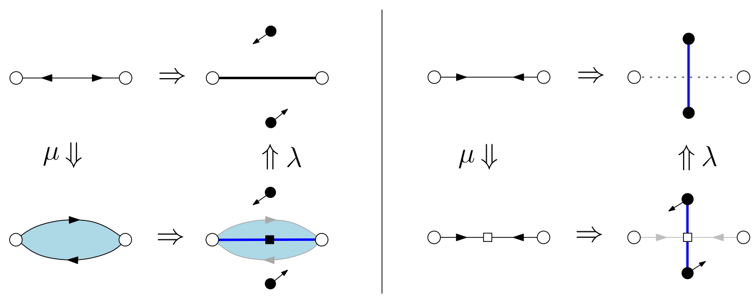


FIGURE 2.10. The effect of the local rule (for oriented maps) on 2-way edges (left-part) and 2-way edges (right-part).

the family $\mathcal{O}_\delta$ of bi-oriented maps and the family of bi-mobiles of excess δ. And, as shown in Figure 2.10, $\Phi(O)$ can be directly obtained from O by applying the extended local rules shown in Figure 2.11 (forgetting weights for now), that is, for each 1-way edge apply the same local rule as before: for each 2-way edge e, keep e unchanged in the bi-mobile, and for each 0-way edge e put the dual edge in the bi-mobile.

In a second step we further extend the master bijection Φ to so-called weighted bi-orientations. A *weighted bi-orientation* of a map M is simply defined as a bi-orientation of M where each edge is assigned a weight in $\mathbb{Z}$, with the condition that weights at ingoing half-edges are positive and weights at outgoing half-edges are nonpositive [1]. For $d \in \mathbb{Z}$, extend the definition of $\mathcal{O}_\delta$ to weighted bi-orientations: a weighted bi-oriented map is said to be in $\mathcal{O}_\delta$ if the underlying unweighted bi-oriented map is in $\mathcal{O}_d$, and with the condition that for $\delta < 0$ the weights on the outer edges are $(0, 1)$ (0 at the outgoing half-edge, 1 at the ingoing half-edge).

[1] It would equally work to assign weights in $\mathbb{R}$, but we will not need it here.

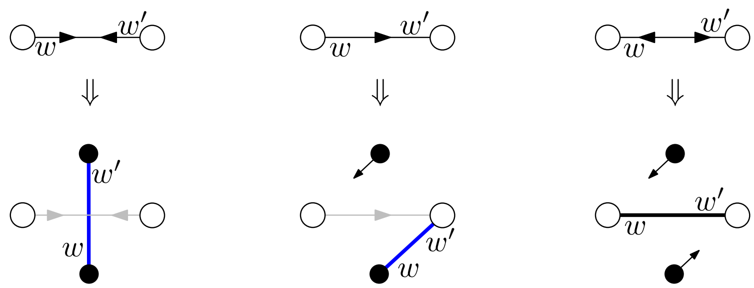


FIGURE 2.11. The local rule performed at each edge (0-way, 1-way or 2-way) in the weighted bi-oriented formulation of the master bijection.

Similarly, define a *weighted bi-mobile* as a bi-mobile where each half-edge (excuding the buds) receives a weight in $\mathbb{Z}$, with the condition that half-edges at black vertices have non-negative weights and half-edges at white vertices have positive weights. Then the master bijection Φ can be extended from bi-orientations to weighted bi-orientations by transfering the weights as shown in Figure 2.11 when applying the local rules of Φ. We obtain (see Figure 2.12 for an example of excess -4):

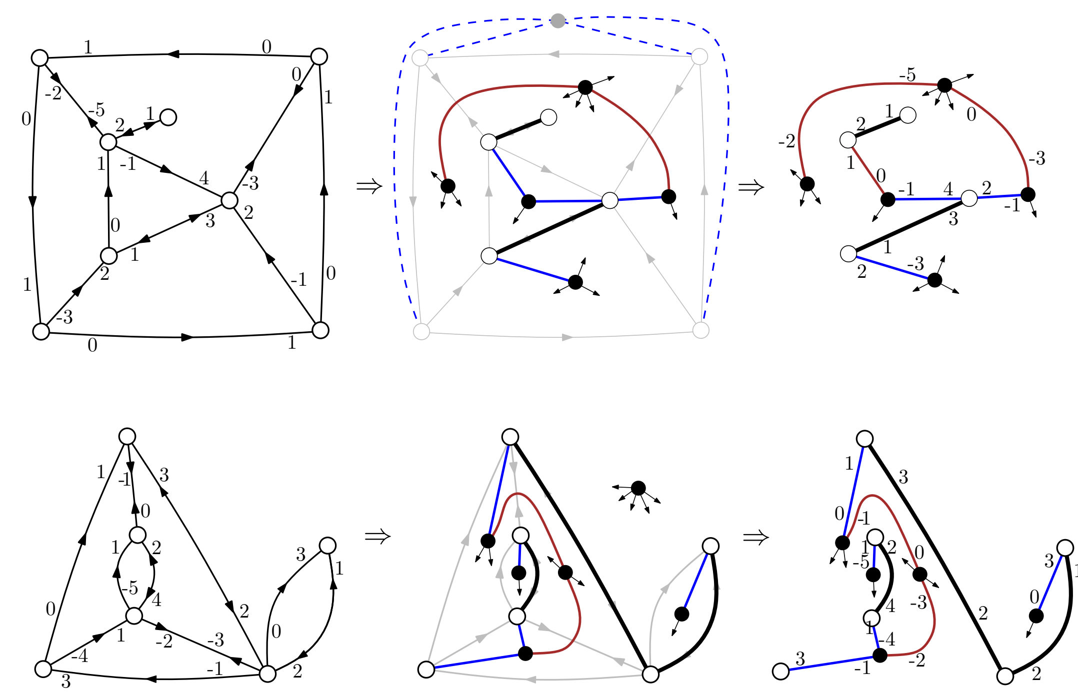


FIGURE 2.12. The master bijection for weighted bi-orientations performed on an example of excess -4.

THEOREM 16. *For $\delta \in \mathbb{Z}$, the extended mapping Φ is a bijection from the family $\mathcal{O}_\delta$ of weighted bi-orientations to the family of weighted bi-mobiles of excess δ.*

Again one can keep track of several weight and degree parameters of the weighted bi-orientation. For a weighted bi-orientation, the *weight* (resp. the *clockwise-degree*) of a face f is defined as the total weight (resp. total number) of outgoing half-edges having f on their right, and the *weight* (resp. the *indegree*) of a vertex v is defined as the total weight (resp. total number) of the ingoing half-edges at v. And for e an edge of O, the *weight* of e is the total weight of the two half-edges of e. For a weighted bi-mobile T and a vertex $v \in T$, the *degree* of v is the number of incident half-edges (including buds) and the weight of v is the sum of the weights of the incident half-edges (excluding buds); and the *weight* of an edge $e \in T$ is the total weight of the two half-edges of e. Then for $\delta \in \mathbb{Z}$, $O \in \mathcal{O}_\delta$, and $T = \Phi(O)$, the parameter-correspondence is as follows:

- For $\delta > 0$ (resp. $\delta < 0$, $\delta = 0$) each vertex (resp. inner vertex, non-marked vertex) v of O corresponds to a white vertex w of T such that the indegree of v equals the degree of w and the weight of v equals the weight of w,
- For $\delta = 0$ (resp. $\delta \neq 0$) each face (resp. inner face) f of O corresponds to a black vertex b of T such that the degree of f equals the degree of b, the clockwise-degree of f equals the number of edges (excluding buds) at b, and the weight of f equals the weight of b,
- For $\delta \geq 0$ (resp. $\delta < 0$) each edge (resp. inner edge) e of O corresponds to an edge ϵ of T such that the weight of e equals the weight of ϵ.

2.5. Proof of Theorem 11

To prove Theorem 11, first in the case $\delta \neq 0$, it is convenient to consider a larger family of orientations that contains all the families $\mathcal{O}_\delta$ for $\delta \neq 0$. Define $\mathcal{F}$ as the family of oriented plane maps with the following conditions:

(i) there is no ccw cycle, except possibly for the outer cycle (when all outer edges are ccw-outer),
(ii) every inner vertex can be accessed by a directed path starting from some outer vertex,
(iii) for every corner $c = (v, e, e')$ in the outer face, with e and e' respectively on the left and on the right at c (looking from v), and such that e is going out of v, then $e' \neq e$ and all the edges (strictly) between e and e' in ccw order around v are inner edges going out of v; the vertex v is called a *floating vertex*.

Note that Property (iii) easily implies that each outer edge of an orientation $O \in \mathcal{F}$ is not incident to the outer face on both sides, hence an outer edge is either cw-outer or ccw-outer, but not both. Note also that, for $\delta > 0$ (resp. $\delta < 0$), $\mathcal{O}_\delta$ is exactly the subfamily of $\mathcal{F}$ where all the $|\delta|$ outer edges are cw-outer (resp. ccw-outer). The *cw-excess* of an orientation in $\mathcal{F}$ is defined as the number of cw-outer edges of O minus the number of ccw-outer edges of O.

We now extend the mapping Φ to orientations from $\mathcal{F}$; for $O \in \mathcal{F}$, define $\Phi(O)$ as the embedded subgraph obtained from O as follows:

(1) insert a black vertex v_f inside each inner face f of O,
(2) for each incidence of an edge $e = (u, v)$ with an inner face f of O, insert a new edge from v_f to v if f is to the right of e, or insert a bud at v_f pointing (without reaching) toward u if f is on the left of e,
(3) delete all the original edges of O, as well as the floating vertices (Property (iii) ensures that these are not incident to any of the new created edges).

Note that in the special cases where $O \in \mathcal{O}_\delta$ for $\delta \neq 0$, the formulation above is equivalent to the formulation of $\Phi(O)$ given in Section 2.1.

The following result easily follows from the definitions:

Claim 17. *Call* cactus-orientation *an orientation from $\mathcal{F}$ with no inner edge. Then, for O a cactus-orientation, $\Phi(O)$ is a mobile whose excess equals the cw-excess of O, and O is the cactus-orientation for T as defined in Section 2.2.*

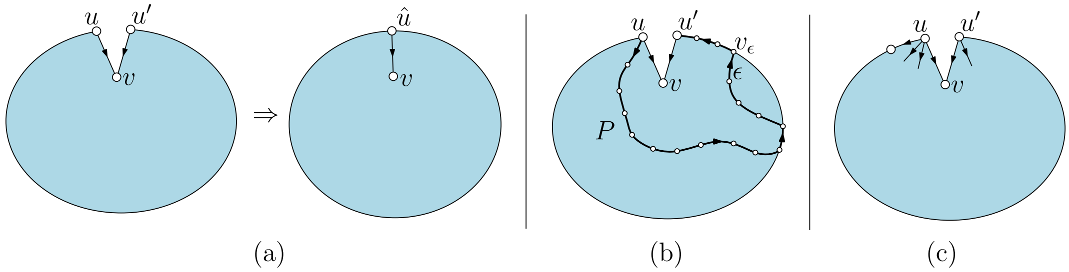


Figure 2.13. (a) A local closure. (b) Proof by contradiction that Property (i) is still satisfied after a local closure. (c) Proof that Property (iii) is still satisfied after a local closure.

For $O \in \mathcal{F}$, two outer edges e, e' of O are said to be *directly matched* if e is cw-outer, e' is ccw-outer, and e' is directly after e in a cw-walk around the outer face of O. Define a *local closure* on $O \in \mathcal{F}$ as the operation of matching a pair of directly matched outer edges into a unique edge, as shown in Figure 2.13(a).

Claim 18. *For O an orientation in $\mathcal{F}$ with outer degree larger than 2, any local closure on O yields an orientation O' that is also in $\mathcal{F}$. In addition, $\Phi(O') = \Phi(O)$ and O' has the same cw-excess as O.*

Proof. Let e, e' be the directly matched pair involved in the local closure, let u and u' be the respective origins of e, e' and let v be their common end, so that the local closure has the effect of merging u and u' into a unique vertex, call it $\hat{u}$.

To show that O' satisfies (i), we have to check that no ccw cycle (except possibly for the outer face contour) can be created by the local closure. Assume such a cycle C is created. Then, before closure, C would form a directed path P from u to u', and P has to traverse at least one inner edge. Let ϵ be the last inner edge traversed by P before reaching u, and let v_ϵ be the end of ϵ. Note that v_ϵ has to be a floating vertex (indeed, u' is a floating vertex, and if $v_\epsilon \neq u'$, then the last portion of P, from v_ϵ to u', only consists of ccw-outer edges), hence we have the contradiction that ϵ is an inner edge going to the floating vertex v_ϵ. To show that O' satisfies (ii), we have to check that an inner vertex w accessible from v in O is still accessible from some outer vertex in O' after the local closure; we easily see that w is actually also accessible from u in O (due to the directed edge from u to v), hence is accessible from $\hat{u}$ in O'. To show that O' satisfies (iii), we just have to check that $\hat{u}$ satisfies (iii) when u is a floating vertex, which is illustrated in Figure 2.13(c). Hence O' is also in $\mathcal{F}$.

Finally, the statements about the cw-excess and the fact that $\Phi(O) = \Phi(O')$ easily follow from the definitions. ☐

To define the inverse operation, we need the following terminology. For $O \in \mathcal{F}$ and $e = (\hat{u}, v)$ an inner edge of O whose origin $\hat{u}$ is an outer vertex, let e' be the next outer edge after e in ccw order around v. Then e is called *admissible* if all the edges strictly between e and e' in ccw order around u are outgoing. The *local opening* of O at e is the operation

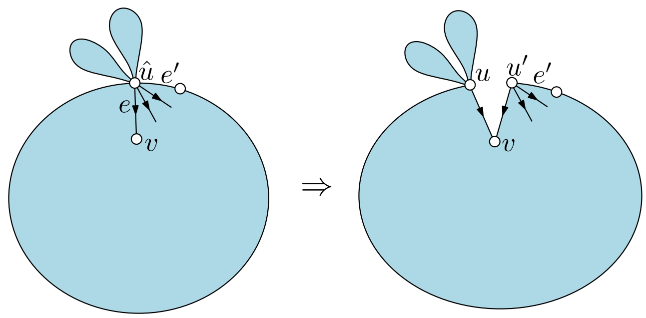


FIGURE 2.14. Local opening at an edge.

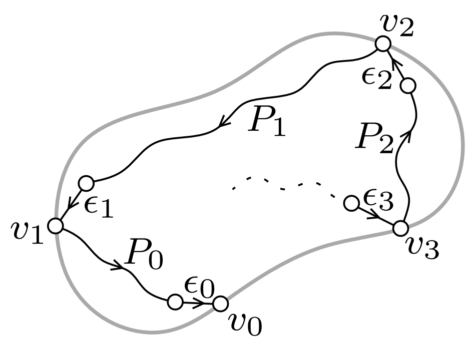


FIGURE 2.15. The situation in the proof of Claim 20.

of splitting e into two edges and $\hat{u}$ into two vertices u, u', so that u' receives the edges of u that are between e and e' (included) in ccw order around $\hat{u}$, see Figure 2.14.

The following result easily follows from the definitions:

CLAIM 19. *For $O \in \mathcal{F}$ and e an admissible inner edge of O, let O' be the orientation obtained from O by a local opening at e. Then $O' \in \mathcal{F}$, $\Phi(O') = \Phi(O)$, and O' has same cw-excess as O.*

We will also need the following result:

CLAIM 20. *Let O be an orientation in $\mathcal{F}$ with at least one inner edge. Then O has at least one admissible inner edge.*

PROOF. Assume O has no admissible inner edge. Then, for each triple v, e, ϵ such that e is an outer edge, ϵ is an inner edge, and ϵ is the next edge after e in cw order around v, ϵ has to be ingoing at v. Since O has at least one inner edge, it is easy to see that such a triple (v_0, e_0, ϵ_0) exists. Property (ii) ensures that there exists a directed path P_0 of inner edges that starts at some outer vertex v_1 of O and ends at ϵ_0, in addition $v_1 \neq v_0$ due to Property (i). Let η_0 be the first edge of P_0, e_1 the next outer edge after η_0 in ccw order around v_1, and ϵ_1 the next edge after e_1 in cw order around v_1, so that ϵ_1 is ingoing at v_1. Again, since O satisfies (ii), there is a directed path P_1 of inner edges that starts at some outer vertex v_2 of O and ends at ϵ_1. In addition v_2 has to strictly avoid the portion between e_0 (included) and e_1 (excluded) in the cw walk around the outer face of O (otherwise P_1 would meet P_0 and create a ccw cycle in O). Continuing iteratively we reach the contradiction that at each step i, the outer vertex v_i has to avoid a strictly growing portion of the outer face contour. $\square$

We can now easily prove Theorem 11 for $\delta \neq 0$. Let $O \in \mathcal{O}_\delta$, with $\delta \neq 0$ (note that O has cw-excess δ). And let $\widetilde{O}$ be obtained from O by a greedy sequence of local openings until there is no inner edge left; note that $\Phi(O) = \Phi(\widetilde{O})$ according to Claim 19. By Claim 17, $\widetilde{O}$ is a cactus-orientation and $T = \Phi(\widetilde{O})$ is a mobile. In addition the excess of T is δ, since it equals (according to Claim 17) the cw-excess of $\widetilde{O}$, which itself equals the cw-excess of O (according to Claim 19). Since a local closure is the inverse of a local opening, O is obtained from $\widetilde{O}$ by a greedy sequence of local closures. Note also that the effect of any greedy sequence of local closures on $\widetilde{O}$ is to match the outer edges according to the cw-matching of $\widetilde{O}$. Hence $\Psi(T) = O$. We have thus proved that $\Phi(O)$ is a mobile T of excess δ such that $\Psi(T) = O$.

Similarly, for T a mobile of excess δ, with $\delta \neq 0$, let $\widetilde{O}$ be the cactus-orientation for T, of cw-excess δ (note that $\Phi(\widetilde{O}) = T$ by Claim 17), and let $O = \Psi(T)$. As already noted, O is obtained from $\widetilde{O}$ by any greedy sequence of local closures, so that, by Claim 18, O is in $\mathcal{O}_\delta$ and $\Phi(O) = \Phi(\widetilde{O}) = T$. This concludes the proof of Theorem 11 for $\delta \neq 0$.

Proof for $\delta = 0$. Let O be a vertex-pointed orientation in $\mathcal{O}_0$, with v_0 the pointed vertex. Consider an edge e incident to v_0, open e into a face of degree 2 to be taken as the outer face. It is easy to see that the obtained orientation $\widehat{O}$ is in $\mathcal{F}$ (of cw-excess 0 since there is one cw-outer edge and one ccw-outer edge), and that $\Phi(O) = \Phi(\widehat{O})$. We can consider the cactus-orientation $\widetilde{O}$ (of cw-excess 0) obtained from $\widehat{O}$ by a greedy sequence of local opening operations. Then $T = \Phi(\widetilde{O})$ is a mobile (of excess 0) with $\widetilde{O}$ as associated cactus. In addition, $\widehat{O}$ is obtained from $\widetilde{O}$ by a greedy sequence of local closures until having outer degree 2, and O is obtained from $\widehat{O}$ by a very last local closure that merges the two outer edges of $\widehat{O}$ and takes their common origin as the pointed vertex of O. Hence $O = \Psi(T)$. Since $\Phi(O) = \Phi(\widehat{O}) = \Phi(\widetilde{O}) = T$, we have thus proved that $T = \Phi(O)$ is a mobile of excess 0 such that $\Psi(T) = O$.

Similarly, for T a mobile of excess 0, let $\widetilde{O}$ be the cactus-orientation for T (of cw-excess 0), and let $\widehat{O} \in \mathcal{F}$ be obtained from $\widetilde{O}$ by a greedy sequence of local closures until the outer degree is 2, so that $\Phi(\widehat{O}) = \Phi(\widetilde{O}) = T$ according to Claim 18. Now let O be the vertex-pointed orientation obtained from $\widehat{O}$ by doing the last local closure, which merges the two outer edges into one edge e, and taking the origin of e as the pointed vertex. Note that O is precisely $\Psi(T)$. In addition it is easy to see that $\widehat{O} \in \mathcal{F}$ implies $O \in \mathcal{O}_0$, and that $\Phi(O) = \Phi(\widehat{O})$. Hence we have proved that $O = \Psi(T)$ is in $\mathcal{O}_0$ and $\Phi(O) = T$. This concludes the proof of Theorem 11 for $\delta = 0$.

CHAPTER 3

A slicing formula for simple bipartite maps

We prove in this chapter the analogue of Tutte's slicing formula for simple bipartite maps (i.e., the counting formula (5) stated in Theorem 2) using the master bijection. First, in Section 3.1, using the bijection for $\delta = -4$ we give a bijection for simple plane quadrangulations, which coincides with the one introduced in [**82**, Sec.2.3.3]. Then, in Section 3.2, we extend this bijection to any bipartite simple plane map with a quadrangular outer face. Then, in Section 3.3 we drop the restriction of having at least one face of degree 4 by considering simple bipartite maps with two marked faces (so-called annular maps) of arbitrary degrees; by the Lagrange inversion formula this yields (5) for $r \geq 2$ faces, which together with the easy verification of the formula for $r = 1$ face (where we are counting rooted plane trees) completes the proof of Theorem 2. We then show in Section 3.4 that, similarly as Tutte's slicing formula, our formula for simple bipartite maps extends to the quasi-bipartite case, i.e., the case where exactly two faces have odd degree. We then briefly present two alternative approaches in Section 3.5.

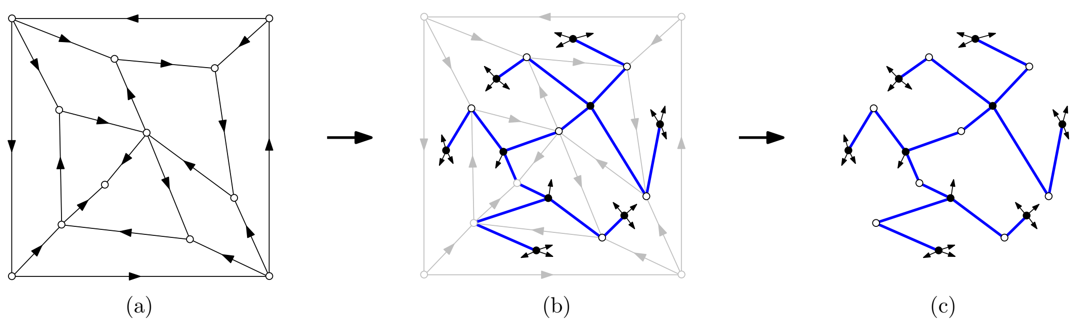


FIGURE 3.1. Left: a simple quadrangulation endowed with its unique 2-orientation in $\mathcal{O}_{-4}$ (obtained from the unique minimal 2-orientation by returning the outer cycle). Right: the corresponding ternary mobile.

3.1. Simple quadrangulations

For Q a plane quadrangulation, define a *2-orientation* of Q as an orientation of Q where all inner vertices have indegree 2 and all outer vertices have indegree 1. The following result is well known (existence can also be proved by constructive algorithms [**75, 43**]):

LEMMA 21. *A plane quadrangulation Q admits a 2-orientation iff Q is simple with at least 2 faces. In the latter case, Q admits a unique 2-orientation in $\mathcal{O}_{-4}$.*

PROOF. We first prove the necessity of Q to be simple. Let C be a cycle of length $2k$ in Q, and let n_v, n_e, n_f be the numbers of vertices, edges, and faces strictly inside C, and let Q_C be the map obtained by keeping all vertices and edges on C or inside C. Then the Euler relation applied to Q_C ensures that $(n_v + 2k) - (n_e + 2k) + (n_f + 1) = 2$, i.e., (i) $n_v - n_e + n_f = 1$. And the fact that all faces inside C are quadrangles ensures that (ii) $2n_e + 2k = 4n_f$. Taking 4∗(i)+(ii) gives $n_e = 2n_v + k - 2$. Note that, if Q is endowed with a 2-orientation, then $n_e - 2n_v$ is also the number of edges inside C having their end on C, hence $n_e - 2n_v \geq 0$. Hence $k \geq 2$, so that a quadrangulation endowed with a 2-orientation has to be simple.

Now let Q be a simple plane quadrangulation, with V the set of vertices and E the set of edges of Q. It is well-known (and can be proved from the Euler formula, as in Section 1.1.3) that for any simple bipartite map M, the number of edges is at most twice the number of vertices minus 4, and equality holds iff M is a quadrangulation. Hence, with the notations of Lemma 9, $|E| = 2|V| - 4$ and for any $S \subseteq V$, $|E_S| \leq 2|S| - 4$. Moreover by definition of 2-orientations, $\alpha(V) = 2|V| - 4$ and for any $S \subseteq V$, $\alpha(S) = 2|S| - |S_{\text{ext}}|$, where S_{ext} is the set of outer vertices of Q that are in S. Hence we have $\alpha(V) = |E|$ and for any $S \subseteq V$, $\alpha(S) \geq |E_S|$ with strict inequality when not all outer vertices are in S. By Lemma 9, Q admits a 2-orientation, and any 2-orientation is accessible from every outer vertex v of Q. And by Lemma 8 Q admits a unique minimal 2-orientation O, and thus the orientation obtained from O by reversing the outer cycle is the unique orientation of Q in $\mathcal{O}_{-4}$. □

Lemma 21 thus gives:

Corollary 22. *Simple plane quadrangulations are in bijection with the subfamily $\mathcal{F}$ of $\mathcal{O}_{-4}$ where each inner vertex has indegree 2 and each inner face has degree 4.*

Now define a *ternary mobile* as a mobile where all white vertices have degree 2 and all black vertices have degree 4 (note that such a mobile corresponds to an unrooted ternary tree where inner nodes are black, the root and the edges at leaves are seen as buds, and a white vertex is inserted in the middle of each edge connecting two inner nodes).

Claim 23. *Any ternary mobile has excess −4.*

Proof. Let $n_\bullet$, $n_\circ$, e, and b be respectively the numbers of black vertices, white vertices, edges, and buds in a ternary mobile T. We have (i) $n_\bullet + n_\circ = e + 1$ (since T is a tree), (ii) $e + b = 4n_\bullet$ (since black vertices have degree 4), (iii) $e = 2n_\circ$ (since white vertices have degree 2). Taking 4∗(i)+(ii)+2∗(iii), we obtain $e - b = -4$. □

Specializing the master bijection Φ for $\delta = -4$ to the subfamily $\mathcal{F}$ we obtain:

Proposition 24. *Simple plane quadrangulations are in bijection with ternary mobiles. For Q a simple plane quadrangulation and T the associated ternary mobile, each inner face of Q corresponds to a black vertex of T.*

We recover here the bijection introduced by Gilles Schaeffer in his PhD [**82**, Sec.2.3.3] between simple plane quadrangulations and unrooted ternary trees.

3.2. Extension to bipartite simple plane maps of outer degree 4

We now extend the bijection of Proposition 24 to bipartite simple plane maps with a quadrangular outer face. Note that an orientation of a map can be seen as a weighted bi-orientation, upon seeing each directed edge as a 1-way edge with weight 0 at the outgoing half-edge and weight 1 at the ingoing half-edge. Hence, a 2-orientation of a plane simple quadrangulation Q can be seen as a weighted bi-orientation such that:

- every inner (resp. outer) vertex has weight 2 (resp. 1),

- every edge has weight 1,
- every face has weight 0.

Now, for M a bipartite plane map with outer degree 4 and no face of degree 2 (as in the previous section it is always assumed that M has at least two faces), we extend the definition as follows. Call *2-orientation* of M a weighted bi-orientation of M such that:

- every inner (resp. outer) vertex has weight 2 (resp. 1),
- every edge has weight 1,
- every inner face of degree $2k$ has weight $-k+2$.

Note that the first condition implies that the half-edge weights are not larger than 2, and then the second condition implies that the edges either have weights $(0,1)$ or $(-1,2)$ (hence all the edges are 1-way). Calling *special* the edges of weights $(-1,2)$, the third condition then implies that each face of degree $2k$ has $k-2$ edges on its contour that are clockwise and special. In view of proving Lemma 25 stated next, we introduce the following definition: for M a plane map (whose vertices are considered as white), the *star-map* of M is the map $\sigma(M)$ obtained from M by inserting a black vertex v_f in each inner face f, and connecting v_f to all corners around f (thus $\sigma(M)$ has two kinds of vertices, black or white, and two kinds of edges, those of M and the new black-white edges).

LEMMA 25. *Let M be a bipartite plane map with a quadrangular outer face and no face of degree* 2*. Then M admits a* 2*-orientation iff M is simple. In that case M admits a unique* 2*-orientation in $\mathcal{O}_{-4}$.*

PROOF. *(Sketch).* First, by similar arguments as in Lemma 3.5 (using the Euler relation) it can be shown that if M is endowed with a 2-orientation, then any cycle of length $2k$ of M must satisfy $k \geq 2$, so that M has to be simple. Now, if M is simple, we can construct a 2-orientation of M as follows, see Figure 3.2. Consider the star-map $\sigma(M)$ of M, and define a 2*-regular orientation* of $\sigma(M)$ as an α-orientation where

- $\alpha(v) = 2$ for each inner white vertex,
- $\alpha(v) = 1$ for each outer white vertex,
- $\alpha(v) = k+2$ for each black vertex of degree $2k$.

Then $\sigma(M)$ can be endowed with a 2-regular orientation by the following procedure, see Figure 3.2(b)-(c):

(1) in each inner face f of M of degree $2k$, with $c_1, \ldots, c_{2k}$ the corners in cw order around f, insert a simple map M_f with a polygonal contour of degree $2k$ and quadrangular inner faces, with $v_1, \ldots, v_{2k}$ the outer vertices of M_f in cw order around its outer contour, and for each $i \in [1..2k]$ insert an edge between c_i and v_i,
(2) the obtained map is a simple quadrangulation Q and can be endowed with a 2-orientation X_Q; by the Euler relation it can be shown that in each inner face f of degree $2k$, exactly $k+2$ of the edges $\{c_i, v_i\}_{1 \leq i \leq 2k}$ are directed from c_i to v_i.
(3) in each inner face f of M contract the inserted map M_f into a black vertex; the obtained orientation X of $\sigma(M)$ is thus a 2-regular orientation.

Note also that, since X_Q is accessible from every outer vertex of Q, then clearly X is accessible from any outer vertex. Hence any 2-regular orientation of M, in particular the minimal one $X_{\min}$, is accessible from every outer vertex v. The minimal 2-regular orientation has also the following crucial local property, see Figure 3.3:

PROPERTY 26. *In $X_{\min}$, for any edge $e = \{b, w\}$ connecting a black vertex b to a white vertex w and directed toward w, the next edge ϵ (which is in M) after e in clockwise order around w is also directed toward w, such an edge ϵ is called* special.

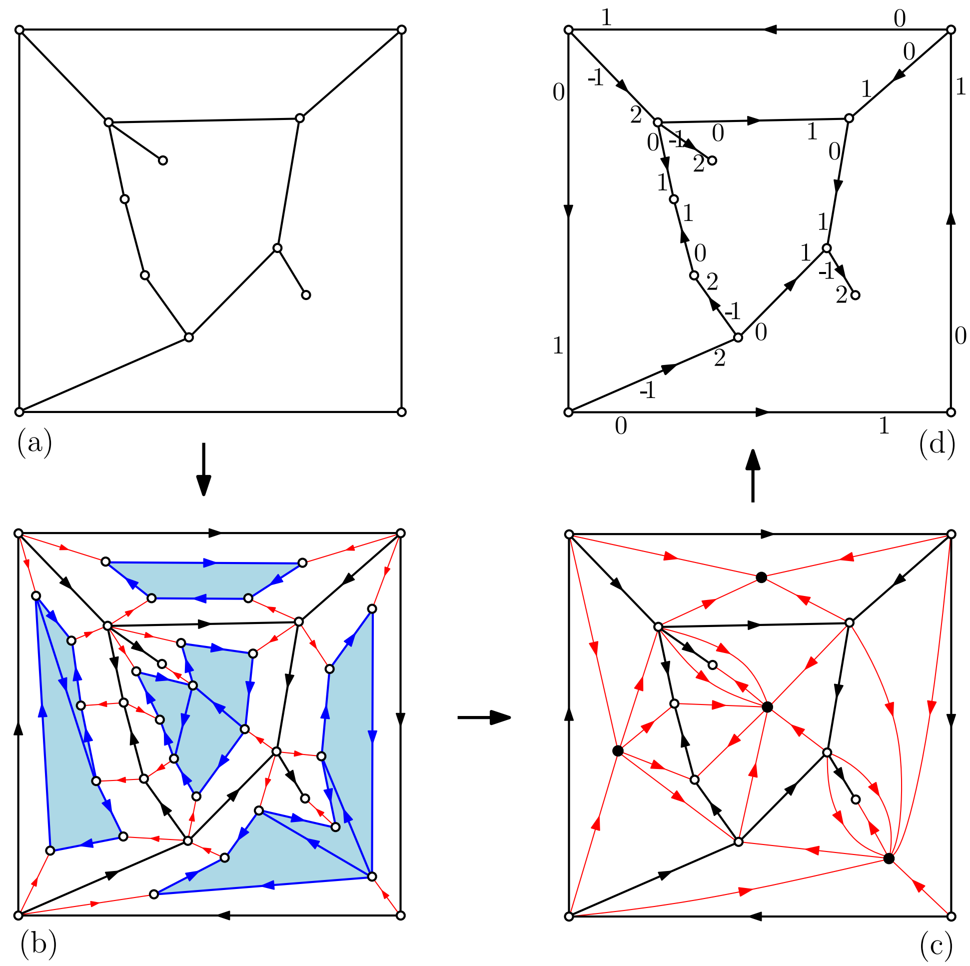


FIGURE 3.2. For M a simple bipartite map with outer degree 4, construction of the unique 2-orientation of M that is in $\mathcal{O}_{-4}$. In (b) M is completed into a simple quadrangulation Q, and Q is endowed with a 2-orientation X_Q. In (c) the orientation X_Q is contracted into a 2-regular orientation X of $\sigma(M)$; if X is not minimal (it is minimal in the example shown) one gets the minimal 2-regular orientation $X_{\min}$ of $\sigma(M)$ by greedily returning ccw cycles. In (d) the orientation induced by $X_{\min}$ on M (with the outer cycle reversed) is the unique 2-orientation of M in $\mathcal{O}_{-4}$ (the edges of weight $(-1, 2)$, are the ones that in $X_{\min}$ were followed by an ingoing edge in ccw order around their end).

Since every black vertex of degree $2k$ has outdegree $k-2$ in $X_{\min}$, Property 26 ensures that every face of M of degree $2k$ has $k-2$ edges on its contour that are clockwise and special. Let Y_0 be the weighted bi-orientation of M induced by $X_{\min}$, where the special edges receive weights $(-1, 2)$ and the non-special edges (which include the outer edges) receive weights $(0, 1)$. Then it is easy to see that with these rules, the weight of every vertex in Y_0 equals its indegree in $X_{\min}$, and moreover the discussion above ensures that every face of degree $2k$

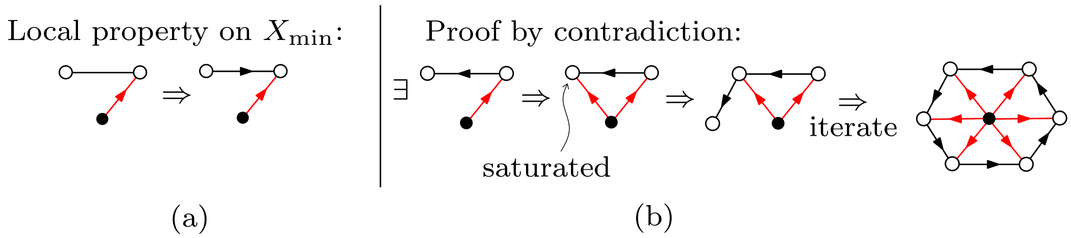


FIGURE 3.3. (a) The local property satisfied by a minimal 2-regular orientation. (b) Proof of the local property by contradiction (failure of the local property would imply the existence of a ccw cycle).

has weight $-k+2$. Hence Y_0 is a 2-orientation. Clearly Y_0 is minimal. Let v_0 be an outer vertex of Y_0 To show that Y_0 is accessible from v_0, it is useful to use the concept of rightmost path [**11**] applied to $X_{\min}$ (it applies more generally to orientations that are minimal and are accessible from a fixed outer vertex v_0). For e an edge of $X_{\min}$, the *rightmost path* of e is the unique oriented path P_e starting from v_0, ending at e, and such that there is no edge pointing to P_e from the left side of P_e (possibly P_e can pass several times by a same vertex). Then it is easy to see that, by Property 26, if e is an edge of M, then all the edges of P_e are also in M, and since every vertex has at least one ingoing edge in M, this guarantees that Y_0 is accessible from v_0. Hence the orientation obtained from Y_0 by reversing the outer cycle is in $\mathcal{O}_{-4}$. This gives the existence of a 2-orientation of M in $\mathcal{O}_{-4}$. Now, if there is another 2-orientation in $\mathcal{O}_{-4}$, let Y_0' be the associated minimal 2-orientation obtained by reversing the outer cycle, and let X_0' be the 2-regular orientation of $\sigma(M)$ that coincides with Y_0' on the edges of M, and such that the edges directed from black to white vertices are exactly those following a special edge ϵ of Y_0' in ccw order around the end of ϵ. Then it can be checked by a similar argument as in Figure 2.15 that X_0' is minimal, hence X_0' is the minimal 2-regular orientation of $\sigma(M)$, hence $X_0' = X_0$, so that $Y_0 = Y_0'$, giving a contradiction. □

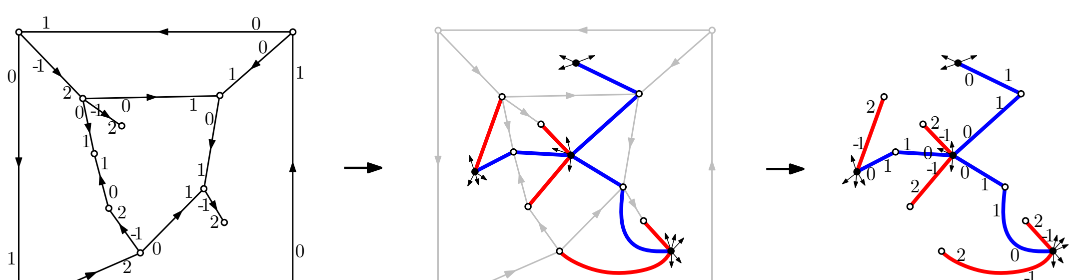


FIGURE 3.4. Left: the bipartite simple map of Figure 3.2 endowed with its unique 2-orientation in $\mathcal{O}_{-4}$. Right: the corresponding 2-branching mobile.

Now define a *2-branching mobile* as a weighted bi-mobile such that all black vertices have even degree, each black vertex of degree $2k$ has weight $-k+2$, each white vertex has weight 2, and each edge has weight 1. Again these conditions imply that the edges of such a bi-mobile T are of two types, those of weights $(0,1)$, and those of weights $(-1,2)$ (in particular all edges are black-white, so that T is a weighted mobile). Note that each white vertex of a 2-branching mobile T has either degree 2, with the two incident edges of weights $(0, 1)$, or has degree 1 with the unique incident edge of weights $(-1, 2)$; such an edge is called a *pending edge* of T. Note that, by the weight condition on black vertices, each black vertex of degree $2k$ is incident to exactly $k-2$ pending edges.

By similar arguments as in Claim 23, we have:

CLAIM 27. *Any 2-branching mobile has excess* -4.

Since simple bipartite plane maps of outer degree 4 correspond bijectively to 2-orientations in $\mathcal{O}_{-4}$, the master bijection for $\delta = -4$ gives:

PROPOSITION 28. *Simple bipartite plane maps of outer degree* 4 *are in bijection with 2-branching mobiles. For* M *a simple bipartite plane map of outer degree* 4 *and* T *the associated 2-branching mobile, each inner face of* M *corresponds to a black vertex of* T *of the same (even) degree.*

Call *rooted 2-branching mobile* a 2-branching mobile with a marked bud (it is convenient to include the mobile with a unique black vertex carrying 2 buds, one of which is the root-bud). Denote by $S \equiv S(t; x_2, x_3, \ldots)$ the series of rooted 2-branching mobiles where t is conjugate to the number of non-root buds and x_i is conjugate to the number of black vertices of degree $2i$, for $i \geq 2$. The bijection of Proposition 28 would make it possible to prove (5) when at least one face has degree 4, by coefficient extraction in S. To avoid too many repetitions (similar calculations will be done in the next section to prove (5) in full generality), we skip the details and give here only the equation that specifies S:

$$S = t + \sum_{i \geq 2} x_i \binom{2i-1}{i-2} S^{i+1},$$

which is classically obtained (similarly as in Figure 2.8) from a decomposition at the root, and where the factor $\binom{2i-1}{i-2}$ gives the number of ways to place the $i-2$ pending edges at a black vertex of degree $2i$.

3.3. Extension to bipartite simple annular maps

The bipartite simple (plane) maps considered in the last section have the restriction that the outer face has degree 4. In this section we drop this restriction by considering bipartite simple maps having two marked faces of arbitrary (even) degrees. Define an *annular map* as a map with two marked faces f_0, f_1, where f_0 is to be considered as the outer face and f_1 as a marked inner face. For M an annular map, a *separating cycle* (resp. *non-separating cycle*) is a cycle containing (resp. not containing) f_1 in its interior, and the *separating girth* (resp. *non-separating girth*) of M is the length of a shortest separating cycle (resp. shortest non-separating cycle). For two separating cycles C, C' we say that C *contains* C' if the region enclosed by C is included in the region enclosed by C' (where possibly the two regions are equal, e.g. C contains itself).

LEMMA 29. *For* M *an annular map of separating girth* s, *and for two separating cycles* C_1, C_2 *each of length* s, *there exists a separating cycle* C_0 *of length* s *and that contains both* C_1 *and* C_2.

PROOF. Let R_1 (resp. R_2) be the region enclosed by C_1 (resp. C_2). And let $R = R_1 \cap R_2$ and $R' = R_1 \bigcup R_2$, C the contour of R and C' the contour of R'. As shown in Figure 3.6, we have $|C| + |C'| \leq |C_1| + |C_2|$. Since one can extract a separating cycle from C (resp. from

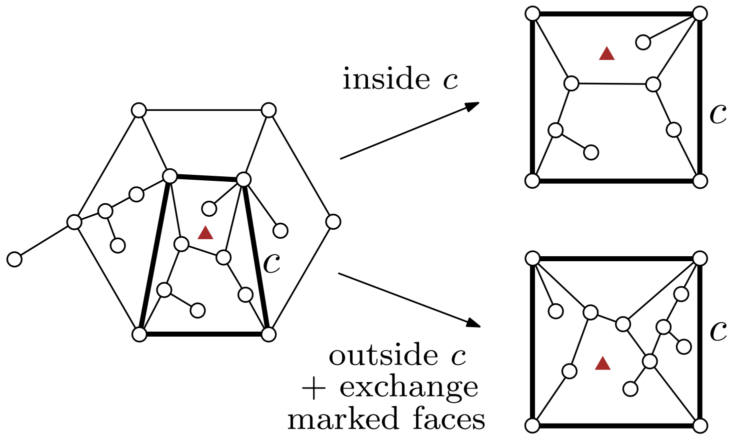


FIGURE 3.5. Left: A bipartite annular map in $\mathcal{A}_{2s}^{(2p,2q)}$ (with $p = 4$, $q = 3$, $s = 2$ in the example), where the marked inner face is indicated by a triangle, and the canonical cycle c is drawn bolder. Right: Cutting along c yields two components, one for the part inside c and the other one for the part outside c (the roles of the two marked faces for the second component being switched), which are respectively in the families $\mathcal{C}_{2s}^{(2q)}$ and $\mathcal{B}_{2s}^{(2p)}$.

C'), we must have $|C| = s$ and $|C'| = s$. Hence we can take C' as a cycle C_0 that fits the stated conditions. □

	∅ ∅	1 1	2 2	1,2 1,2	∅ 1	∅ 2	1 2	∅ 1,2	1 1,2	2 1,2
$\in C_1$					×		×	×		×
$\in C_2$						×	×	×	×	
$\in C$								×	×	×
$\in C'$					×	×		×		

FIGURE 3.6. The table shows all the possible cases for the status of the area on each side of an edge e (either in R_1 only, or in R_2 only, or in both, or in none). We see that in all cases, the contribution to $|C_1| + |C_2|$ is at least as large as the contribution to $|C| + |C'|$ (with C_1, C_2, C, C' the contours of $R_1, R_2, R_1 \cap R_2, R_1 \cup R_2$, respectively).

Thus, for M an annular map of separating girth s, M has a separating cycle of length s that contains all separating cycles of length s. This cycle, called the *canonical cycle* of M, can be seen as the *outermost* separating cycle of length s. Our strategy is to use the canonical cycle to split any bipartite simple annular map M into components that can be characterized by certain weighted bi-orientations and are amenable to a specialization of the master bijection.

For $s, p, q \geq 1$, denote by $\mathcal{A}_{2s}^{(2p,2q)}$ the family of bipartite annular maps with separating girth $2s$ and non-separating girth at least 4, where the outer face has degree $2p$ and the marked inner face has degree $2q$; let $\mathcal{C}_{2s}^{(2q)} := \mathcal{A}_{2s}^{(2s,2q)}$, and denote by $\mathcal{B}_{2s}^{(2q)}$ the subfamily of $\mathcal{C}_{2s}^{(2q)}$ such that the outer face contour is the unique separating cycle of length $2s$. Finally denote respectively by $\vec{\mathcal{A}}_{2s}^{(2p,2q)}$, $\vec{\mathcal{C}}_{2s}^{(2q)}$, $\vec{\mathcal{B}}_{2s}^{(2q)}$, the corresponding families where there is a distinguished corner in each of the two marked faces.

Let $M \in \mathcal{A}_{2s}^{(2p,2q)}$, with f_0, f_1 the outer face and the marked inner face, and let c be the canonical cycle of M. If we cut along c, we obtain two maps $M_0, M_1 \in \mathcal{C}_{2s}^{(2q)} \times \mathcal{B}_{2s}^{(2p)}$, where M_0 is the part inside c (with the face delimited by c as outer face and f_1 as marked inner face) and M_1 is the part outside c (with the face delimited by c as outer face and f_0 as marked inner face). Conversely, in a rooted formulation, take two maps $M_0, M_1 \in \vec{\mathcal{C}}_{2s}^{(2q)} \times \vec{\mathcal{B}}_{2s}^{(2p)}$, with f_0 the marked inner face of M_0 and f_1 the marked inner face of M_1. Let c_0 be the outer face contour of M_0 and v_0 the outer vertex incident to the marked corner in the outer face; define similarly c_1 and v_1 for M_1. Then paste M_0, M_1 at their outer contours (both of length $2s$) so that v_0 and v_1 get identified, and c_0, c_1 are merged into a unique cycle c, where we distinguish the vertex resulting from the identification of v_0 and v_1. Let M be the obtained map with f_0 as the outer face and f_1 as the marked inner face. We easily check that c is the canonical cycle of M and that $M \in \vec{\mathcal{A}}_{2s}^{(2p,2q)}$. The decomposition along the canonical cycle thus yields the isomoprhism:

$$2s \times \vec{\mathcal{A}}_{2s}^{(2p,2q)} \simeq \vec{\mathcal{C}}_{2s}^{(2q)} \times \vec{\mathcal{B}}_{2s}^{(2p)}. \tag{11}$$

Define respectively $\vec{A}_{2s}^{(2p,2q)} \equiv \vec{A}_{2s}^{(2p,2q)}(x_2, x_3, \ldots)$, $\vec{C}_{2s}^{(2q)} \equiv \vec{C}_{2s}^{(2q)}(x_2, x_3, \ldots)$, and $\vec{B}_{2s}^{(2p)} \equiv \vec{B}_{2s}^{(2p)}(x_2, x_3, \ldots)$ as the generating functions of $\vec{\mathcal{A}}_{2s}^{(2p,2q)}$, $\vec{\mathcal{C}}_{2s}^{(2p)}$, and $\vec{\mathcal{B}}_{2s}^{(2q)}$, where x_i is conjugate to the number of non-marked inner faces of degree $2i$ for $i \geq 2$. Then (11) gives $2s \cdot \vec{A}_{2s}^{(2p,2q)} = \vec{C}_{2s}^{(2p)} \cdot \vec{B}_{2s}^{(2q)}$, and in the particular case $p = s$, gives $2s \cdot \vec{C}_{2s}^{(2q)} = \vec{C}_{2s}^{(2s)} \cdot \vec{B}_{2s}^{(2q)}$. Hence

$$\vec{A}_{2s}^{(2p,2q)} = \frac{\vec{C}_{2s}^{(2p)} \cdot \vec{C}_{2s}^{(2q)}}{\vec{C}_{2s}^{(2s)}}. \tag{12}$$

We now introduce weighted bi-orientations for annular maps in $\mathcal{C}_{2s}^{(2p)}$. For $s, p \geq 1$ and M a bipartite annular map of outer degree $2s$ and marked inner face of degree $2p$, define a $(2, 2s)$-*orientation* of M as a weighted bi-orientation such that:

- every inner (resp. outer) vertex has weight 2 (resp. 1),
- every edge has weight 1,
- every non-marked inner face of degree $2k$ has weight $-k+2$, and the marked inner face, of degree $2p$, has weight $-p+s$.

Again these conditions imply that the edges are of two types, those of weights $(0,1)$ and those of weights $(-1,2)$, which are called special. This time every inner face of degree $2k$ has $k-2$ clockwise special edges on its contour, except for the marked inner face, which has $k-s$ clockwise special edges; note that for $s = 2$ the marked inner face plays no special role and we recover the definition of the previous section.

LEMMA 30. *For $s, p \geq 1$, let M be a bipartite annular map of outer degree $2s$ and marked inner face of degree $2p$. Then M admits a $(2, 2s)$-orientation iff $M \in \mathcal{C}_{2s}^{(2p)}$. In the latter case M admits a unique $(2, 2s)$-orientation in $\mathcal{O}_{-2s}$.*

PROOF. (Sketch.) By similar arguments as in Lemma 21 using the Euler relation, it can be shown that if M is endowed with a 2-orientation then any cycle C of length $2k$ in M satisfies $k \geq 2$ if C does not enclose the marked inner face, and satisfies $k \geq s$ if C encloses

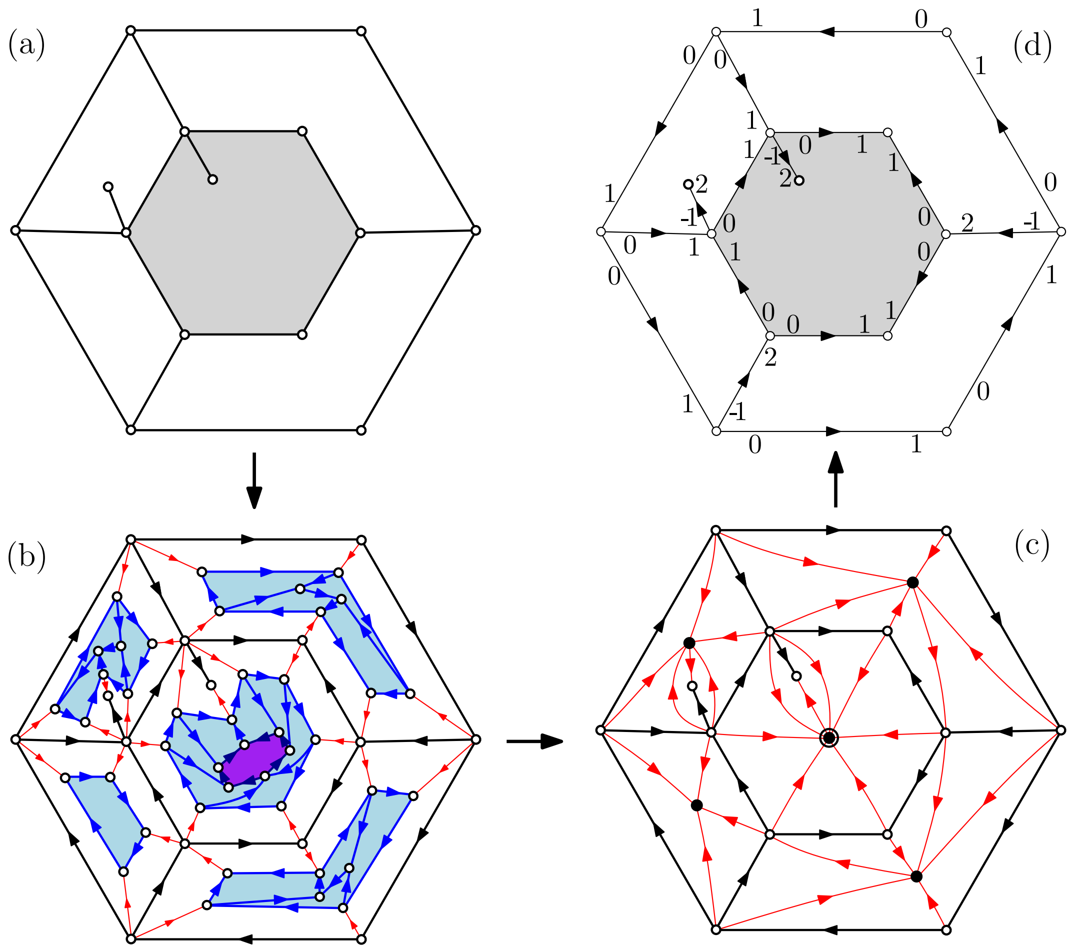


FIGURE 3.7. For M a simple bipartite map in $\mathcal{C}_{2s}^{(2p)}$ ($s = 3$ and $p = 4$ here), construction of the unique $(2, 2s)$-orientation of M that is in $\mathcal{O}_{-2s}$. In (b) M is completed into a map $Q \in \mathcal{E}_s$, and Q is endowed with a $(2, 2s)$-orientation (with no special edge, all inner vertices have indegree 2). In (c) the $(2, 2s)$-orientation of Q is contracted into a $(2, 2s)$-regular orientation X of $\sigma(M)$, where the black vertex inside the marked inner face f_1 of M gets indegree $\deg(f_1)/2+s$; the obtained orientation (if not already minimal) can then be made minimal by greedily returning ccw cycles. In (d) the induced orientation of M with the outer cycle reversed is the unique 2-orientation of M in $\mathcal{O}_{-2s}$ (special edges, indicated by a double arrow, are the ones that in X were followed by an ingoing edge in ccw order around their end).

the marked inner face, so that M must be in $\mathcal{C}_{2s}^{(2p)}$. To show existence of such an orientation for $M \in \mathcal{C}_{2s}^{(2p)}$ (and accessibility from the outer cycle), the base-case is for the subfamily $\mathcal{E}_s \subset \mathcal{C}_{2s}^{(2s)}$ where all non-marked inner faces have degree 4. In that case, a 2-orientation of M has no special edge, and existence (and accessibility from the outer contour) can be proved using Lemma 9 and the Euler relation, in a similar way as done in Lemma 21 for simple quadrangulations. Then the general case can be proved very similarly as in Lemma 25 with the following adjustements: (i) in the so-called $(2, 2s)$-regular orientation of $\sigma(M)$, the black vertex inserted in the marked inner face f has indegree $\deg(f)/2+s$ (instead of $\deg(f)/2+2$ for the other inner faces), (ii) to prove the existence of a regular orientation of $\sigma(M)$ one completes M (adding vertices and edges inside each face f of M) into a map $M' \in \mathcal{E}_s$ (instead of completing it into a simple quadrangulation as in Lemma 25), where the marked face of M' is in the area of the marked inner face of M, see Figure 3.7. □

Define now a $(2, 2s)$-branching mobile as a weighted bi-mobile where all black vertices, one of which is marked, have even degree, such that every unmarked (resp. the marked) black vertex of degree $2k$ has weight $-k + 2$ (resp. weight $-k + s$), every edge has weight 1, and every vertex has weight 2. Again the edges are of two types, those of weights $(0, 1)$ and those of weights $(-1, 2)$ called *pending edges*, each white vertex is incident either to two non-pending edges or to one pending edge; and the weight conditions at black vertices then imply that every non-marked (resp. the marked) black vertex of degree $2k$ is incident to $k - 2$ (resp. $k - s$) pending edges. Similarly as in Claims 23 and 27 we have

CLAIM 31. *Each $(2, 2s)$-branching mobile has excess $-2s$.*

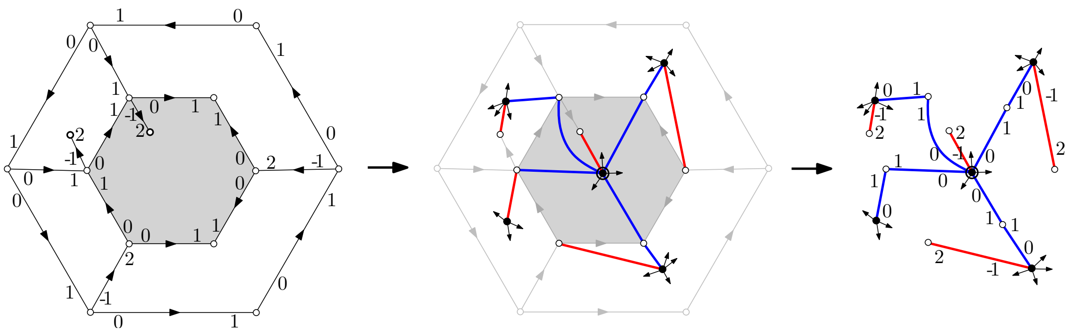


FIGURE 3.8. Left: A bipartite simple annular map in $\mathcal{C}_6^{(8)}$ endowed with its unique $(2, 6)$-orientation in $\mathcal{O}_{-6}$. Right: the corresponding $(2, 6)$-branching mobile.

By Lemma 30 the master bijection Φ for $\delta = -2s$ thus gives:

PROPOSITION 32. *The family $\mathcal{C}_{2s}^{(2p)}$ of annular maps is in bijection with the family of $(2, 2s)$-branching mobiles where the marked black vertex has degree $2p$. For $M \in \mathcal{C}_{2s}^{(2p)}$ and T the associated weighted mobile, each non-marked inner face in M corresponds to a non-marked black vertex of the same (even) degree in T.*

We can then use (12) and Proposition 32 to obtain an explicit expression for the series $\vec{A}^{(2p,2q)} := \sum_{s\geq 2} \vec{A}_{2s}^{(2p,2q)}$, i.e., the generating function of bipartite simple annular maps with prescribed degrees of the two marked faces (and a distinguished corner in each of the two marked faces).

PROPOSITION 33. *For $p, q \geq 2$, the generating function $\vec{A}^{(2p,2q)}$ is given by*

$$\vec{A}^{(2p,2q)} = \frac{4pq}{p+q}\binom{2p-1}{p-2}\binom{2q-1}{q-2}S^{p+q}|_{t=1}, \quad \text{where } S = t + \sum_{i\geq 2} x_i \binom{2i-1}{i-2} S^{i+1}. \tag{13}$$

PROOF. Let $T_{2s}^{(2p)} \equiv T_{2s}^{(2p)}(t; x_2, x_3, \ldots)$ be the series counting $(2, 2s)$-branching mobiles whose marked black vertex has degree $2p$, with a distinguished corner at the marked black vertex, where t is conjugate to the number of buds and x_i is conjugate to the number of non-marked black vertices of degree $2i$, for $i \geq 2$. Then Proposition 32 gives $\vec{C}_{2s}^{(2p)} = 2s \cdot T_{2s}^{(2p)}|_{t=1}$ (where the factor $2s$ is due to the marked corner in the outer face of annular maps in $\vec{\mathcal{C}}_{2s}^{(2p)}$). As in the previous section, call *pending edge* an edge incident to a white leaf. In a $(2, 2s)$-branching mobile, the marked black vertex b carries $\deg(b)/2 - s$ pending edges (edges leading to a white leaf) and the other $\deg(b)/2 + s$ adjacencies are with "rooted mobiles" such as considered in the previous section, which are counted by the series S specified by $S = t + \sum_{i \geq 2} x_i \binom{2i-1}{i-2} S^{i+1}$. Hence $T_{2s}^{(2p)} = \binom{2p}{p-s} S^{p+s}$, so that $\vec{C}_{2s}^{(2p)} = 2s \binom{2p}{p-s} S^{p+s}|_{t=1}$. By (12), it gives (note that the power of S does not depend on s):

$$\vec{A}_{2s}^{(2p,2q)} = 2s \binom{2p}{p-s} \binom{2q}{q-s} S^{p+q}|_{t=1}. \tag{14}$$

Hence, defining $\beta(p, q, s) := 2s\binom{2p}{p-s}\binom{2q}{q-s}$, we have $\vec{A}^{(2p,2q)} = \gamma(p, q, 2)\ S^{p+q}$, where $\gamma(p, q, d) := \sum_{s=d}^{\min(p,q)} \beta(p, q, s)$. It is then easy to check by a decreasing induction on d (starting from $d = \min(p, q)$) that $\gamma(p, q, d)$ is equal to $\eta(p, q, d) := \frac{4pq}{p+q}\binom{2p-1}{p-d}\binom{2q-1}{q-d}$, noticing that $\eta(p, q, d) - \eta(p, q, d + 1) = \beta(p, q, d)$. □

We can then easily extract the coefficients of $\vec{A}^{(2p,2q)}$ using the Lagrange inversion formula. First, note that, since $S = t/(1 - \sum_{i \geq 2} x_i \binom{2i-1}{i-2} S^i)$, then $\tilde{S} := S/t$ satisfies $\tilde{S} = 1/(1 - \sum_{i \geq 2} x_i t^i \binom{2i-1}{i-2} \tilde{S}^i)$, so that, for any $k \geq 1$, and for any nonnegative numbers $n_2, \ldots, n_h$, we have,

$$\begin{aligned}
[x_2^{n_2} \cdots x_h^{n_h}] S^k|_{t=1} &= [x_2^{n_2} \cdots x_h^{n_h}] \tilde{S}^k|_{t=1} \\
&= [t^{\sum_i i n_i} x_2^{n_2} \cdots x_h^{n_h}] \tilde{S}^k \\
&= [t^{k + \sum_i i n_i} x_2^{n_2} \cdots x_h^{n_h}] S^k.
\end{aligned}$$

Now, by the Lagrange inversion formula, we have, with $e = p + q + \sum_i i n_i$ (note that e gives the number of edges in the associated annular map):

$$\begin{aligned}
[x_2^{n_2} \cdots x_h^{n_h}] S^{p+q}|_{t=1} &= [t^e x_2^{n_2} \cdots x_h^{n_h}] S^{p+q} \\
&= \frac{p+q}{e} [x_2^{n_2} \cdots x_h^{n_h}][y^{e-p-q}] \Big(1 - \sum_{i \geq 2} x_i \binom{2i-1}{i-2} y^i\Big)^{-e} \\
&= \frac{p+q}{e} [x_2^{n_2} \cdots x_h^{n_h}] \Big(1 - \sum_{i \geq 2} x_i \binom{2i-1}{i-2}\Big)^{-e} \\
&= \frac{p+q}{e} \binom{e - 1 + n_2 + \cdots + n_h}{e - 1, n_2, \ldots, n_h} \prod_{i=2}^{h} \binom{2i-1}{i-2}^{n_i} \\
&= (p+q) \frac{(e + r - 3)!}{e!} \prod_{i=2}^{h} \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i},
\end{aligned}$$

with $r = 2 + \sum_i n_i$ the associated number of faces. Hence

$$[x_2^{n_2} \cdots x_h^{n_h}] \vec{A}^{(2p,2q)} = \frac{(e + r - 3)!}{e!} 4pq \binom{2p-1}{p-2} \binom{2q-1}{q-2} \prod_{i=2}^{k} \frac{1}{n_i!} \binom{2i-1}{i-2}^{n_i},$$

which proves (5) for $r \geq 2$ faces (upon multiplying by $\prod_{i=2}^{h} n_i!(2i)^{n_i}$ to account for numbering the non-marked faces and marking a corner in each of these faces). Finally, the case of one face in (5) is just the fact that Catalan numbers count rooted plane trees, which concludes the proof of Theorem 2.

REMARK 34. *Another possible way to prove Theorem 2 is via a substitution approach. Indeed the bipartite model and the simple bipartite model are well-known to be linked by the following substitution scheme* [**9, 29**]*: any rooted bipartite map M is uniquely obtained from a rooted simple bipartite map C where every edge is either left untouched or is doubled into a 2-cycle inside which a rooted bipartite map is patched; C is called the* simple core *of M. Denote by $F \equiv F(t; x_1, x_2, x_3, \ldots)$ (resp. $G \equiv G(t; x_2, x_3, \ldots)$) the series of rooted bipartite maps (resp. rooted simple bipartite maps) where t is conjugate to the number of edges and x_i is conjugate to the number of faces of degree $2i$ for $i \geq 1$. Then the above substitution scheme gives:*

$$F(t; x_1, x_2, x_3, \ldots) = G(u, x_2, x_3, \ldots), \quad \text{where } u = t + t\,F(t; x_1, x_2, x_3, \ldots).$$

From this equation it is then in principle possible to obtain a "Lagrangean" expression for G from a Lagrangean expression for F (which itself can be obtained from the BDG bijection [**26**] *using an integration-step to unmark the pointed vertex, or more directly from the bijection in* [**81**]*). Such calculations in the univariate case can be found for instance in* [**55**, Sec.2.9]*, also in* [**29**] *for quadrangulations and $2p$-angulations, and more generally in* [**31**] *for multivariate series to count maps of girth at least d with a control on the degrees of the faces.*

3.4. The case of two odd faces

It is known that Tutte's slicing formula can be extended to *quasi-bipartite* maps, i.e., maps with exactly two faces of odd degree, the first combinatorial proof was given in [**40**, Theo.VI p.75] (by a reduction to the bipartite case); it can also be proved using an extension [**26**] of the bijection of Section 2.3 to arbitrary maps with control on the face degrees, see e.g. [**34**, Prop.7.5] and [**A29**]. The unified statement is as follows, with the notations of Section 1.2.2:

THEOREM 35 (Tutte). *Let $\ell_1, \ldots, \ell_r$ be positive numbers. Then, when 0 (bipartite case) or 2 (quasi-bipartite case) of the ℓ_i are odd we have*

$$|\mathcal{A}(\ell_1, \ldots, \ell_r)| = \frac{(e-1)!}{v!} \prod_{i=1}^{r} \ell_i \binom{\ell_i - 1}{\lceil \ell_i/2 - 1 \rceil}, \tag{15}$$

where $e = \frac{1}{2}\sum_i \ell_i$ and $v = e - r + 2$.

As we show here, our formula for simple bipartite maps has a similar extension:

THEOREM 36. *Let $\ell_1, \ldots, \ell_r$ be positive numbers. Then, when 0 (bipartite case) or 2 (quasi-bipartite case) of the ℓ_i are odd we have*

$$|\mathcal{S}(\ell_1, \ldots, \ell_r)| = \frac{(e+r-3)!}{e!} \prod_{i=1}^{r} \ell_i \binom{\ell_i - 1}{\lceil \ell_i/2 - 2 \rceil}, \tag{16}$$

where $e = \frac{1}{2}\sum_i \ell_i$.

The approach follows the very same lines as the one in the previous section (relying on annular maps), so that we only list the main arguments. For $p, q, s \geq 0$, denote by $\mathcal{A}_{2s+1}^{(2p+1,2q+1)}$ the family of annular maps where the outer face has degree $2p+1$, the marked inner face has degree $2q+1$, and the separating girth is $2s+1$ (it has to be odd since all cycles separating the two odd degree faces have odd length) and the non-separating girth is at least 4. Let $\mathcal{C}_{2s+1}^{(2q+1)} := \mathcal{A}_{2s+1}^{(2s+1,2q+1)}$, and denote by $\mathcal{B}_{2s+1}^{(2q+1)}$ the subfamily of $\mathcal{C}_{2s+1}^{(2q+1)}$ such that the outer cycle is the unique separating cycle of length $2s+1$. And let $\vec{\mathcal{A}}_{2s+1}^{(2s+1,2q+1)}$, $\vec{\mathcal{C}}_{2s+1}^{(2q+1)}$, $\vec{\mathcal{B}}_{2s+1}^{(2q+1)}$ be the associated families with a distinguished corner in the outer face and in the marked inner face. Then by the same arguments as to obtain (11) we have

$$(2s+1)\vec{\mathcal{A}}_{2s+1}^{(2p+1,2q+1)} \simeq \vec{\mathcal{C}}_{2s+1}^{(2q+1)} \times \vec{\mathcal{B}}_{2s+1}^{(2p+1)}.$$

Considering now the generating functions $\vec{A}_{2s+1}^{(2p+1,2q+1)} \equiv \vec{A}_{2s+1}^{(2p+1,2q+1)}(x_2,x_3,\ldots)$ (resp. $\vec{C}_{2s+1}^{(2q+1)} \equiv \vec{C}_{2s+1}^{(2q+1)}(x_2,x_3,\ldots)$) of maps from $\vec{\mathcal{A}}_{2s+1}^{(2p+1,2q+1)}$ (resp. $\vec{\mathcal{C}}_{2s+1}^{(2q+1)}$), where x_i is conjugate to the number of non-marked inner faces of degree $2i$, we have similarly as in (12)

$$\vec{A}_{2s+1}^{(2p+1,2q+1)} = \frac{\vec{C}_{2s+1}^{(2p+1)} \cdot \vec{C}_{2s+1}^{(2q+1)}}{\vec{C}_{2s+1}^{(2s+1)}}.$$

The next step is to charaterize maps in $\mathcal{C}_{2s+1}^{(2q+1)}$ by certain orientations. For M a quasi-bipartite annular map with outer face degree $2s+1$ and marked inner face degree $2q+1$, define a $(2,2s+1)$-orientation of M as weighted bi-orientation where each inner (resp. outer) vertex has weight 2 (resp. 1), each edge has weight 1, each non-marked inner face of degree $2k$ has weight $-k+2$, the marked inner face has weight $-q+s$, and the outer face has weight 0. Then, similarly as in Lemma 30, M admits a 2-orientation iff M is in $\mathcal{C}_{2s+1}^{(2q+1)}$, and in that case, M admits a unique 2-orientation in $\mathcal{O}_{-2s-1}$ (we can also make it a corollary of Lemma 30; indeed let M' be the unique annular map with a π-rotation symmetry ρ around the marked inner face, and such that M is the quotient of M' by ρ; then M' admits a unique $(2,4s+2)$-orientation in $\mathcal{O}_{-4s-2}$, and by uniqueness, this orientation is invariant under ρ, hence by taking the quotient, M inherits a $(2,2s+1)$-orientation in $\mathcal{O}_{-2s-1}$, which has to be unique).

The master bijection then ensures that $\mathcal{C}_{2s+1}^{(2q+1)}$ is in bijection with the family of weighted bi-mobiles with a marked black vertex of degree $2q+1$, non-marked black vertices of even degree, such that each non-marked black vertex of degree $2k$ has weight $-k+2$, and the marked black vertex has weight $-q+s$, every edge has weight 1 and every white vertex has weight 2. This bijection and a decomposition of such mobiles with a distinguished corner at the marked black vertex then ensure that

$$\vec{C}_{2s+1}^{(2q+1)} = (2s+1)\binom{2q+1}{q-s} S^{q+s+1}|_{t=1}, \quad \text{with } S = t + \sum_{i\geq 2} x_i \binom{2i-1}{i-2} S^{i+1},$$

so that

$$\vec{A}_{2s+1}^{(2p+1,2q+1)} = (2s+1)\binom{2p+1}{p-s}\binom{2q+1}{q-s} \cdot S^{p+q+1}|_{t=1}.$$

Hence, if we define $\bar{\beta}_{p,q,s} = (2s+1)\binom{2p+1}{p-s}\binom{2q+1}{q-s}$, then the generating function $\vec{A}^{(2p+1,2q+1)} := \sum_{s\geq 1} \vec{A}_{2s+1}^{(2p+1,2q+1)}$ satisfies

$$\vec{A}^{(2p+1,2q+1)} = \bar{\gamma}(p,q,1) S^{p+q+1}|_{t=1},$$

where $\bar{\gamma}(p,q,d) := \sum_{s=d}^{\min(p,q)} \bar{\beta}(p,q,s)$, and similarly as in Proposition 33 it can be checked by a decreasing induction on d (starting at $\min(p,q)$) that $\bar{\gamma}(p,q,d) = \frac{(2p+1)(2q+1)}{p+q+1}\binom{2p}{p-d}\binom{2q}{q-d}$, so that we obtain

$$\vec{A}^{(2p+1,2q+1)} = \frac{(2p+1)(2q+1)}{p+q+1}\binom{2p}{p-1}\binom{2q}{q-1}S^{p+q+1}|_{t=1}. \tag{17}$$

Then, for any non-negative integers $n_2, \ldots, n_h$, with $e := p+q+1+\sum_i in_i$, the Lagrange inversion formula gives

$$[x_2^{n_2}\ldots x_k^{n_h}]\vec{A}^{(2p+1,2q+1)} = \frac{(e+r-3)!}{e!}(2p+1)(2q+1)\binom{2p}{p-1}\binom{2q}{q-1}\left[\prod_{i=2}^{h}\binom{2i-1}{i-2}^{n_i}\right],$$

which yields (16) in the quasi-bipartite case, upon multiplying the above coefficient by the factor $\prod_{i=2}^{h} n_i!(2i)^{n_i}$ to account for numbering the non-marked inner faces and distinguishing a corner in each of these faces.

3.5. Alternative approaches

3.5.1. Orientations for the annular maps in $\mathcal{B}_s^{(q)}$. In the two previous sections we have seen that for $1 \leq s \leq q$ with s and q of the same parity, every map $M \in \mathcal{C}_s^{(q)}$ admits a unique weighted orientation in $\mathcal{O}_{-s}$ such that every inner (resp. outer) vertex has weight 2 (resp. 1), every edge has weight 1, every non-marked inner face of degree $2k$ has weight $-k+2$, and the marked inner face (of degree q) has weight $(-q+s)/2$.

One can actually prove the existence of closely related orientations, with the difference that this time all vertices (including the outer ones) have weight 2, and the marked inner face has weight $(-q-s)/2$ instead of $(-q+s)/2$. It can be proved that the existence of such an orientation also characterizes maps in $\mathcal{C}_s^{(q)}$. However, (using the Euler relation), one can check that if the outer face contour is not the unique separating cycle of length s, then there is an outer vertex v such that any such orientation is not accessible from v. Hence it is necessary that $M \in \mathcal{B}_s^{(q)}$ to have accessibility. Conversely one can prove that any $M \in \mathcal{B}_s^{(q)}$ has a unique such orientation in $\mathcal{O}_s$, called the *canonical orientation* of M. Specializing the master bijection (case $\delta = s$), such maps are in bijection with weighted mobiles having the same conditions as the $(2,q)$-branching mobiles considered in the previous sections, with the only difference that the marked black vertex of degree q has weight $(-q-s)/2$ instead of $(-q+s)/2$, thus is incident to $(q+s)/2$ pending edges (instead of $(q-s)/2$ for $(2,q)$-branching mobiles). It then implies that

$$\vec{B}_s^{(q)} = s\binom{q}{(q+s)/2}S^{(q-s)/2}|_{t=1}$$

which is consistent with $s\vec{C}_s^{(q)} = \vec{B}_s^{(q)} \cdot \vec{C}_s^{(s)}$ and $\vec{C}_s^{(q)} = \binom{q}{(q-s)/2}S^{(q+s)/2}$.

3.5.2. A simplified annular approach in the bipartite case. For $q \geq 1$, consider a rooted simple bipartite maps with a secondary marked face of degree $2q$ having a distinguished corner (we take here the equivalent convention that rooted means "with a marked directed edge"). Blowing the root-edge e into a face of degree 2 taken as the outer face, and where the corner at the origin of e is marked, we obtain a map that is in $\vec{\mathcal{B}}_2^{(2q)}$, and the mapping is clearly a bijection. Hence

$$\vec{B}_2^{(2q)} = 2\binom{2q}{q-1}S^{q-1}|_{t=1}$$

is the series of rooted simple bipartite maps with a secondary marked face of degree $2q$ having a distinguished corner, where x_i is conjugate to the number of non-marked faces of

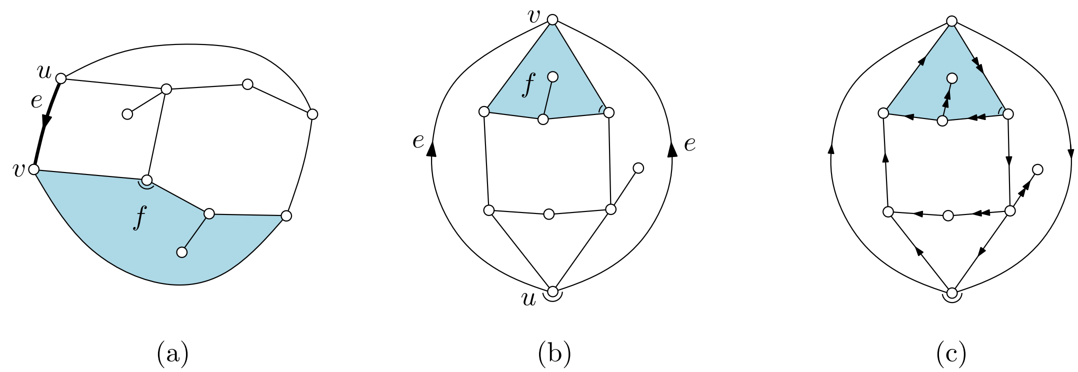


FIGURE 3.9. (a) A rooted bipartite simple map M with a secondary marked face of degree $2q = 6$. (b) Blowing the root-edge into a face of degree 2 taken as the outer face, we obtain a map $\widehat{M}$ in $\mathcal{B}_2^{(2q)}$. (c) The canonical orientation (in $\mathcal{O}_2$) for $\widehat{M}$.

degree $2i$ and t is conjugate to the number of non-root edges. By coefficient extraction in this series using the Lagrange inversion formula, we again recover (5). Note that this proof gives the formula also in the case of one face (as opposed to the approaches of Section 3.3 and Section 3.5.1, where the map is required to have at least two faces). This approach is simpler but has some limitations when compared to the approach of Sections 3.3 and Section 3.5.1: it does not seem well adapted to an extension to the quasi-bipartite case, and it does not extend well to higher girth (the most generalizable approach is the one of Section 3.5.1, which extends to hypermaps with control on a so-called *ingirth* parameter). However it is nice to have such a simplified approach when dealing with the bipartite case only.

We will somehow rely on a combination of the two approaches presented in this section (from Section 3.5.2 the idea of marking an edge to be opened into a face of degree 2, and from Section 3.5.1 the idea of cutting along a canonical cycle into two components that are to be endowed with canonical weighted bi-orientations in $\mathcal{O}_{-2}$ and $\mathcal{O}_2$, respectively) to handle bipartite quadrangulations with boundaries in the next chapter.

CHAPTER 4

An analogue of Krikun's formula for quadrangulations with boundaries

In this chapter we prove an analogue of Krikun's formula for (bipartite) quadrangulations with boundaries, as stated in Theorem 4, using the master bijection adapted to maps with boundaries. In the maps we consider, the boundaries are self-avoiding and vertex-disjoint, hence can be considered as "big" vertices; we first (Section 4.1) adapt the theory of orientations and the master bijection to this setting. Then in Section 4.2 we show how to apply the master bijection strategy to bipartite quadrangulations with boundaries; similarly as in Section 3.2, we start with a subcase where the orientations are simpler to handle, here the case where at least one boundary has length 2 (in Section 4.2.1). We introduce bi-orientations for such maps, and then characterize the associated bi-mobiles and their generating function. Then in Section 4.2.2 we extend the bijective encoding to allow for an outer boundary face of arbitrary even degree, which allows us to prove (in Section 4.2.3) Theorem 4, by counting the corresponding weighted bi-mobiles (via generating functions). We then explain in Section 4.3 how to apply a similar technique to triangulations with boundaries, and thereby recover Krikun's counting formula (Theorem 3). Finally in Section 4.4 we observe that these formulas make it possible to solve the dimer model on rooted quadrangulations and rooted triangulations (upon seeing dimers as boundaries of length 2).

4.1. Preliminaries on maps with boundaries

4.1.1. Maps with boundaries. A face f of a map is said to be *simple* if the number of vertices incident to f is equal to the degree of f (in other words there is no pair of corners of f incident to the same vertex). A *map with boundaries* is a map M where the set of faces is partitioned into two subsets: *boundary faces* and *internal faces*, with the constraint that the boundary faces are simple, and the contours of any two boundary faces are vertex-disjoint; these contour-cycles are called the *boundaries* of M. Edges (and similarly vertices) are called *boundary edges* or *internal edges* whether they are on a boundary or not; half-edges are called *boundary* or *internal* whether they belong to an internal edge or a boundary edge. If M is a *plane* map with boundaries, whose outer face is a boundary face, then the contour of the outer face is called the *outer boundary* and the contours of the other boundary faces are called *inner boundaries.* A *quadrangulation with boundaries* (resp. *triangulation with boundaries*) is a map with boundaries where every internal face has degree 4 (resp. degree 3).

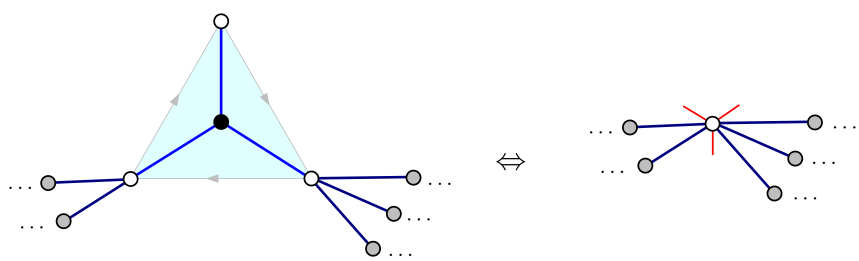
4.1.2. Orientations for maps with boundaries. For M a map with boundaries, an orientation of M is called *consistent* if every boundary edge has the incident boundary face on its right. Define the *indegree* of a boundary C as the number of internal edges with their end-vertex on C. Let $\mathcal{V}_{\text{int}}$ be the set of internal vertices of M and Γ the set of boundaries of M. Given α a mapping from $\mathcal{V}_{\text{int}} \cup \Gamma$ to $\mathbb{N}$, define a *boundary-α-orientation* of M as a consistent orientation of M such that each boundary C has indegree $\alpha(C)$, and each internal vertex v has indegree $\alpha(v)$. For M a plane map with boundaries whose outer face is a boundary-face, a consistent orientation of M is called *almost-minimal* if the outer face contour is the unique counterclockwise cycle. As an easy consequence of Lemmas 8 and 9 (upon seeing boundaries as "big" vertices), we have

LEMMA 37. *Let M be a plane map with boundaries, and let X be a boundary-α-orientation of M. If the outer face of M is an internal face, then M admits a unique minimal boundary-α-orientation X_0. If the outer face of M is a boundary face and the outer boundary C_0 satisfies $\alpha(C_0) = 0$, then M admits a unique almost-minimal α-orientation X_0. In addition, in both cases, X is accessible from a vertex v iff X_0 is accessible from v.*

More generally we call a weighted bi-orientation of M *consistent* if every boundary edge is 1-way of weights $(0, 1)$ with the incident boundary face on its right. And such a weighted bi-orientation is called *almost-minimal* if the outer face is a boundary-face and the outer contour is the unique ccw cycle.

4.1.3. Adaptation of the master bijection to maps with boundaries. We will only need here to adapt the master bijection for $\delta \in \mathbb{Z}\backslash\{0\}$ (the case $\delta = 0$ would correspond to vertex-pointed maps with boundaries such that the pointed vertex is internal). Let $\delta \in \mathbb{Z}\backslash\{0\}$, and denote by $\widehat{\mathcal{O}}_\delta$ the family of plane maps with boundaries endowed with a consistent weighted bi-orientation, such that the outer face is a boundary face for $\delta < 0$ and an internal face for $\delta > 0$, and when forgetting which faces are boundary faces, the underlying weighted bi-oriented plane map is in $\mathcal{O}_\delta$.

Define now a (weighted) *boundary bi-mobile* as a (weighted) bi-mobile where every corner at a white corner might carry additional dangling half-edges called *legs* (as buds, legs carry no weight); white vertices having at least one leg are called *special*. The *degree* of a white vertex v is the number of non-leg half-edges incident to v. Define the *excess* of a boundary bi-mobile as the number of half-edges incident to a white vertex (including the legs) minus the number of buds. For $\delta \in \mathbb{Z}$, denote by $\widehat{\mathcal{B}}_\delta$ the set of weighted boundary bi-mobiles of excess δ (again with the constraint that half-edges at white vertices have positive weight, while half-edges at black vertices have non-positive weight).



(a) (b)

FIGURE 4.1. Reduction operation at the black vertex corresponding to a boundary face in $O \in \widehat{\mathcal{O}}_\delta$.

We can now specialize the master bijection. For $O \in \widehat{\mathcal{O}}_\delta$, let $T = \Phi(O)$ be the associated weighted bi-mobile. Note that each inner boundary face f of O, of some degree k, yields a black vertex b of degree k in T, such that b has no bud and the k neighbours $w_1, \ldots, w_k$ of b are the white vertices corresponding to the vertices around f. We perform the following operations: insert k legs at b, one toward each of the edges around f; then pull the k

neighbours $w_1, \ldots, w_k$ to b, and finally recolor b as white, see Figure 4.1(a)-to-(b). Doing this for each inner boundary we obtain (without loss of information) a weighted boundary bi-mobile T' of the same excess as T, called the *reduction* of T. Let $\widehat{\Phi}$ be the mapping such that $\widehat{\Phi}(O) = T'$.

Conversely, for T' a (weighted) boundary bi-mobile, the *expansion* of T' is the (weighted) bi-mobile T obtained from T' by applying Figure 4.1(b)-to-(a) to every special white vertex of T', such a white vertex with k legs yielding in T a distinguished black vertex b of degree k with no buds, and with only white neighbours (and with half-edge weights $(0, 1)$ on the edges from b to each of the white neighbours). Note that, if T' has non-zero excess δ and if $O \in \mathcal{O}_\delta$ denotes the weighted bi-oriented plane map associated to T by the master bijection, then each distinguished face $f \in O$ (i.e., a face associated to a distinguished black vertex of T) is simple; indeed if $k \geq 1$ denotes the degree of f, the corresponding black vertex $b \in T$ has k white neighbours, which thus correspond to k distinct vertices incident to f. In addition the contours of the distinguished inner faces are disjoint since the expansions of any two distinct special white vertices of T' are vertex-disjoint in T. And for $\delta < 0$, the outer face is simple and disjoint from the contours of the inner distinguished faces (indeed the vertices around an inner distinguished face of O are all present in T, hence are inner vertices of O). We thus conclude that $O \in \widehat{\mathcal{O}}_\delta$, upon seeing the distinguished faces (including the outer face for $\delta < 0$) as boundary faces. The following statement summarizes the discussion:

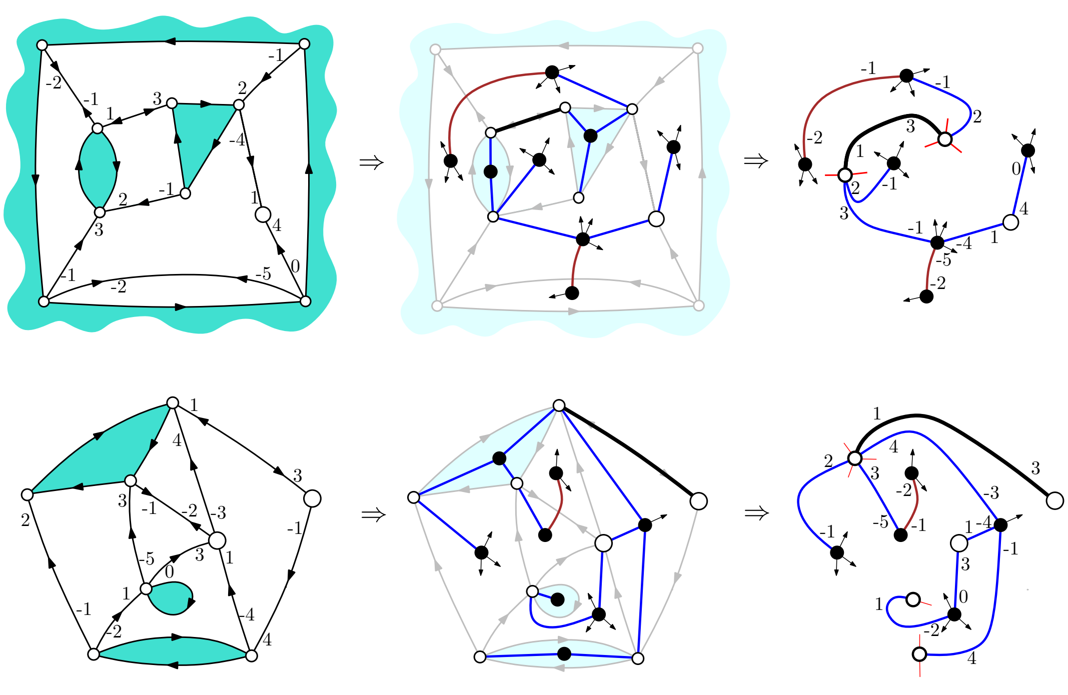


FIGURE 4.2. The master bijection from a consistent weighted plane bi-orientation in $\widehat{\mathcal{O}}_\delta$ to a weighted boundary bi-mobile of excess δ (the top example has $\delta = -4$, the bottom-example has $\delta = 5$, the weights of boundary edges, which are always $(0, 1)$ by definition, are not indicated).

THEOREM 38. *The master bijection $\widehat{\Phi}$ adapted to consistent weighted bi-orientations is a bijection between $\widehat{\mathcal{O}}_\delta$ and $\widehat{\mathcal{B}}_\delta$ for each $\delta \in \mathbb{Z}\backslash\{0\}$.*

Again several parameters (weight and degree) of a consistent weighted bi-orientation correspond to parameters of the associated weighted boundary bi-mobile. For a map M with boundaries endowed with a consistent weighted bi-orientation, define the *weight* (resp. the *indegree*) of a boundary C as the total weight (resp. total number) of ingoing half-edges incident to a vertex of C but not lying on an edge of C. And for a weighted boundary bi-mobile, define the *weight* of a white vertex v as the total weight of the half-edges (excluding legs) incident to v.

Let $O \in \widehat{\mathcal{O}}_\delta$ and $T = \widehat{\Phi}(O)$. Then every internal inner face of O corresponds to a black vertex in T of same degree and same weight; every internal vertex $v \in O$ corresponds to a non-special white vertex $v' \in T$ of the same weight and such that the indegree of v equals the degree of v'; and every inner boundary of length k, indegree r, and weight j in O corresponds to a special white vertex in T with k legs, degree r, and weight j.

We finally state a useful parity lemma for orientations in $\widehat{\mathcal{O}}_\delta$:

LEMMA 39. *Let O be a consistent weighted bi-orientation in $\widehat{\mathcal{O}}_\delta$ (for some $\delta \in \mathbb{Z}\backslash\{0\}$), such that every internal edge, internal vertex, internal face, boundary, has even weight. Then every internal half-edge also has even weight.*

PROOF. Let T be the boundary bi-mobile associated with O by the master bijection (Theorem 38). The parity conditions of O imply that all edges and vertices of T have even weight. In particular an edge e of T either has its two half-edges of odd weight, in which case e is called *odd*, or has its two half-edges of even weight, in which case e is called *even*. Let F be the subforest of T formed by the odd edges. Since every vertex of T has even weight, it is incident to an even number of edges in F. Hence F has no leaf, so that F has no edge. Thus all edges of T are even, and by the local rules of the master bijection it implies that all internal half-edges of O have even weight. □

4.2. Application to quadrangulations with boundaries

4.2.1. The case where at least one boundary has length 2. Define a *2-outer quadrangulation with boundaries* as a quadrangulation with boundaries, with a distinguished boundary face of degree 2 taken as the outer face. Let M be a 2-outer quadrangulation with boundaries. A *boundary-2-orientation* of M is a consistent weighted bi-orientation of M such that:

- each internal edge has weight 0,
- each internal vertex has weight 2,
- each internal face (of degree 4) has weight -2,
- each inner boundary of length i has weight $i + 2$,
- the outer boundary has weight 0.

Note that the weight-conditions at internal vertices and edges easily imply that all weights on internal half-edges are in $[-2..2]$.

LEMMA 40. *Let M be a 2-outer quadrangulation with boundaries. If M is loopless, then M admits a unique boundary-2-orientation in $\widehat{\mathcal{O}}_{-2}$.*

PROOF. Let M_2 be the map with boundaries obtained from M by inserting a vertex (considered as a *white square vertex*, whereas the vertices of M are considered as round and white) in the middle of each edge, and with same boundaries (upon subdividing by 2)

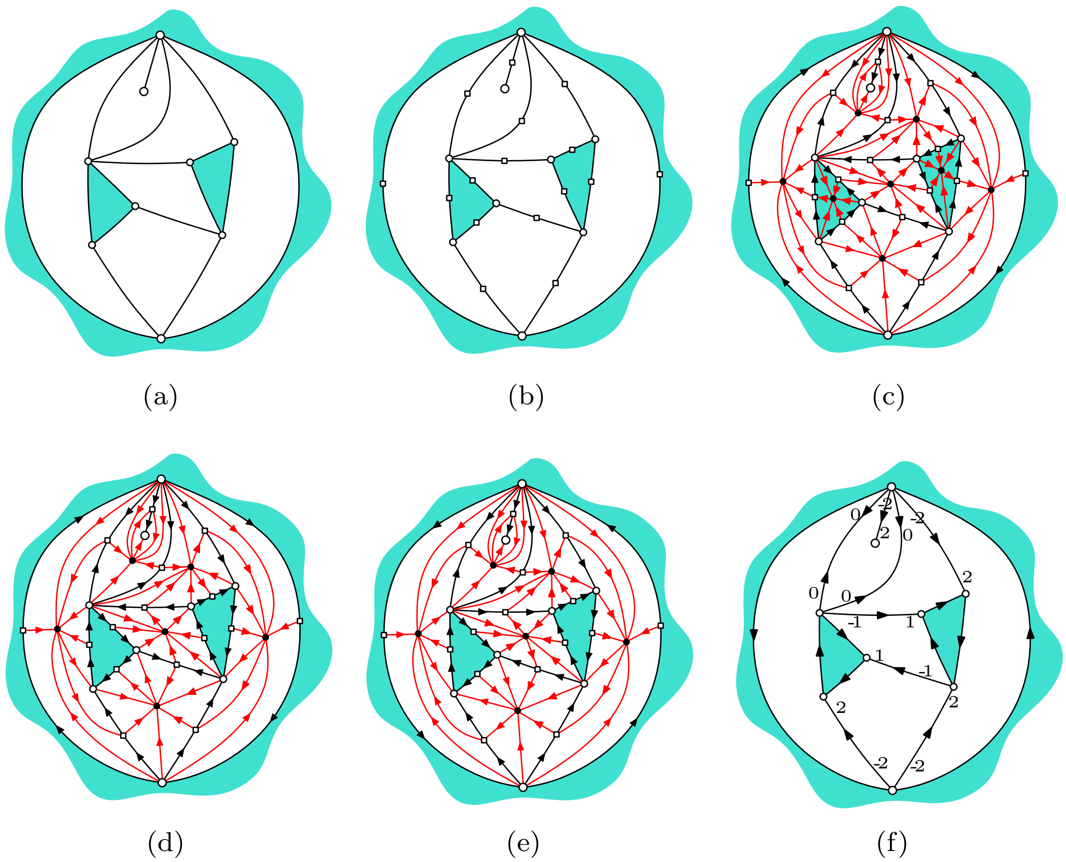


FIGURE 4.3. (a) A 2-outer quadrangulation M with boundaries. (b) The 2-subdivided map M_2. (c) The map $\sigma(M_2)$ (obtained from M_2 by inserting a star in each inner face) endowed with a 2-regular orientation X, which can be obtained by the technique of Figure 3.2. (d) The map $\tau(M_2)$ (same as $\sigma(M_2)$ except that the stars in the inner boundary faces are erased) endowed with the induced boundary-α-orientation X' (which is not minimal here). (e) The map $\tau(M_2)$ endowed with its unique almost-minimal boundary-α-orientation. (f) The induced boundary-2-orientation in $\widehat{\mathcal{O}}_{-2}$, obtained by applying the transfer rules of Figure 4.5.

as in M. Note that if M is loopless then M_2 is a bipartite simple map of outer degree 4 (precisely, all internal faces have degree 8 and each boundary face originally of degree i in M has degree $2i$ in M_2). Hence, with the notations of Lemma 25, $\sigma(M_2)$ admits a 2-regular orientation X. Let $\tau(M_2)$ be the map with boundaries obtained from $\sigma(M_2)$ by deleting the stars inserted inside the boundary faces of M_2, the boundary faces of $\tau(M_2)$ being the same as in M_2 (in other words, stars are inserted only in the internal faces of M_2, not in the boundary faces). It is easy to check that the orientation X' induced by X on $\tau(M_2)$ (upon re-orienting the boundary edges so that they have the incident boundary face on their right) is a boundary-α-orientation for the following α:

- for each internal white vertex (round or square) v of $\tau(M_2)$, $\alpha(v) = 2$,

- for each black vertex b of $\tau(M_2)$ (of degree 8 since the internal faces of M_2 have degree 8), $\alpha(b) = 6$, i.e., b has 2 outgoing and 6 ingoing edges,
- for each inner boundary C of length $2i$ in $\tau(M_2)$ (originally an inner boundary of length i in M), $\alpha(C) = i + 2$. The outer boundary C_{ext} has $\alpha(C_{\text{ext}}) = 0$.

In addition X' inherits from X the property of being accessible from every outer vertex of $\tau(M_2)$. By Lemma 37, $\tau(M_2)$ admits a unique almost-minimal boundary-α-orientation, denoted $X'_{\min}$, which is accessible from every outer vertex of $\tau(M_2)$. Similarly as in Property 26 we have the following property illustrated and proved [1] in Figure 4.4:

PROPERTY 41. *In $X'_{\min}$, for any edge $e = \{b, w\}$ connecting a black vertex b to a white vertex w (either round or square) and directed toward w, the next edge ϵ (which is in M_2) after e in clockwise order around w is also directed toward w.*

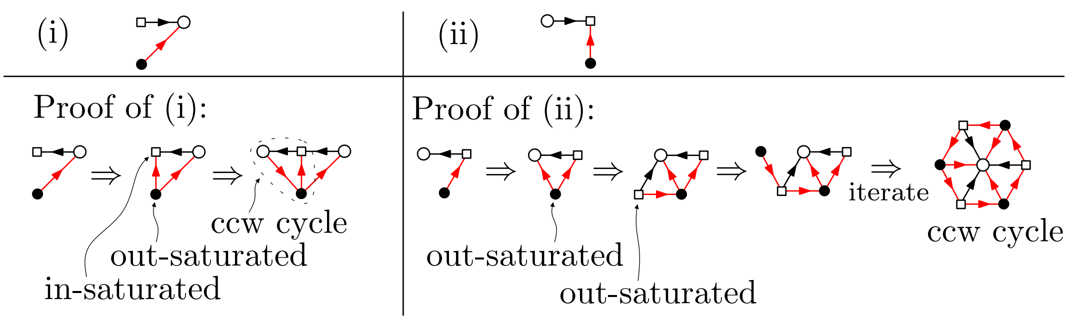


FIGURE 4.4. (i) the local property at a round white vertex, below the proof by contradiction, (ii) the local property at a square white vertex, below the proof by contradiction. At each step the possible configurations are constrained either due to ccw-cycle avoidance or due to a vertex outdegree or indegree being saturated.

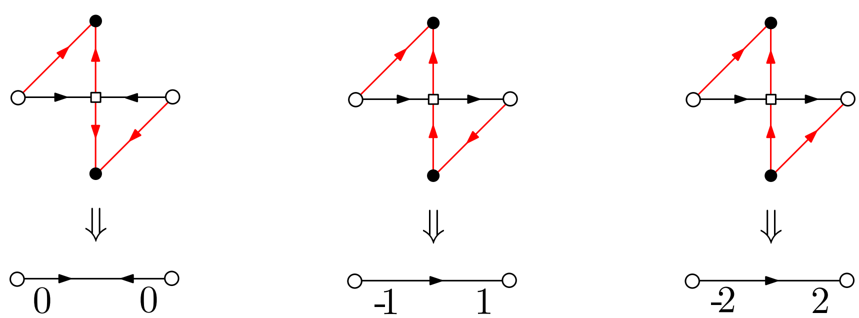


FIGURE 4.5. Top: the 3 possible configurations at an internal edge of M in the minimal boundary-α-orientation of $\tau(M_2)$. Bottom: transfering the configurations into weights on the half-edges.

[1]Note that in $X'_{\min}$, a vertex of M on a boundary might have indegree larger than 2, so that we can not recycle the proof of Property 26, it is actually crucial here that the internal faces of M have degree 4.

Property 41 and the fact that each square vertex has indegree 2 implies that each internal edge of M (subdivided by 2 in $\tau(M_2)$) is of the 3 possible types shown in the top-part of Figure 4.5. Let Y_0 be the *induced* consistent weighted bi-orientation of M, which is obtained by applying the transfer rules of Figure 4.5, see Figure 4.3(e)-(f) for an example. It is easy to see that Y_0 is a boundary-2-orientation of M. In addition, since X'_{min} is almost-minimal, Y_0 is also almost-minimal. And similarly as in Lemma 25, the fact that Property 41 is satisfied ensures that Y_0 is accessible from the 2 outer vertices. Hence Y_0 is in $\mathcal{O}_{-2}$. Conversely if there was another boundary-2-orientation Y_1 in $\mathcal{O}_{-2}$, then applying the transfer rules of Figure 4.5 the other way would yield a boundary-α-orientation X_1 of $\tau(M_2)$, and a similar argument as in Figure 2.15 would imply that X_1 is almost-minimal, implying $X_1 = X'_{\mathrm{min}}$ and thus $Y_0 = Y_1$. □

It is tempting to directly use Lemma 40 in order to count loopless 2-outer quadrangulations via the master bijection. But a crucial part is missing, namely we do not have the property that being loopless is necessary in order to admit a boundary-2-orientation in $\widehat{\mathcal{O}}_{-2}$. However, in the bipartite case (i.e., when all faces, including the boundary faces, have even degree), being loopless is automatically granted, and in addition the orientations simplify.

Denote by $\mathcal{D}_\diamond$ the family of *bipartite* 2-outer quadrangulations with boundaries. For $M \in \mathcal{D}_\diamond$, define a *boundary-1-orientation* of M as a consistent weighted bi-orientation of M such that:

- every internal vertex has weight 1,
- every internal edge has weight 0,
- every internal face (of degree 4) has weight -1,
- every inner boundary of length $2i$ has weight $i+1$, and the outer boundary, of length 2, has weight 0.

Note that the weight-condition at internal vertices implies that the half-edge weights are at most 1, hence the internal edges either are 0-way of weights $(0,0)$ or 1-way of weights $(-1,1)$. Then the weight condition at internal faces implies that every internal face has a unique 1-way clockwise edge on its contour.

LEMMA 42. *Every $M \in \mathcal{D}_\diamond$ has a unique boundary-1-orientation in $\widehat{\mathcal{O}}_{-2}$, which is called the* canonical bi-orientation *of M.*

PROOF. By Lemma 40, M has a unique boundary-2-orientation X_0 in $\widehat{\mathcal{O}}_{-2}$. Clearly, every internal face, internal vertex, and edge has even weight. And, since M is bipartite, every boundary face also has even weight. Hence, by Lemma 39 every internal half-edge has even weight. Dividing all these half-edge weights by 2 we obtain a boundary-1-orientation X'_0 that is in $\widehat{\mathcal{O}}_{-2}$. In addition if there was another boundary-1-orientation X'_1 in $\widehat{\mathcal{O}}_{-2}$, then doubling the weights of internal half-edges would yield a boundary-2-orientation X_1 in $\widehat{\mathcal{O}}_{-2}$ and different from X_0, contradicting the uniqueness in Lemma 40. □

The corresponding (via the master bijection for maps with boundaries, Theorem 38) weighted boundary bi-mobiles thus satisfy the following properties (there is also the fact that the excess is -2, but it can easily be checked to be a consequence of these properties):

- every edge has weight 0, either black-black of weights $(0,0)$, or black-white of weights $(-1,1)$,
- every black vertex has degree 4 and weight -1, hence has a unique white neighbour,
- for $i \geq 0$, every white vertex of degree $i+1$ carries $2i$ legs.

Denote by $\mathcal{T}_\diamond$ the family of such weighted boundary bi-mobiles. To summarize, Theo-

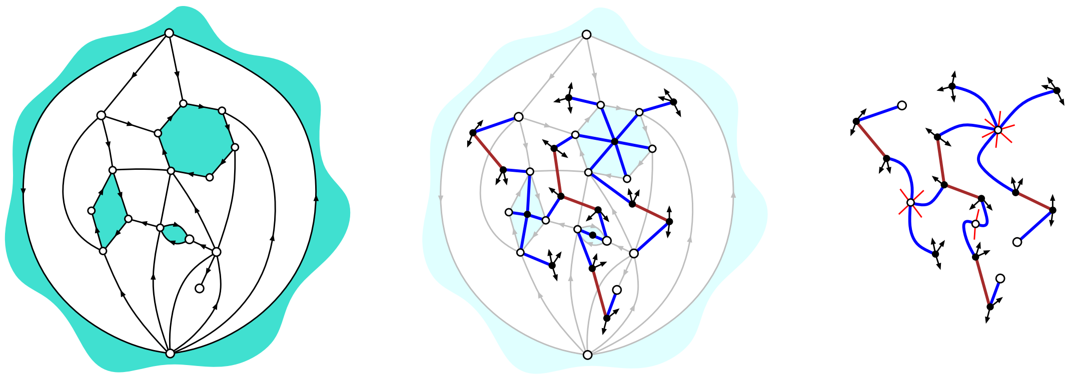


FIGURE 4.6. (a) A bipartite 2-outer quadrangulation with boundaries endowed with its canonical bi-orientation (where the 1-way edges are indicated as directed edges, the 0-way edges are indicated as undirected edges, and the weights, which are uniquely induced by the bi-orientation, are not indicated). (b) The dissection superimposed with the corresponding bi-mobile. (c) The reduced boundary bi-mobile (with $2i$ legs at each white vertex of degree $i+1$), where again the weights (which are uniquely induced by the bi-mobile) are not indicated.

rem 38 and Lemma 42 yield the following bijective statement for bipartite 2-outer quadrangulations with boundaries, see Figure 4.6 for an example:

PROPOSITION 43. *The family $\mathcal{D}_\diamond$ is in bijection with the family $\mathcal{T}_\diamond$. For $M \in \mathcal{D}_\diamond$ and $T \in \mathcal{T}_\diamond$ the associated weighted boundary bi-mobile, each inner boundary of length $2i$ in M corresponds to a white vertex in T of weight (and degree) $i+1$; and each internal vertex of M corresponds to a white leaf in T.*

4.2.2. The general case. For $a \geq 1$, denote by $\mathcal{D}_\diamond^{(2a)}$ the family of bipartite quadrangulations with boundaries, with a distinguished boundary face of degree $2a$ taking as the outer face. In the last section we have described a bijection for $\mathcal{D}_\diamond = \mathcal{D}_\diamond^{(2)}$. We describe here more generally a bijection for $\mathcal{D}_\diamond^{(2a)}$ for any $a \geq 1$; more precisely we give a a bijection for $\overline{\mathcal{D}}_\diamond^{(2a)}$, the family of objects from $\mathcal{D}_\diamond^{(2a)}$ where an arbitrary edge (which can be a boundary edge or an internal edge) is distinguished. Similarly as in Section 3.5.2, the first step is to turn a map $M \in \overline{\mathcal{D}}_\diamond^{(2a)}$ into an *annular map*, by blowing the distinguished edge e into an internal face f_1 of degree 2, taken as a marked inner face, to obtain a so-called $(2a)$-*annular quadrangulation with boundaries*, i.e., an annular map with boundaries, where the outer face f_0 is a boundary face of degree $2a$, the marked inner face f_1 is an internal face of degree 2, and all the other internal faces have degree 4. Let $\mathcal{G}_\diamond^{(2a)}$ be the family of bipartite $(2a)$-annular quadrangulations with boundaries, and let $\mathcal{E}_\diamond := \mathcal{G}_\diamond^{(2)}$. Then the above edge blowing process yields, for any $a \geq 1$:

$$\overline{\mathcal{D}}_\diamond^{(2a)} \simeq \mathcal{G}_\diamond^{(2a)}. \tag{18}$$

For $M \in \mathcal{G}_\diamond^{(2a)}$, an *admissible 2-cycle* of M is a 2-cycle c of M such that f_1 is inside c and such that any face inside c and incident to a vertex on c is an internal face (note

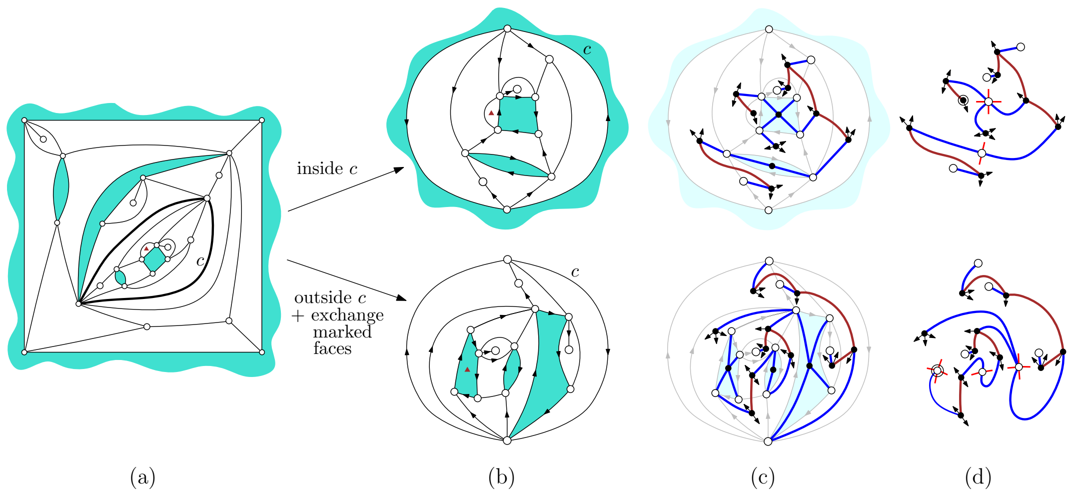


FIGURE 4.7. (a) A bipartite 4-annular quadrangulation with boundaries, where the internal face of degree 2 (marked inner face) is indicated by a triangle, and the outermost admissible 2-cycle c is drawn bold. (b) The two parts M_0, M_1 resulting from cutting along c, each endowed with its canonical bi-orientation (the marked inner face in each case is indicated by a triangle, 1-way and 0-way edges are shown as directed and undirected edges, respectively; the weights, which are uniquely induced by the bi-orientation, are not indicated). (c) The bi-mobiles associated to the two parts. (d) The reduced boundary bi-mobiles, where the marked vertex (corresponding to the marked inner face) in each case is surrounded; the weights, which are uniquely induced by the bi-mobile, are not indicated.

that the contour of f_1 is an admissible 2-cycle). A $(2a)$-annular quadrangulated dissection is called *reduced* if the unique admissible 2-cycle is the contour of f_1. Let $\mathcal{H}_{\diamond}^{(2a)}$ (resp. $\mathcal{F}_{\diamond}^{(2a)}$) be the family of annular maps obtained from maps in $\mathcal{G}_{\diamond}^{(2a)}$ (resp. from reduced maps in $\mathcal{G}_{\diamond}^{(2a)}$) by exchanging the roles of the two marked faces (so that the outer face is a degree 2 internal face, while the marked inner face is a boundary face of degree $2a$). Finally, denote by $\vec{\mathcal{G}}_{\diamond}^{(2a)}$, $\vec{\mathcal{E}}_{\diamond}$, and $\vec{\mathcal{F}}_{\diamond}^{(a)}$ the families of maps respectively from $\mathcal{G}_{\diamond}^{(2a)}$, $\mathcal{E}_{\diamond}$, and $\mathcal{F}_{\diamond}^{(2a)}$, where there is a distinguished corner in the outer face and in the marked inner face. Note that each $M \in \mathcal{G}_{\diamond}^{(2a)}$ has an "outermost" admissible 2-cycle c; if we cut M (seen as embedded on the sphere) along c we obtain two annular maps $M_0, M_1 \in \mathcal{F}_{\diamond}^{(2a)} \times \mathcal{E}_{\diamond}$ called the *two parts* of M: M_1 is the part "inside" c with the outer face (delimited by c) considered as a boundary face and with f_1 as the marked inner face (an internal face); M_0 is the part "outside" c, where the face delimited by c is taken as the outer face (considered as an internal face) and f_0 is taken as the marked inner face (a boundary face), see Figure 4.7(a)-(b) for an example. Conversely, for $M_0, M_1 \in \mathcal{F}_{\diamond}^{(2a)} \times \mathcal{E}_{\diamond}$, we obtain a map $M \in \mathcal{G}_{\diamond}^{(2a)}$ by pasting the outer contours of M_0 and M_1 (there are 2 ways to do it since the outer contours have length 2), and then taking the marked inner face of M_0 as the outer face of M, and the marked inner face of M_1 as the marked inner face of M.

At the level of rooted objects, this decomposition yields the isomorphism:

$$2 \cdot \vec{\mathcal{G}}_{\diamond}^{(2a)} \simeq \vec{\mathcal{F}}_{\diamond}^{(2a)} \times \vec{\mathcal{E}}_{\diamond}, \tag{19}$$

where the factor 2 accounts for the choice of a vertex v on the outermost admissible 2-cycle c of M (so that after cutting along c, in each component M_0, M_1, the distinguished corner in the face delimited by c is the corner at v; under this constraint there is now just one way to paste the outer contours of M_0 and M_1, for $M_0, M_1 \in \vec{\mathcal{F}}_\diamond^{(2a)} \times \vec{\mathcal{E}}_\diamond$).

Similarly as in the previous section, the maps in the respective families $\mathcal{E}_\diamond$ and $\mathcal{F}_\diamond^{(2a)}$ can be endowed with canonical bi-orientations from which we can apply the master bijection. For $M \in \mathcal{E}_\diamond$, a *1-orientation* of M is a consistent weighted bi-orientation of M such that:

- every internal vertex has weight 1,
- every internal edge has weight 0,
- every internal face of degree 4 has weight -1, and the (unique) internal face of degree 2 has weight 0,
- every inner boundary of length $2i$ has weight $i+1$, and the outer boundary has weight 0.

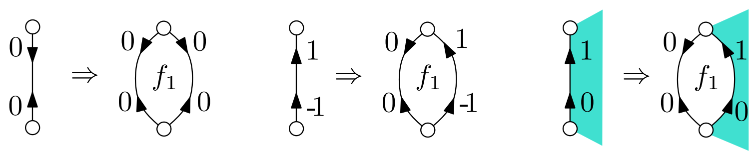


FIGURE 4.8. Transferring bi-orientations and weights when blowing an edge into an internal face of degree 2.

LEMMA 44. *Every $M \in \mathcal{E}_\diamond$ admits a unique 1-orientation in $\widehat{\mathcal{O}}_{-2}$, which is called the* canonical bi-orientation *of M.*

PROOF. This is a corollary of Lemma 42. Indeed, seeing M as a map $D \in \mathcal{D}_\diamond$ where an edge e is opened into an internal face f_1 of degree 2, the canonical bi-orientation of M is directly derived from the canonical bi-orientation of D, using the rules shown in Figure 4.8. □

For $M \in \mathcal{H}_\diamond^{(2a)}$ (recall that the outer face is internal of degree 2 while the marked inner face is a boundary face of degree $2a$), define a *1-orientation* of M as a consistent weighted bi-orientation such that:

- every internal vertex has weight 1,
- every internal edge has weight 0,
- every internal inner face (of degree 4) has weight -1, and the outer face (internal of degree 2) has weight 0,
- every non-marked boundary of length $2i$ has weight $i+1$,
- the marked boundary, of length $2a$, has weight $a-1$.

LEMMA 45. *Let $M \in \mathcal{H}_\diamond^{(2a)}$. Then M has a 1-orientation in $\widehat{\mathcal{O}}_2$ iff $M \in \mathcal{F}_\diamond^{(2a)}$, and in that case M has a unique 1-orientation in $\widehat{\mathcal{O}}_2$, which is called the* canonical bi-orientation *of M.*

As in Section 3.5.1 the proof is omitted, see **[S5]** for details.

The weighted boundary bi-mobiles corresponding to maps in $\mathcal{E}_\diamond$ via the master bijection are specified by the following properties (which imply that the excess is -2):

- every edge has weight 0, either black-black of weights $(0,0)$, or black-white of weights $(-1,1)$,

- every black vertex has degree 4 and weight -1 (hence has a unique white neighbour), except for a unique black vertex of degree 2 and weight 0,
- for $i \geq 0$, every white vertex of degree $i+1$ carries $2i$ legs.

Denote by $\mathcal{U}_\diamond$ the family of these weighted boundary bi-mobiles.

And for $a \geq 1$, the weighted boundary bi-mobiles corresponding to maps in $\mathcal{F}_\diamond^{(2a)}$ are specified by the following properties (which imply that the excess is 2):

- every edge has weight 0, either black-black of weights $(0,0)$, or black-white of weights $(-1,1)$,
- every black vertex has degree 4 and weight -1 (hence has a unique white neighbour),
- there is a marked white vertex of degree $a-1$ that additionally carries $2a$ legs,
- for $i \geq 0$, every non-marked white vertex of degree $i+1$ carries $2i$ legs.

Denote by $\mathcal{V}_\diamond^{(2a)}$ the family of these weighted boundary bi-mobiles. Lemmas 44 and 45 together with the master bijection (Theorem 38) then yield:

PROPOSITION 46. *The family $\mathcal{E}_\diamond$ is in bijection with the family $\mathcal{U}_\diamond$, such that for $M \in \mathcal{E}_\diamond$ and $T \in \mathcal{U}_\diamond$ the associated weighted boundary bi-mobile, each inner boundary of length $2i$ in M corresponds to a white vertex in T of weight (and degree) $i+1$; and each internal vertex of M corresponds to a white leaf in T.*

And for $a \geq 1$, the family $\mathcal{F}_\diamond^{(2a)}$ is in bijection with the family $\mathcal{V}_\diamond^{(2a)}$, such that for $M \in \mathcal{F}_\diamond^{(2a)}$ and $T \in \mathcal{V}_\diamond^{(2a)}$ the associated weighted boundary bi-mobile, each non-marked boundary of length $2i$ in M corresponds to a non-marked white vertex in T of weight (and degree) $i+1$; and each internal vertex of M corresponds to a non-marked white leaf in T.

4.2.3. Proof of Theorem 4. We now prove the counting formula (7) in Theorem 4, using Proposition 46 and counting the associated weighted bi-mobiles (via generating functions). Define a *planted bi-mobile of quadrangulated type* as one of the two connected components $P \in \{T_1, T_2\}$ obtained after cutting a bi-mobile $T \in \mathcal{T}_\diamond$ in the middle of an edge e; the half-edge h of e that belongs to P is called the *root half-edge* of P, and the vertex incident to h is called the *root-vertex* of P. The *root-weight* of P is the weight of h in T. For $j \in \{-1,0,1\}$, denote by $R_j \equiv R_j(t; x_0, x_1, x_2, \ldots)$ the generating function of planted bi-mobiles of quadrangulated type and of root-weight j, where t is conjugate to the number of buds, and x_i is conjugate to the number of white vertices of degree $i+1$ (with $2i$ additional legs) for $i \geq 0$. Define also $R := t + R_0$. Then a decomposition at the root easily implies that $\{R_{-1}, R_0, R_1\}$ are specified by the equation-system

$$(20)\qquad \begin{cases} R_{-1} &= R^3, \\ R_0 &= 3R_1R^2, \\ R_1 &= \sum_{i\geq 0} x_i \binom{3i}{i} {R_{-1}}^i. \end{cases}$$

For instance the factor $\binom{3i}{i}$ in the 3rd line accounts for the number of ways to place the $2i$ legs when the root-vertex has degree $i+1$ (the root half-edge plus i children), and the factor 3 in the second line accounts for choosing which of the 3 children of the root-vertex is white.

This implies that $R = t + 3\sum_{i\geq 0} x_i \binom{3i}{i} R^{3i+2}$, or equivalently,

$$(21)\qquad R = t\phi(R), \quad \text{with } \phi(y) = \Big(1 - 3\sum_{i\geq 0} x_i \binom{3i}{i} y^{3i+1}\Big)^{-1}.$$

Let $U_\diamond \equiv U_\diamond(t; x_0, x_1, \ldots)$ be the generating function of bi-mobiles from $\mathcal{U}_\diamond$ where one of the 2 corners at the marked black vertex is distinguished, with t conjugate to the number of buds and x_i conjugate to the number of white vertices of degree $i+1$ for $i \geq 0$. And for $a \geq 1$, let $V_\diamond^{(2a)} \equiv V_\diamond^{(2a)}(t; x_0, x_1, \dots)$ be the generating function of bi-mobiles from $\mathcal{V}_\diamond^{(2a)}$ where one of the $2a$ legs at the marked white vertex is distinguished, with t conjugate to the number of buds and x_i conjugate to the number of non-marked white vertices of degree $i+1$ for $i \geq 0$. A decomposition at the black vertex of degree 2 for $U_\diamond$ (resp. at the marked white vertex of degree $a-1$ for $V_\diamond^{(2a)}$) gives

$$U_\diamond = R^2, \qquad V_\diamond^{(2a)} = \binom{3a-2}{a-1} R_{-1}{}^{a-1} = \binom{3a-2}{a-1} R^{3a-3}.$$

Let $\vec{G}_\diamond^{(2a)} \equiv \vec{G}_\diamond^{(2a)}(x_0, x_1, \dots)$, $\vec{E}_\diamond \equiv \vec{E}_\diamond(x_0, x_1, \dots)$, $\vec{F}_\diamond^{(2a)} \equiv \vec{F}_\diamond^{(2a)}(x_0, x_1, \dots)$ be the respective generating functions of $\vec{\mathcal{G}}_\diamond^{(2a)}$, $\vec{\mathcal{E}}_\diamond$, and $\vec{\mathcal{F}}_\diamond^{(2a)}$, where x_0 is conjugate to the number of internal vertices and for $i \geq 1$, x_i is conjugate to the number of boundaries of length i that delimit an inner boundary face that is unmarked (i.e., the outer boundary face is discarded for $\vec{\mathcal{E}}_\diamond$ and $\vec{\mathcal{G}}_\diamond^{(2a)}$, and the marked inner boundary face of degree $2a$ is discarded for $\vec{\mathcal{F}}_\diamond^{(2a)}$). Proposition 46 implies that (the factor 2 accounts for marking a corner in the outer face)

$$\vec{E}_\diamond = 2U_\diamond|_{t=1}, \qquad \vec{F}_\diamond^{(2a)} = 2V_\diamond^{(2a)}|_{t=1},$$

and Equation 19 implies that $2\,\vec{G}_\diamond^{(2a)} = \vec{E}_\diamond \cdot \vec{F}_\diamond^{(2a)}$, so that we obtain

$$\vec{G}_\diamond^{(2a)} = 2\binom{3a-2}{a-1} R^{3a-1}|_{t=1}. \tag{22}$$

Now denote by $\beta_a(m; n_1, \dots, n_h)$ the number of maps in $\mathcal{D}_\diamond^{(2a)}$ with a distinguished corner in the outer boundary face, with m internal vertices, n_i inner boundaries of length $2i$ for $1 \leq i \leq h$, and no inner boundary of length larger than $2h$. The half total boundary length is $b = a + \sum_i i n_i$, the total number of boundaries is $r = 1 + \sum_i n_i$, and the number of edges is (by the Euler relation) $e = 3b + 2r + 2m - 4$, which is $3b + 2k$ with $k := r + m - 2$. Then (18) yields (with the factor 2 in front of e due to marking a corner in the new opened face of degree 2)

$$2e\beta_a(m; n_1, \dots, n_h) = [x_0^m x_1^{n_1} \cdots x_h^{n_h}] \vec{G}_\diamond^{(2a)} = 2\binom{3a-2}{a-1} [x_0^m x_1^{n_1} \cdots x_h^{n_h}] R^{3a-1}|_{t=1}.$$

Since R is specified by $R = t\phi(R)$, with $\phi(y) = (1 - \sum_{i\geq 0} x_i \binom{3i}{i} y^{3i+1})^{-1}$, the Lagrange inversion formula (recalled in Theorem 5) gives for any positive integers n, q,

$$[t^n]R^q = \frac{q}{n}[y^{n-q}]\phi(y)^n.$$

Note that $\widetilde{R} := R/t$ satisfies $\widetilde{R} = 1/(1 - \sum_{i\geq 0} x_i t^{3i+1} \binom{3i}{i} \widetilde{R}^{3i+1})$, hence for any $k \geq 1$,

$$\begin{aligned} [x_0^m x_1^{n_1} \cdots x_h^{n_h}] R^k|_{t=1} &= [x_0^m x_1^{n_1} \cdots x_h^{n_h}] \widetilde{R}^k|_{t=1} \\ &= [t^{m + \sum_{i=1}^h (3i+1)n_i} x_0^m x_1^{n_1} \cdots x_h^{n_h}] \widetilde{R}^k \\ &= [t^{k + m + \sum_{i=1}^h (3i+1)n_i} x_0^m x_1^{n_1} \cdots x_h^{n_h}] R^k. \end{aligned}$$

Hence, denoting $p := 3a - 1 + m + \sum_{i=1}^h (3i+1)n_i = m + r + 3b - 2 = k + 3b$, we have (using the Lagrange inversion formula from the 1st to the 2nd line):

$$\begin{aligned}
[x_0^m x_1^{n_1} \cdots x_h^{n_h}] R^{3a-1}|_{t=1} &= [t^p x_0^m x_1^{n_1} \ldots x_h^{n_h}] R^{3a-1} \\
&= \frac{3a-1}{p} [x_0^m \cdots x_h^{n_h}][y^{p-3a+1}] \Big(1 - 3\sum_{i=0}^{h} x_i \binom{3i}{i} y^{3i+1}\Big)^{-p} \\
&= \frac{3a-1}{p} [x_0^m \cdots x_h^{n_h}] \Big(1 - 3\sum_{i=0}^{h} x_i \binom{3i}{i}\Big)^{-p} \\
&= \frac{3a-1}{p} 3^{m+r-1} \binom{p-1+m+r-1}{p-1, m, n_1, \ldots, n_h} \prod_{i=1}^{h} \binom{3i}{i}^{n_i}
\end{aligned}$$

so that we obtain (using $k = m + r - 2$, $p = k + 3b$, $e = p + k$, and $(3a-1)\binom{3a-2}{a-1} = \frac{2}{3}a\binom{3a}{a}$)

$$\beta_a(m; n_1, n_2, \ldots, n_h) = 3^k \frac{(e-1)!}{m!(k+3b)!} 2a \binom{3a}{a} \prod_{i=1}^{h} \frac{1}{n_i!} \binom{3i}{i}^{n_i}, \tag{23}$$

which, multiplied by $\prod_{i=1}^{h} n_i!(2i)^{n_i}$ (to account for numbering the inner boundary faces and marking a corner in each of these faces), gives (7).

4.3. Application to triangulations with boundaries

We can follow a very similar approach for triangulations with boundaries, and derive a bijective proof of Krikun's formula (Theorem 3). We start with the case of *1-outer triangulations with boundaries*, i.e., triangulations with boundaries with a distinguished boundary face of degree 1, taken as the outer face. Let $\mathcal{D}_\triangle$ be the family of these maps. For $M \in \mathcal{D}_\triangle$, a *boundary-1-orientation* of M is a consistent weighted bi-orientation of M such that:

- every internal vertex has weight 1,
- every internal edge has weight -1,
- every internal face has weight -2,
- every inner boundary of length i has weight $i + 1$, and the outer boundary has weight 0.

Note that the weight condition at internal vertices implies that the half-edge weights are at most 1, hence the internal edges are either 1-way of weights $(-2, 1)$ or 0-way of weights $(-1, 0)$. Similarly as in Section 4.2.1 we have

Lemma 47. *Every $M \in \mathcal{D}_\triangle$ has a unique boundary-1-orientation in $\widehat{\mathcal{O}}_{-1}$, which is called the* canonical bi-orientation *of M.*

Proof. The proof follows similar lines as the proof of Lemma 40 (the main difference being that edges have to be subdivided into 4 edges instead of 2). For M a 1-outer triangulated dissection, define a *boundary-2-orientation* of M as a consistent weighted bi-orientation of M such that:

- every internal vertex has weight 2,
- every internal edge has weight -2,
- every internal face has weight -4,
- every inner boundary of length i has weight $2i + 2$, and the outer boundary has weight 0.

Note that this exactly amounts to doubling the specifications for boundary-1-orientations. And by Lemma 39 showing that any $M \in \mathcal{D}_\Delta$ admits a unique boundary-1-orientation in $\widehat{\mathcal{O}}_{-1}$ amounts to showing that M admits a unique boundary-2-orientation in $\widehat{\mathcal{O}}_{-1}$, which we are going to prove now (without all details).

Let M_4 be the map with boundaries obtained from M by subdividing each edge e (boundary or internal) into 4 edges, i.e., by inserting 3 vertices (of degree 2) on e, these 3 vertices being considered as white square vertices (whereas the vertices of M are considered as white round vertices). Then M_4 is a plane bipartite simple map with a quadrangular outer face. Hence $\sigma(M_4)$ can be endowed with a 2-regular orientation, and it induces a boundary-α-orientation of $\tau(M_4)$ for the following α:

- each internal vertex v has $\alpha(v) = 2$,
- each black vertex b (of degree 12) has $\alpha(b) = 8$ (i.e., indegree 8 and outdegree 4),
- each inner boundary C of length $4i$ (originally of length i in M) has $\alpha(C) = 2i+2$; and the outer boundary C_0 (of length 4) has $\alpha(C_0) = 0$.

Let $X_{\min}$ be the unique almost-minimal boundary-α-orientation of $\tau(M_4)$. Similarly as in the proof of Lemma 40 (but with a slightly more involved proof, omitted here), we have:

PROPERTY 48. *In $X_{\min}$, for any edge $\{b, w\}$ connecting a black vertex b to a white vertex w (either round or square) and directed toward w, the next edge (which is in M_4) after $\{b, w\}$ in clockwise order around w is also directed toward w.*

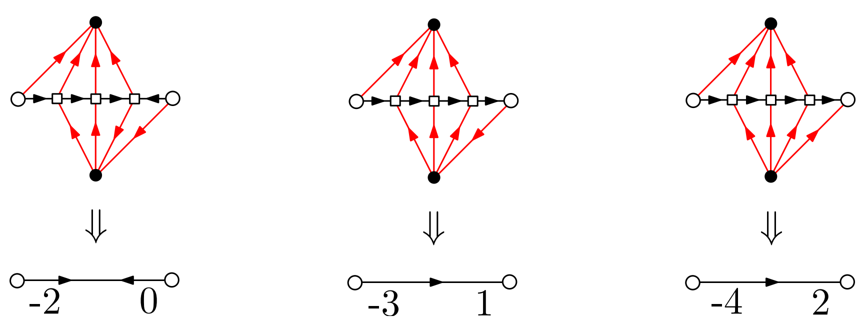


FIGURE 4.9. Top: the 3 possible configurations at an internal edge e of M in the minimal 2-regular orientation of $\tau(M_4)$. Bottom: transferring the configurations into weights on e.

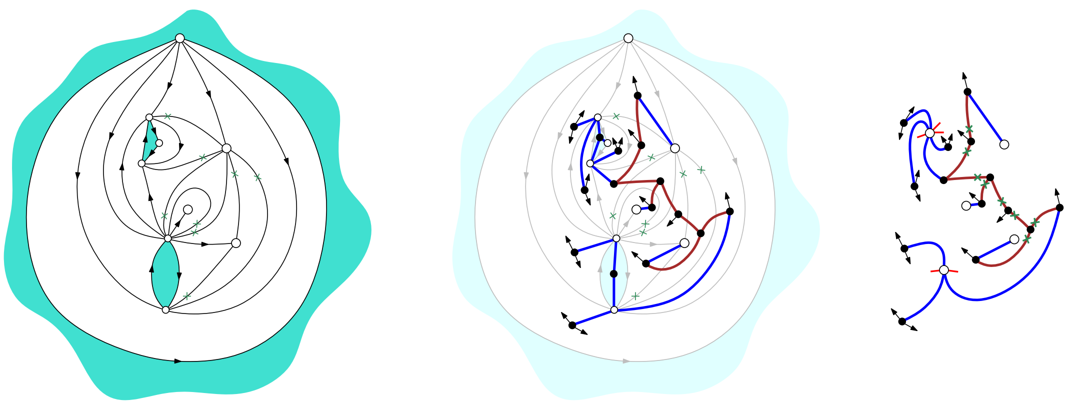
Property 48 then easily implies that any internal edge of M (subdivided into 4 edges in M_4) is of the 3 possible types shown in the top-part of Figure 4.9 (actually the 2nd case can be excluded a posteriori, using Lemma 39). Let Y_0 be the *induced* consistent weighted bi-orientation of M, which is obtained by applying the transfer rules of Figure 4.9. It is easy to see that Y_0 is a boundary-1-orientation of M, and similarly as in the proof of Lemma 40, it is in $\widehat{\mathcal{O}}_{-1}$ and it has to be unique. □

The weighted boundary bi-mobiles associated via the master bijection are characterized by the following properties (which easily implies that the excess is -1):

- every edge has weight -1, either black-black of weights $(-1, 0)$, or black-white of weights $(-2, 1)$,
- every black vertex has degree 3 and weight -2,

- for $i \geq 0$, every white vertex of degree $i+1$ carries i legs.

Call $\mathcal{T}_{\vartriangle}$ the family of these weighted boundary bi-mobiles. We obtain from Lemma 47 and Theorem 38 the following bijective statement, see Figure 4.10 for an example:



(a) (b) (c)

FIGURE 4.10. (a) A 1-outer triangulation with boundaries endowed with its canonical bi-orientation (oriented edges have weights $(-1,2)$ if internal and weights $(0,1)$ if boundary, unoriented edges have weights $(-1,0)$, the half-edges of weight -1 being indicated by a cross). (b) The triangulation superimposed with the corresponding bi-mobile. (c) The reduced boundary bi-mobile (with i legs at each white vertex of degree $i+1$, and with again the convention that half-edges of weight -1 are indicated by a cross).

PROPOSITION 49. *The family $\mathcal{D}_{\vartriangle}$ is in bijection with the family $\mathcal{T}_{\vartriangle}$. For $M \in \mathcal{D}_{\vartriangle}$ and $T \in \mathcal{T}_{\vartriangle}$ the associated weighted boundary bi-mobile, each inner boundary of length i in M corresponds to a white vertex in T of degree $i+1$; and each internal vertex of M corresponds to a white leaf in T.*

Then the general case is treated similarly as in Section 4.2.2; we consider, for any $a \geq 1$, the family $\mathcal{D}_{\vartriangle}^{(a)}$ of triangulations with boundaries with a distinguished boundary face of degree a taken as the outer face, and define $\overline{\mathcal{D}}_{\vartriangle}^{(a)}$ as the family of maps from $\mathcal{D}_{\vartriangle}^{(a)}$ where an arbitrary half-edge (either on a boundary edge or on an internal edge) is distinguished. Given $D \in \overline{\mathcal{D}}_{\vartriangle}^{(a)}$, we can blow a new face of degree 1 at the marked half-edge h, this face itself surrounded by a new face of degree 3 (both faces being considered as internal faces), as shown in Figure 4.11. The new face f_1 of degree 1 is taken as the marked inner face, and the resulting annular map is called an *a-annular triangulation with boundaries*, that is, an a-annular triangulation with boundaries is a map with boundaries where the outer face f_0 is a boundary face of degree a, and all internal faces have degree 3 except for one internal face f_1 of degree 1. For $a \geq 1$ the family of a-annular triangulations with boundaries is denoted by $\mathcal{G}_{\vartriangle}^{(a)}$, and the half-edge opening operation yields

$$\overline{\mathcal{D}}_{\vartriangle}^{(a)} \simeq \mathcal{G}_{\vartriangle}^{(a)}. \tag{24}$$

Next we can follow an annular approach (similarly as in Section 4.2.2) to encode a map $M \in \mathcal{G}_{\vartriangle}^{(a)}$ by two mobiles (the approach is illustrated in Figure 4.12). Define an *admissible*

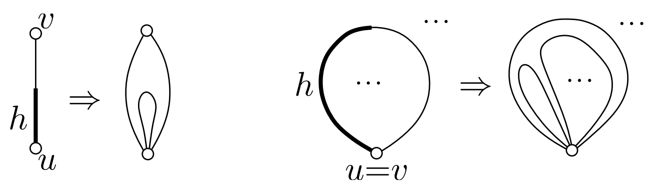


FIGURE 4.11. The operation of opening an half-edge h (on an internal or boundary edge) in a triangulated dissection, which yields a new face of degree 1 surrounded by a new face of degree 3 (both cases, whereas the half-edge is on a loop or not, can be seen as the same operation of opening the edge into a face of degree 2 inside which we place a loop on h's side).

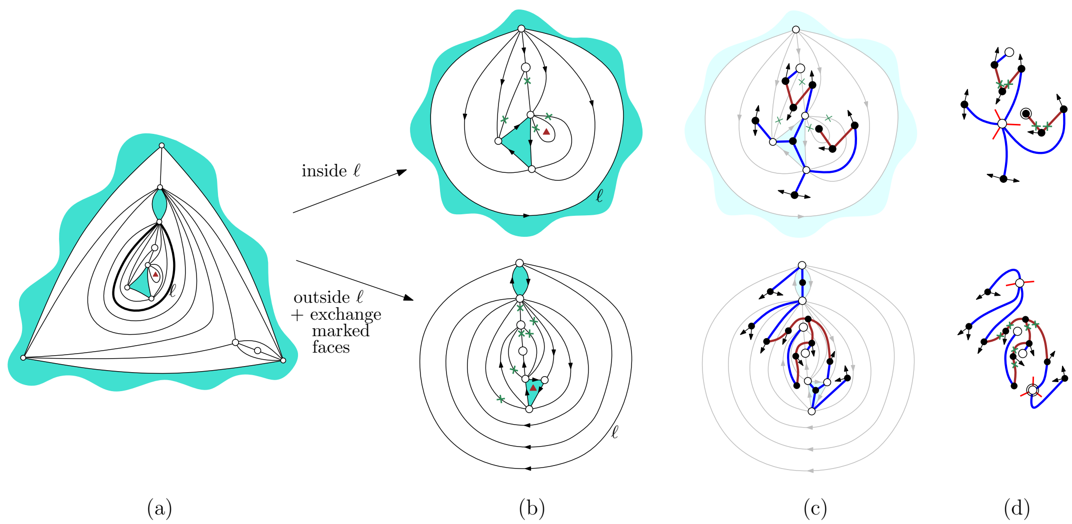


FIGURE 4.12. (a) a 3-annular triangulation with boundaries, where the internal face of degree 1 (marked inner face) is indicated by a triangle, and the outermost admissible loop ℓ is drawn bold. (b) The two parts resulting from cutting along ℓ, each endowed with its canonical bi-orientation (the marked inner face in each case is indicated by a triangle, each directed internal edge has weights $\{-2, 1\}$ and each undirected edge has weights $\{-1, 0\}$, with a cross on the half-edge of weight -1). (c) The bi-mobiles associated to the two parts. (d) The reduced boundary bi-mobiles, where the marked vertex in each case is surrounded (black-white edges have weights $\{-2, 1\}$, and black-black edges have weights $\{-1, 0\}$, with a cross on the half-edge of weight -1).

loop of M as a loop ℓ such that f_1 is inside ℓ and such that any face inside ℓ and incident to the vertex at ℓ is an internal face. Cutting M along the "outermost" admissible loop, yields two annular maps M_0, M_1 called the *two parts* of M: M_1 is the part "inside" ℓ with the outer face (delimited by ℓ) considered as a boundary face and with f_1 as the marked inner face (an internal face); M_0 is the part "outside" ℓ, where the face delimited by ℓ is taken as the outer face (considered as an internal face) and f_0 is taken as the marked inner face (a boundary face), see Figure 4.12(a)-(b) for an example.

Then (as an easy consequence of Lemma 47) one can show that M_1 can be endowed with a unique consistent weighted bi-orientation in $\widehat{\mathcal{O}}_{-1}$, called its *canonical bi-orientation*, such that:

- every internal vertex has weight 1,
- every internal edge has weight -1, either 1-way of weights $(-2, 1)$ or 0-way of weights $(-1, 0)$,
- every internal face of degree 3 has weight -2, and the (unique) internal face of degree 1 has weight 0,
- every inner boundary of length i has weight $i + 1$, and the outer boundary has weight 0.

And M_0 can be endowed (the proof is omitted) with a unique consistent weighted bi-orientation in $\widehat{\mathcal{O}}_1$, called its *canonical bi-orientation*, such that

- every internal vertex has weight 1,
- every internal edge has weight -1, either 1-way of weights $(-2, 1)$ or 0-way of weights $(-1, 0)$,
- every internal inner face (of degree 3) has weight -2, and the internal outer face (of degree 1) has weight 0,
- every non-marked boundary of length i has weight $i + 1$,
- the marked boundary, of length a, has weight $a - 1$.

The weighted boundary bi-mobiles associated to M_1 is thus specified by the following properties (which imply that the excess is -1):

- every internal edge has weight -1, either black-black of weights $(-1, 0)$, or black-white of weights $(-2, 1)$,
- every black vertex has degree 3 and weight -2, except for a unique black vertex of degree 1 and weight 0,
- for $i \geq 0$, every white vertex of degree $i + 1$ carries i legs.

And the boundary weighted bi-mobile associated with M_0 is specified by the following properties (which imply that the excess is 1):

- every internal edge has weight -1, either black-black of weights $(-1, 0)$, or black-white of weights $(-2, 1)$,
- every black vertex has degree 3 and weight -2,
- there is a marked white vertex of degree $a - 1$ that additionally carries a legs,
- for $i \geq 0$, every non-marked white vertex of degree $i + 1$ carries i legs.

We can now sketch how to prove Theorem 3 by counting the encoding mobiles (via generating functions), proceeding similarly as in Section 4.2.3, with slightly more involved calculations. Define a *planted bi-mobile of triangulated type* as one of the two connected components $P \in \{T_1, T_2\}$ obtained from some $T \in \mathcal{T}_\Delta$ by cutting an edge e in its middle; the half-edge h of e belonging to P is called the *root half-edge* of P, and the weight of h in T is called the *root-weight* of P. For $j \in \{-2, -1, 0, 1\}$, denote by $S_j \equiv S_j(t; x_0, x_1, \ldots)$ the generating function of planted bi-mobiles of triangulated type and root-weight j, with t conjugate to the number of buds and x_i conjugate to the number of white vertices of degree $i + 1$ for $i \geq 0$. And let $S := t + S_{-1}$. A decomposition at the root easily implies that

$\{S_{-2}, S_{-1}, S_0, S_1\}$ are specified by the following equation-system:

$$\left\{\begin{array}{rcl} S_{-2} &=& S^2, \\ S_{-1} &=& 2SS_0, \\ S_0 &=& 2SS_1 + S_0^2, \\ S_1 &=& \sum_{i\geq 0} x_i \binom{2i}{i} {S_{-2}}^i. \end{array}\right. \tag{25}$$

The second line (and $S = t + S_{-1}$) gives $S = t/(1-2S_0)$, so that the 3rd line gives $S_0(1-S_0)(1-2S_0) = 2tS_1$. If we now define $A = 1 - 2S_0$, we have $S_0 = (1-A)/2$ and $1-S_0 = (1+A)/2$, so that $A(1-A^2) = 8tS_1$. Hence $A = 8tS_1 + A^3$, hence $X := 1/A$ satisfies $X^2 = 1 + 8tS_1X^3$. Since $S = tX$, S satisfies $S^2 = t^2 + 8S_1S^3$, hence $S = t(1-8S_1S)^{-1/2}$, i.e., S satisfies the equation:

$$S = t\phi(S), \quad \text{where } \phi(y) = \Big(1 - 8\sum_{i\geq 0} x_i \binom{2i}{i} y^{2i+1}\Big)^{-1/2}. \tag{26}$$

Denote by $\vec{G}_{\triangle}^{(a)} \equiv \vec{G}_{\triangle}^{(a)}(x_0, x_1, \ldots)$ the generating function of maps from $\mathcal{G}_{\triangle}^{(a)}$ with a distinguished corner in the outer face, where x_0 is conjugate to the number of internal vertices and for $i \geq 1$, x_i is conjugate to the number of inner boundaries of length i. Then

$$\vec{G}_{\triangle}^{(a)} = \binom{2a-2}{a-1} {S_{-2}}^{a-1}|_{t=1} \cdot S|_{t=1} = \binom{2a-2}{a-1} S^{2a-1}|_{t=1}. \tag{27}$$

where the two factors (separated by $\cdot$) account for the contributions given by M_0 and M_1, respectively.

Define now $\eta_a(m; n_1, n_2, \ldots, n_h)$ as the number of triangulations with boundaries, with a marked boundary of length a having a distinguished corner, with m internal vertices, n_i non-marked boundaries of length n_i for $1 \leq i \leq h$, and no non-marked boundary of length larger than h. The total boundary length is $b := a + \sum_i in_i$, the number of boundaries is $r = 1 + \sum_i n_i$, and the associated number of edges is (by the Euler relation) $e = 2b + 3r + 3m - 6$, which is $2b + 3k$ with $k := r + m - 2$. Then (24) yields

$$2e\,\eta_a(m; n_1, n_2, \ldots, n_h) = [x_0^m x_1^{n_1} \cdots x_h^{n_h}]\vec{G}_{\triangle}^{(a)} = [x_0^m x_1^{n_1} \cdots x_h^{n_h}]\binom{2a-2}{a-1} S^{2a-1}|_{t=1}.$$

And a similar argument as in Section 4.2.2 (using $\widetilde{S} = S/t$) ensures that for $k \geq 1$,

$$[x_0^m x_1^{n_1} \cdots x_h^{n_h}]S^k|_{t=1} = [t^{k+m+\sum_{i=1}^h (2i+1)n_i} x_0^m x_1^{n_1} \cdots x_h^{n_h}]S^k$$

Hence, by the Lagrange inversion formula (used in the transition from the 1st to the 2nd line), and using the notations $p := 2a - 1 + m + \sum_{i=1}^h (2i+1)n_i = 2b + k$, $s = m + \sum_{i\geq 1} n_i =$

$k+1$ and $B = 8\sum_{i=0}^{h} x_i\binom{2i}{i}$:

$$\begin{aligned}
[x_0^m x_1^{n_1}\dots x_h^{n_h}]S^{2a-1}|_{t=1} &= [t^p x_0^m x_1^{n_1}\dots x_h^{n_h}]S^{2a-1}\\
&= \frac{2a-1}{p}[y^{p-2a+1}x_0^m x_1^{n_1}\dots x_h^{n_h}]\Big(1-8\sum_{i=0}^{h} x_i\binom{2i}{i}y^{2i+1}\Big)^{-p/2}\\
&= \frac{2a-1}{p}[x_0^m x_1^{n_1}\dots x_h^{n_h}]\Big(1-8\sum_{i=0}^{h} x_i\binom{2i}{i}\Big)^{-p/2}\\
&= \frac{2a-1}{p}[x_0^m x_1^{n_1}\dots x_h^{n_h}][u^s](1-Bu)^{-p/2}\\
&= \frac{2a-1}{p}[x_0^m x_1^{n_1}\dots x_h^{n_h}]B^s\cdot[u^s](1-u)^{-p/2}\\
&= \frac{2a-1}{p}8^s\binom{s}{m,n_1,\dots,n_h}\Big[\prod_{i=1}^{h}\binom{2i}{i}^{n_i}\Big]\cdot\frac{(p+2s-2)!!}{(p-2)!!s!2^s}\\
&= \frac{2a-1}{m!}4^s\Big[\prod_{i=1}^{h}\frac{1}{n_i!}\binom{2i}{i}^{n_i}\Big]\cdot\frac{(p+2s-2)!!}{p!!},
\end{aligned}$$

so that we obtain, using $e = p+2s-2$,

$$\eta_a(m;n_1,n_2,\dots,n_h) = \frac{(2a-1)!}{(a-1)!^2}4^{k+1}\left[\prod_{i=1}^{h}\frac{1}{n_i!}\binom{2i}{i}^{n_i}\right]\cdot\frac{(e-2)!!}{2m!\,p!!}.$$

Using $\frac{(2a-1)!}{(a-1)!^2} = \frac{1}{2}a\binom{2a}{a}$, and $p = 2b+k$, this rewrites as

$$\eta_a(m;n_1,n_2,\dots,n_h) = 4^k\frac{(e-2)!!}{m!\,(2b+k)!!}a\binom{2a}{a}\left[\prod_{i=1}^{h}\frac{1}{n_i!}\binom{2i}{i}^{n_i}\right].$$

Multiplying this last expression by $\prod_{i=1}^{h} n_i!i^{n_i}$ (to account for numbering the inner boundary faces and marking a corner in each of these faces) finally gives (6).

4.4. Solution of the dimer model on quadrangulations and triangulations

A *dimer-configuration* on a map M is a subset X of the non-loop edges of M such that every vertex of M is incident to at most one edge in X. The edges of X are called *dimers*, and the vertices not incident to a dimer are called *free*. The *partition function* of the *dimer model* on a class $\mathcal{C}$ of rooted maps[2] is the generating function of maps in $\mathcal{C}$ endowed with a dimer configuration, counted according to the number of dimers and free vertices; and *solving* the dimer model on $\mathcal{C}$ means deriving an explicit expression for the partition function. The partition function of the dimer model is well-known for rooted 4-valent maps [**86, 25**] (and more generally p-valent maps).

We observe that counting rooted maps with dimer configurations is a special case of counting rooted maps with boundaries. More precisely, upon blowing each dimer into a boundary face of degree 2, a rooted map with a dimer-configuration can be seen as a rooted map with boundaries, such that all boundaries have length 2, and the root-corner is in an internal face. Based on this observation we easily obtain from Theorem 4 that, for all $m, r \geq 0$

[2]Recall that rooted means: with a distinguished corner c_0 called the root-corner. The root-vertex v_0 is the vertex incident to c_0, and the root-edge is the edge after c_0 in clockwise order around v_0.

with $m+2r\geq 3$, the number $q_{m,r}$ of dimer-configurations on rooted quadrangulations with r dimers and $m+2r$ vertices is

$$q_{m,r} = 4(m+2r-2)\frac{3^{2r+m-2}(5r+2m-5)!}{r!m!(4r+m-2)!}. \tag{28}$$

Similarly, Theorem 3 implies that, for all $m,r\geq 0$ with $m+2r\geq 3$, the number $t_{m,r}$ of dimer-configurations on rooted triangulations with r dimers and $m+2r$ vertices is

$$t_{m,r} = (m+2r-2)\frac{2^{2m+3r-3}3^{r+1}(7r+3m-8)!!}{r!m!(5r+m-2)!!}. \tag{29}$$

We now aim at getting an explicit expression for the partition function, that is, the generating function of the coefficients $q_{m,r}$ or $t_{m,r}$. It should be possible to lift the expressions in (28) and (29) to generating function expressions, however we find it easier to obtain directly an exact expression from the bijections of Proposition 43 (for quadrangulations) and Proposition 49 (for triangulations), without a possibly technical lift from the coefficient expressions. Here this works by considering generating functions for the model with a slight restriction at the root.

For quadrangulations, we consider the generating function $Q(x,w)$ of rooted quadrangulations endowed with a dimer-configuration, with the constraint that both extremities of the root-edge are free, where x is conjugate to the number of free vertices minus 2, and w is conjugate to the number of dimers. These objects are clearly in bijection (by opening the root-edge and every dimer into a boundary face of degree 2) with the set $\mathcal{Q}$ of rooted quadrangulation with boundaries all of length 2, such that the root-corner is in a boundary face. So $Q(x,w)$ is the generating function of maps in $\mathcal{Q}$, where x is conjugate to the number of internal vertices and w is conjugate to the number of inner boundaries. Note that $\mathcal{Q}$ can be seen as a subset of $\mathcal{D}_\diamond$, except that we are marking a corner in the outer face. Thus, applying the bijection of Proposition 43, we can interpret $Q(x,w)$ in terms of the set $\mathcal{T}_\diamond'$ of mobiles from $\mathcal{T}_\diamond$ such that every boundary vertex has 2 legs. More precisely, upon remembering that mobiles in $\mathcal{T}_\diamond$ have excess -2, it is not hard to see that $Q(x,w)=Q_1-Q_2$, where Q_1 (resp. Q_2) is the generating function of mobiles from $\mathcal{T}_\diamond'$ with a marked bud (resp. with a marked leg or half-edge at a white vertex) with x counting white leaves, and w counting boundary vertices. From the series expressions obtained in Section 4.2.3 we get $Q_1=R_0=R-1$ and $Q_2=x\,R_{-1}+6w\,{R_{-1}}^2$, under the specialization $\{t=1, x_0=x, x_1=w, x_i=0\ \forall i\geq 2\}$. Hence

$$Q(x,w)=R-1-x\,R^3-6w\,R^6,\quad \text{where } R=1+3xR^2+9w\,R^5. \tag{30}$$

Note that $\widetilde{Q}(z,w):=Q(z,z^2w)$ is the generating function for the same objects, with z conjugate to the number of vertices minus 2 (which by the Euler relation is also the number of faces) and w conjugate to the number of dimers. Now, if we are interested in the *phase transition* of this model, we need to determine how the asymptotic behavior of the coefficients $c_n=[z^n]\widetilde{Q}(z,w)$ (for $n\to\infty$) depends on the parameter w. This amounts to studying [**51**] the dominant singularities of $\widetilde{Q}(z,w)$ *considered as a function of* z. (A companion maple worksheet can be found on the webpage of the author.) Denote by $\sigma(w)$ the dominant singularity of $\widetilde{Q}(z,w)$, and let $Z=\sigma(w)-z$. For all $w\geq 0$, the singularity type of $\widetilde{Q}(z,w)$ is $Z^{3/2}$ (as for maps without dimers), and no phase-transition occurs. However we find a singular value of w at $w_0=-3/125$, where $\sigma(w_0)=4/45$ and the singularity of $\widetilde{Q}(z,w_0)$ is of type $Z^{4/3}$ (as a comparison, it is shown in [**25**, Sec.6.2] that for the dimer model on rooted 4-valent maps endowed the critical value of the dimer-weight is $w_0=-1/10$ and the singularity type is the same: $Z^{4/3}$).

For triangulations we consider the generating function $T(x,w)$ of rooted triangulations endowed with a dimer-configuration, with the constraint that the root-vertex is free, where x is conjugate to the number of free vertices minus 1, and w is conjugate to the number of dimers. These objects are in bijection (up to opening the dimers into boundaries and opening the root half-edge as in Figure 4.11) with the set $\mathcal{T}$ of triangulations with boundaries, with one boundary of degree 1 taken as the outer face and all the other boundaries (inner boundaries) of length 2, and such that there are at least two inner faces. Let τ be the unique triangulation with one boundary face of length 1 (the outer face) and one inner face. By the preceding, $T(x,w)+x$ is the generating function of maps in $\mathcal{T}' = \mathcal{T} \cup \{\tau\}$. The bijection of Proposition 49 applies to the set $\mathcal{T}'$ and allows us to express $T(w,x)$ in terms of the set $\mathcal{T}'_\Delta$ of mobiles from $\mathcal{T}_\Delta$ such that every boundary vertex has 2 legs. More precisely, upon remembering that mobiles in $\mathcal{T}'_\Delta$ have excess -1, this bijection gives $T(x,w)+x = T_1 - T_2$, where T_1 (resp. T_2) is the generating function of mobiles from $\mathcal{T}'_\Delta$ with a marked bud (resp. a marked leg or half-edge incident to a white vertex) with x counting white leaves and w counting boundary vertices. From the series expressions obtained in Section 4.3, we get $T_1 = S_0 = (S-1)/(2S)$ and $T_2 = x\, S_{-2} + 10w\, {S_{-2}}^3$, under the specialization $\{t=1, x_0 = x, x_2 = w, x_i = 0\ \forall i \notin \{0,2\}\}$. Hence

$$T(x,w) = \frac{S-1}{2S} - x - x\,S^2 - 10\,w\,S^6, \quad \text{where } S^2 = 1 + 8xS^3 + 48\,w\,S^7. \tag{31}$$

Again we note that $\widetilde{T}(z,w) := T(z, z^2 w)$ is the generating function for the same objects, with z conjugate to the number of vertices minus 1 (which by the Euler relation is also one plus half the number of faces) and w conjugate to the number of dimers. We now discuss the phase transition. We use the notations $\sigma(w)$ for the dominant singularity of $\widetilde{T}(z,w)$, and $Z = \sigma(w) - z$. We find that for all $w \geq 0$, the singularity of $\widetilde{T}(z,w)$ is of type $Z^{3/2}$, so that no phase-transition occurs. However, we find a singular value $w_0 = -8\sqrt{105}/5145 \approx -0.0159$, for which $\sigma(w_0) = 5\sqrt{105}/1008 \approx 0.0508$ and $\widetilde{T}(z,w_0)$ has singularity type $Z^{4/3}$.

CHAPTER 5

Extensions, perspectives, and other results

5.1. Extension of Chapter 3 to higher girth

5.1.1. Bijection for $2b$-angulations of girth $2b$. Call a weighted bi-orientation *non-negative* if the weight of every half-edge is non-negative, i.e., every outgoing half-edge has weight 0. For M a map, with V the vertex-set and E the edge-set, and for $\alpha : V \to \mathbb{N}$ and $\beta : E \to \mathbb{N}$, define an *α/β-orientation* of M as a non-negative bi-orientation of M such that every vertex $v \in V$ has weight $\alpha(v)$ and every edge $e \in E$ has weight $\beta(e)$. For $V' \subseteq V$ and $E' \subseteq E$ define $\alpha(V') := \sum_{v\in V'} \alpha(v)$ and $\beta(E') := \sum_{e\in E'} \beta(e)$. Let M' be the map obtained from M after replacing every edge e by a bunch of $\beta(e)$ parallel edges, called the *edge-group* of M. Using the transfer rules shown in Figure 5.1, every α/β-orientation of M is equivalent to an α-orientation of M' with no ccw cycle inside an edge-group. This observation easily yields the following extension of Lemmas 8 and 9 to weighted bi-orientations:

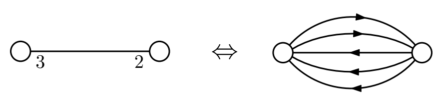


FIGURE 5.1. Rule to transfer a non-negative weighted bi-orientation to an orientation, so that the total weight at a vertex is mapped to the indegree at the same vertex.

LEMMA 50. *For $M = (V, E)$ a map, and for $\alpha : V \to \mathbb{N}$ and $\beta : E \to \mathbb{N}$, M admits an α/β-orientation iff:*

- $\alpha(V) = \beta(E)$,
- $\forall S \subseteq V$, $\alpha(S) \geq \beta(E_S)$.

In addition, for $v_0 \in V$, either all α/β-orientations are non-accessible from v_0 or all α/β-orientations are accessible from v_0; and the latter case occurs iff $\forall S \subseteq V\backslash\{v_0\}$, $\alpha(S) > \beta(E_S)$.

If M is a plane map and admits an α/β-orientation, then M admits a unique minimal α/β-orientation.

Lemma 50 makes it possible to extend Lemma 21 from simple quadrangulations to $2b$-angulations of girth $2b$, for any $b \geq 2$. The crucial point is the Euler relation (2), which guarantees that, for $M = (V, E)$ a (non-tree) bipartite map of girth at least $2b$, we have $(b-1)|E| \leq b|V| - 2b$, with equality iff M is a $2b$-angulation. For $M = (V, E)$ a $2b$-angulation, define a *$b/(b-1)$-orientation* of M as an α/β-orientation where $\alpha(v) = b$ for every inner vertex v, and $\beta(e) = b-1$ for every inner edge e, $\alpha(v) = 1$ for every outer vertex

v, and $\beta(e) = 1$ for every outer edge e. Then Lemma 50 and the remark above on the Euler relation easily give the following extension [1] of Lemma 21:

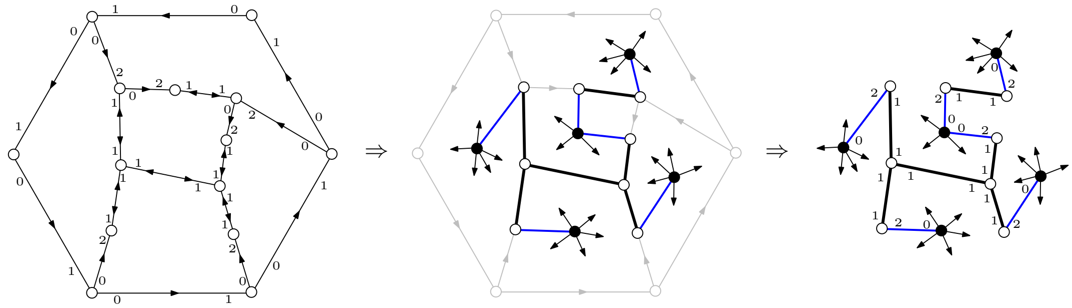


FIGURE 5.2. Left: a $2b$-angulation of girth $2b$ ($b = 3$ in the example) endowed with its unique $b/(b-1)$-orientation in $\mathcal{O}_{-2b}$. Right: the associated weighted bi-mobile.

LEMMA 51. *For $b \geq 2$ and M a plane $2b$-angulation, M admits a $b/(b-1)$-orientation iff M has girth $2b$. In the latter case, $b/(b-1)$-orientations are accessible from every outer vertex of M, and M admits a unique $b/(b-1)$-orientation in $\mathcal{O}_{-2b}$.*

Using the master bijection this implies that plane $2b$-angulations of girth $2b$ with n inner faces are in bijection with weighted bi-mobiles with n black vertices all of degree $2b$ and weight 0, and where every white vertex has weight b and every edge has weight $b-1$, see Figure 5.2 for an example.

5.1.2. Extension to bipartite plane maps of outer degree $2b$ and girth $2b$. We can now extend the result to (non-necessarily $2b$-angulated) bipartite plane maps of outer degree $2b$ and girth $2b$. For $b \geq 2$ and M a bipartite plane map of outer degree $2b$, define a $b/(b-1)$-*orientation* of M as a weigted bi-orientation of M such that:

- every inner (resp. outer) vertex of M has weight b (resp. weight 1),
- every inner (resp. outer) edge of M has weight $b-1$ (resp. weight 1),
- every inner face of degree $2k$ has weight $-k+b$ (hence $k \geq b$).

Then, building on Lemma 51 similarly as Lemma 25 builds on Lemma 21 in Chapter 3, we obtain the following statement that extends both Lemma 25 and Lemma 51:

LEMMA 52. *For $b \geq 2$ and M a bipartite plane map of outer degree $2b$, M admits a $b/(b-1)$-orientation iff M has girth $2b$. In the latter case, M admits a unique $b/(b-1)$-orientation in $\mathcal{O}_{-2b}$.*

Now if we define a $2b$-branching mobile as a weighted bi-mobile where:

- every white vertex has degree b,
- every edge has weight $b-1$,
- every black vertex has even degree, and every black vertex of degree $2k$ has weight $-k+b$,

then the master bijection gives:

[1] When carrying out the existence proof using Lemma 50, it is more convenient to temporarily modify the definition to $\alpha(v) = b-1$ at outer vertices and $\beta(e) = b-1$ at outer edges.

PROPOSITION 53. *For $b \geq 2$, plane bipartite maps of outer degree $2b$ and girth $2b$ are in bijection with $2b$-branching mobiles, such that every inner face of degree $2k$ of the map corresponds to a black vertex of degree $2k$ in the associated weighted bi-mobile.*

Regarding the shape of a $2b$-branching mobile T, note that the vertex weight condition implies that the weights on half-edges are not larger than b, hence every edge either has weights (i, j) with $i, j \geq 0$ and $i + j = b - 1$, or has weights $(-1, b)$. An edge e of weights $(-1, b)$ is called a *pending edge* (the white extremity of e has its weight saturated to b by e, hence is a leaf); note that the weight condition at black vertices ensures that any black vertex of degree $2k$ is incident to $k - b$ pending edges (the case with no pending edge thus corresponds to $2b$-angulations).

Define a *planted* $2b$*-branching mobile* as one of the two connected components $P \in \{T_1, T_2\}$ after cutting an edge of a $2b$-branching mobile in its middle. The half-edge $h \in e$ that belongs to P is called the *root half-edge* of P, and its weight is called the *root-weight* of P. For $i \in [-1..b]$, let $T_i \equiv T_i(t, x_b, x_{b+1}, \ldots)$ be the generating function of planted $2b$-branching mobiles of root-weight $b - 1 - i$, where t is conjugate to the number of buds, and x_r is conjugate to the number of black vertices of degree $2r$. Then a decomposition at the root easily implies that $\{T_{-1}, \ldots, T_b\}$ are specified by the equation-system (see Figure 5.3 for the case of $i \in [0..b - 2]$):

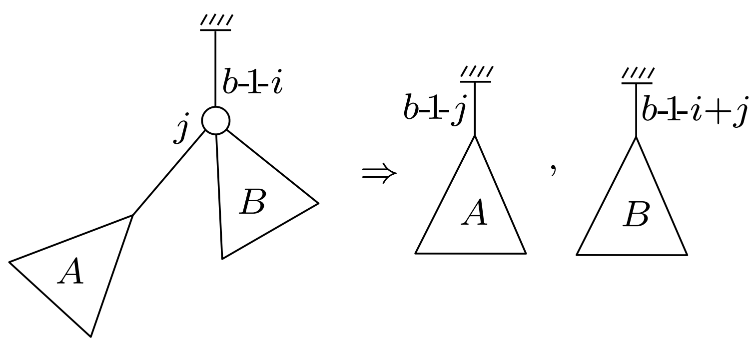


FIGURE 5.3. For $i \in [0..b - 2]$, decomposition of a mobile counted by T_i into two mobiles counted respectively by T_j and T_{i-j}, where j is the weight of the half-edge incident to the root-vertex v_0 and on the edge leading to the leftmost child of v_0.

$$(32)\qquad \begin{cases} T_{-1} &= 1, \\ T_i &= \displaystyle\sum_{j=1}^{i+1} T_j T_{i-j} \quad \text{for } i \in [0..b-2], \\ T_{b-1} &= \displaystyle\sum_{i \geq b} x_i \binom{2i-1}{i-b} (t + T_0)^{\,i+b-1}, \\ T_b &= \displaystyle\sum_{i \geq b+1} x_i \binom{2i-1}{i-b-1} (t + T_0)^{\,i+b}. \end{cases}$$

To our knowledge, these series have nice factorized coefficients only for $b = 2$. For $b > 2$ and a fixed integer N, the system is algebraic whenever the variables x_i are set to 0 for i larger than N, and the coefficients can be extracted iteratively from the system. Note also that for any $b \geq 2$ one has $T_0 = T_1$ and one can first extract the coefficients of $T_0, \ldots, T_{b-1}$, and then the coefficients of T_b from the coefficients of T_0. For instance, for $b = 3$ the system reads:

$$\begin{cases} T_0 &= T_1, \\ T_1 &= T_1 T_0 + T_2, \\ T_2 &= \sum_{i \geq 3} x_i \binom{2i-1}{i-3} (t + T_0)^{i+2}, \\ T_3 &= \sum_{i \geq 4} x_i \binom{2i-1}{i-4} (t + T_0)^{i+3}. \end{cases}$$

5.1.3. Extension to annular bipartite maps of non-separating girth at least $2b$. Similarly, the results of Chapter 3 on annular maps can be extended to higher girth. For $b \geq 2$ and $s, p \geq 1$, let $\mathcal{C}_{2b,2s}^{(2q)}$ be the family of bipartite annular maps of non-separating girth at least $2b$, separating girth $2s$, outer face degree $2s$, and marked inner face degree $2q$. And for M a bipartite annular map with outer face degree $2s$ and marked inner face degree $2q$, define a $b/(b-1)$-*orientation* of M as a weighted bi-orientation such that:

- every inner (resp. outer) vertex has weight b (resp. 1),
- every inner (resp. outer) edge has weight $b - 1$ (resp. 1),
- every non-marked inner face of degree $2k$ has weight $-k + b$ (hence $k \geq b$), and the marked inner face, of degree $2q$, has weight $-q + s$.

Again, Lemma 29 can be generalized to any $b \geq 2$ (with similar proof arguments, in particular one starts with the case where all non-marked inner faces have degree $2b$, and the marked inner face has degree $2s$), and one obtains

LEMMA 54. *For $b \geq 2$ and $s, q \geq 1$, a bipartite annular map M with outer face degree $2s$ and marked inner face degree $2q$ admits a $b/(b-1)$-orientation iff $M \in \mathcal{C}_{2b,2s}^{(2q)}$. In the latter case, M admits a unique $b/(b-1)$-orientation in $\mathcal{O}_{-2s}$.*

The master bijection then gives the following result, which extends both Proposition 32 (case $b = 2$) and Proposition 53 (case $s = b$):

PROPOSITION 55. *For $b \geq 2$ and $s, q \geq 1$, the family $\mathcal{C}_{2b,2s}^{(2q)}$ is in bijection with the family of weighted bi-mobiles with white vertices of weight b, edges of weight $b - 1$, black vertices of even degree, one of which is marked, where every non-marked black vertex of degree $2k$ has weight $-k + b$, and the marked black vertex has degree $2q$ and weight $-q + s$.*

For $b \geq 2$ and $r, s \geq 1$, let $\vec{\mathcal{A}}_{2b,2s}^{(2p,2q)}$ be the family of rooted annular maps (rooted meaning with a marked corner in the outer face and in the marked inner face) of outer face degree $2p$, marked inner face degree $2q$, non-separating girth at least $2b$ and separating girth at least $2s$; define similarly the family $\vec{\mathcal{A}}_{2b,=2s}^{(2p,2q)}$, with the difference that the separating girth is required to be exactly $2s$. And let $\vec{A}_{2b,2s}^{(2p,2q)} \equiv \vec{A}_{2b,2s}^{(2p,2q)}(x_b, x_{b+1}, \ldots)$ (resp. $\vec{A}_{2b,=2s}^{(2p,2q)} \equiv \vec{A}_{2b,=2s}^{(2p,2q)}(x_b, x_{b+1}, \ldots)$) be the corresponding generating functions, with x_i conjugate to the number of non-marked inner faces of degree $2i$, for $i \geq b$; and let $\vec{C}_{2b,2s}^{(2q)} := \vec{A}_{2b,2s}^{(2s,2q)}$. Then, similarly as in (14), we have

$$\vec{A}_{2b,=2s}^{(2p,2q)} = \frac{\vec{C}_{2b,2s}^{(2p)} \cdot \vec{C}_{2b,2s}^{(2q)}}{\vec{C}_{2b,2s}^{(2s)}}, \tag{33}$$

and moreover it follows from Proposition 55 (and decomposing the associated mobiles at the marked black vertex) that

$$\vec{C}_{2b,2s}^{(2q)} = 2s\binom{2q}{q-s}(t+T_0)^{q+s}|_{t=1}.$$

Hence we have

$$\vec{A}_{2b,=2s}^{(2p,2q)} = 2s\binom{2p}{p-s}\binom{2q}{q-s}(t+T_0)^{p+q}|_{t=1},$$

where we note that the power of $(t+T_0)$ does not depend on s. Moreover, as we have seen in the proof of Proposition 33, the coefficient $\gamma(p,q,d) = \sum_{s\geq d} 2s\binom{2p}{p-s}\binom{2q}{q-s}$ is equal to $\frac{4pq}{p+q}\binom{2p-1}{p-d}\binom{2q-1}{q-d}$. Hence we obtain the following generalization of Proposition 33:

PROPOSITION 56. *For $b \geq 2$ and $s,p,q \geq 1$, the generating function $\vec{A}_{2b,2s}^{(2p,2q)}$ is given by*

$$\vec{A}_{2b,2s}^{(2p,2q)} = \frac{4pq}{p+q}\binom{2p-1}{p-s}\binom{2q-1}{q-s}(t+T_0)^{p+q}|_{t=1}, \quad (34)$$

where T_0 is specified by the system (32).

The following consequence of Proposition 56 is in agreement with the strong belief that for any 'reasonable' family $\mathcal{F} = \bigcup_n \mathcal{F}_n$ of planar maps, indexed by a size-parameter n (such as the number of edges or the number of faces), there holds the universal asymptotic behaviour $|\mathcal{F}_n| \sim c\cdot\gamma^n\cdot n^{-5/2}$, for positive constants c,γ depending on the family considered (similarly as the universal law $|\mathcal{F}_n| \sim c\cdot\gamma^n\cdot n^{-3/2}$ for tree-families).

COROLLARY 57. *For $b \geq 2$, and for any finite set $\Delta \subset \{2b, 2b+2, 2b+4, \ldots\}$, let $a_{b,\Delta}(n)$ be the number of rooted bipartite maps of girth at least $2b$ and whose faces have degree in Δ. Then, there exist computable positive constants c,γ (depending on Δ) such that*

$$a_{b,\Delta}(n) \sim c\,\gamma^n n^{-5/2}.$$

PROOF. (Sketch.) We specialize the series $T_0,\ldots,T_{b-1},T_b$ by setting $x_i = t$ for $2i \in \Delta$ and $x_i = 0$ otherwise. Then the system (32) restricted to $T_0,\ldots,T_{b-1}$ is easily checked to satisfy the hypotheses of the Drmota-Lalley-Wood theorem [**51**, VII.6], so that these series have square-root singularities at their unique dominant singularity ρ. Therefore the same applies to $\vec{A}_{2b,2b}^{(2p,2q)}$ for any p,q with $p \geq b$ and $q \geq b$, implying $[t^n]\vec{A}_{2b,2b}^{(2p,2q)} \sim \kappa n^{-3/2}\gamma^n$, with $\gamma = \rho^{-1}$ and some computable constant $\kappa > 0$. Now observe that $\frac{1}{2q}[t^n]\vec{A}_{2b,2b}^{(2p,2q)}$ counts rooted bipartite maps of girth at least $2b$, with a root-face of degree $2p$, a marked inner face of degree $2q$, and n additional inner faces having degrees in Δ. Therefore

$$(n+1)a_{b,\Delta}(n+2) = \sum_{p,q\in\Delta}\frac{1}{2q}[t^n]\vec{A}_{2b,2b}^{(2p,2q)},$$

which gives the claimed asymptotic form of $a_{b,\Delta}(n)$. □

5.1.4. Extension to (non-necessarily bipartite) maps. It turns out that one can easily build on the results obtained for bipartite maps and extend these results to arbitrary maps, with control on the girth and face-degrees. We directly give the results at the most general level, i.e., for annular maps. For $d \geq 2$ and $s \geq 1$, let M be an annular map of outer face degree s. Define a $d/(d-2)$*-orientation* of M as a weighted bi-orientation such that:

- every inner (resp. outer) vertex has weight d (resp. 1),
- every inner (resp. outer) edge has weight $d-2$ (resp. 1),

- every non-marked inner face of degree k has weight $-k+d$ (hence $k \geq d$), and the marked inner face, denote by q its degree, has weight $-q+s$ (hence $q \geq s$).

Note that, since the weights on half-edges are at most d (due to the vertex weight condition), the inner edges are of 3 possible types: either of weights (i,j) with $i,j \geq 0$ and $i+j=d-2$, or of weights $(-1,d-1)$, or of weights $(-2,d)$. We have:

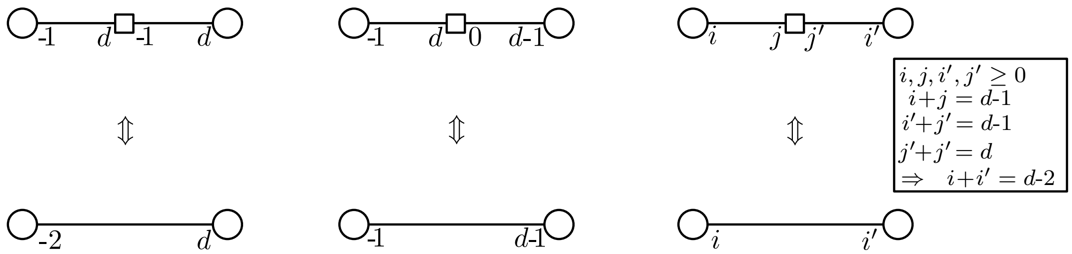


FIGURE 5.4. Rules to transfer the weights of a $d/(d-1)$-orientation of M_2 to a $d/(d-2)$-orientation of M.

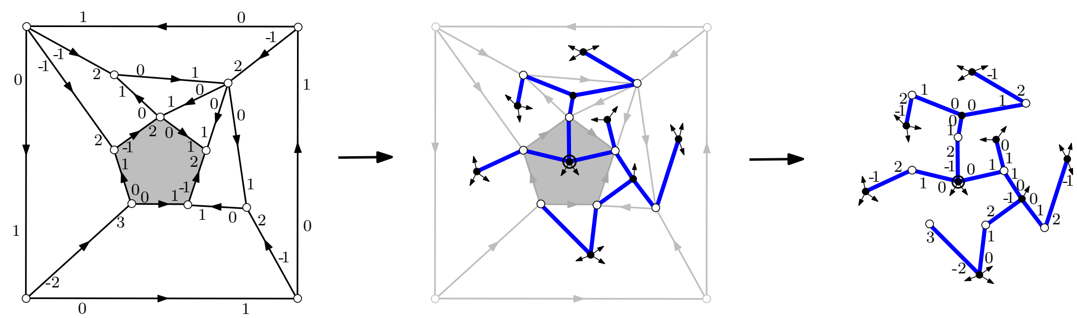


FIGURE 5.5. Bijection for annular maps. Left: an annular map of separating girth and outer face degree s, and non-separating girth at least d ($s=4$ and $d=3$ in the example), endowed with its unique $d/(d-2)$-orientation. Right: the associated weighted mobile.

LEMMA 58. *Let $d \geq 2$, $s \geq 1$, and let M be an annular map of outer degree s. Then M admits a $d/(d-2)$-orientation iff M has separating girth s and non-separating girth at least d. In that case M admits a unique $d/(d-2)$-orientation in $\mathcal{O}_{-s}$.*

PROOF. First, one can check that if there is a cycle c that is either a separating cycle of length smaller than s or a non-separating cycle of length smaller than d then such an orientation can not exist (by using the Euler relation applied to the map restricted to c and to the interior of c).

Now assume this is not the case, i.e., that M has separating girth s and non-separating girth at least d. Similarly as in Chapter 4, we consider the map M_2 obtained from M (whose vertices are considered as round) by inserting a vertex (considered as a square vertex) in the middle of each edge. Note that M_2 is a bipartite map of outer degree $2s$, separating girth $2s$ and non-separating girth at least $2d$, hence by Lemma 54 M_2 admits a unique $d/(d-1)$-orientation in $\mathcal{O}_{-2s}$. Now, as illustrated in Figure 5.4 there are simple transfer rules for weights, ensuring that $d/(d-1)$-orientations of M_2 are in bijection with $d/(d-2)$-orientations of M, and the bijection preserves the property of being minimal and being accessible from an outer vertex. Hence M has a unique $d/(d-2)$-orientation in $\mathcal{O}_{-s}$. $\square$

By similar arguments as in Section 5.1.3, it is then possible to obtain an expression for the generating function $A_{d,s}^{(p,q)}(x_d, x_{d+1}, \cdots)$ of rooted (non-necessarily bipartite) annular maps of non-separating girth at least d, separating girth at least s, outer-face degree p and marked inner face degree q, where x_i is conjugate to the number of non-marked inner faces of degree i for $i \geq d$. We omit the details and expressions (see [**A24**]), which are a bit more involved than the ones for bipartite maps. In turn, these expressions make it possible to extend Corollary 57 to any finite subset of integers (not necessarily even).

5.1.5. Getting the smaller girth cases. We have deduced Lemma 58 from Lemma 54 by 2-subdividing the edges. It is actually also possible to deduce Lemma 54 from Lemma 58 (supposing we are given Lemma 58 without knowing about Lemma 54). Indeed, for $b, s \geq 1$, and for M a bipartite annular map of outer degree $2s$, separating girth $2s$ and non-separating girth at least $d = 2b$, the $d/(d-2)$-orientation of M in $\mathcal{O}_{-2s}$ has all its inner vertices, inner edges, and inner faces of even weight. Hence, the associated mobile T has all its black vertices, white vertices, and edges of even weight, from which it is easy to deduce that all half-edges in the mobile have even weight (similarly as in Lemma 39 in Chapter 4, we denote by F the subforest formed by the edges of T whose two half-edges have odd weights, then the even weight condition guarantees that all vertices are incident to an even number of edges from F, hence F has no leaf and has thus to be empty). Hence all inner half-edges of M have even weight, and dividing all the weights by 2, we obtain a $b/(b-1)$-orientation of M in $\mathcal{O}_{-2s}$, which has to be unique.

The argument works for any $d \geq 2$, and in particular it works for $d = 2$, so that we can extend the statement of Lemma 54 (and also the associated bijective statement in Proposition 56) to $b = 1$. For $s = 1$, this gives a bijection for unconstrained bipartite plane maps with an outer face of degree 2 (which are equivalent to bipartite maps with a marked edge, upon seeing the marked edge as opened into a face of degree 2). It can be checked (see [**A24**] for details) that the bijection obtained this way coincides with the one in [**81**]. In a second step, we note that we can exploit once more the interplay between Lemma 54 and Lemma 58. Indeed the transfer rules of Figure 5.4 work for $d = 1$ as well, hence Lemma 54 extended to $b = 1$ yields Lemma 58 extended to $d = 1$. For $s = 1$, the bijection we obtain can be checked (see [**A24**]) to be equivalent to the bijection introduced by Bouttier, Di Francesco, and Guitter [**24**] for arbitrary maps with a marked vertex of degree 1, with control on the vertex degrees (dually, in our setting, for arbitrary maps with an outer loop, with control on the face degrees).

5.1.6. Extension to hypermaps. A *hypermap* (see [**41**] for a survey) is a face-bicolored map, with dark faces and light faces, where every edge has a light face on one side and a dark face on the other side (dark faces are sometimes called *hyperedges*); this generalizes the concept of map, since a map M can be turned into a hypermap by blowing each edge into a dark face of degree 2. Encouraged by the fact that the BDG bijection for vertex-pointed bipartite maps (reviewed in Section 2.3) extends to (vertex-pointed) hypermaps [**26**], we have been able to extend the master bijection to the context of hypermaps, with applications to counting hypermaps constrained by a certain generalization of the concept of girth.

For H a hypermap, a *hyperorientation* of H is a bi-orientation of H where every edge is either 0-way or 1-way with a dark face on its right. And a *weighted* hyperorientation is a hyperorientation where every edge is assigned a weight in $\mathbb{Z}$, with the constraint that

0-way edges have non-positive weight, and 1-way edges have positive weight. Again, this generalizes the concept of bi-orientation and weighted bi-orientation, as shown in Figure 5.7.

The master bijection can then be extended to the setting of hypermaps (in **[S1]**). For $\delta < 0$ (resp. $\delta > 0$) we define $\mathcal{H}_\delta$ as the family of plane maps with a dark (resp. light) outer face, endowed with a weighted hyperorientation such that, when forgetting weights and face-colors, the underlying bi-oriented plane map is in $\mathcal{O}_\delta$. And we define $\mathcal{H}_0$ as the family of vertex-pointed hypermaps endowed with a weighted hyperorientation such that the underlying bi-oriented vertex-pointed map is in $\mathcal{O}_0$. A *hypermobile* is defined as a plane tree with 3 types of vertices: light square, dark square, round, with possibly buds attached at corners of light square vertices, and with the constraint that each (non-bud) edge has exactly one dark square extremity, i.e., is either connecting a dark square vertex to a round vertex, or connecting a dark square vertex to a light square vertex. A hypermobile is *weighted* by assigning a weight to each edge, positive if the edge is incident to a round vertex, and non-positive if the edge is incident to a light square vertex. Again this extends the concept of (weighted) bi-mobile, since a bi-mobile can be turned into a hypermobile by considering white vertices as round vertices, black vertices as light square vertices, and by inserting a dark square vertex of degree 2 in the middle of each edge.

Then the master bijection is performed in the same way as for the underlying bi-oriented map, with the only adaptation that, instead of calling 'black vertices' the vertices inserted inside faces, we call these vertices respectively 'dark square vertices' (resp. 'light square vertices') if inserted inside a dark (resp. light) face; the local rules at each edge (1-way or 0-way, with the obvious weight-transfer rule) are shown in Figure 5.6. Note also that, as shown in Figure 5.7, when all inner dark faces have degree 2, the local rules become equivalent (up to the identification of bi-mobiles with hypermobiles where dark square vertices have degree 2) to the local rules for bi-oriented maps (Figure 2.11).

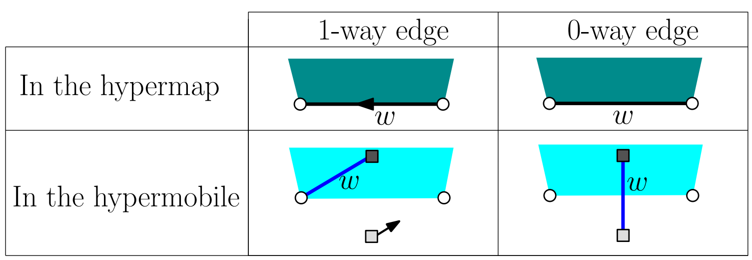


FIGURE 5.6. Local rule performed at each edge when applying the master bijection for hyperorientations.

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Regarding how the notion of girth can be extended to hypermaps (there are several possibilities), we have found a parameter (depending on the embedding in the plane) that can be well handled by our bijective method: the *ingirth* of a plane hypermap H is defined as the length of a shortest cycle c such that all faces inside c and sharing at least one edge with c are light. This notion provides an extension of the girth (when all inner dark faces have degree 2 the ingirth coincides with the girth), and we again have the nice property that the ingirth can be captured by the existence and uniqueness of certain weighted hyperorientations. Precisely, for H a plane hypermap with an outer face of degree d and ingirth d, it can be shown that H admits a unique hyperorientation in $\mathcal{H}_{-d}$ such that

- every inner (resp. outer) vertex has weight d,

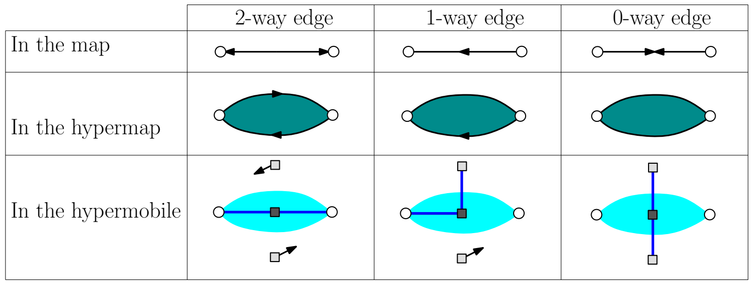

FIGURE 5.7. Under the identification of edges in maps with dark faces of degree 2 in hypermaps, the local rules of Figure 5.6 for hyperorientions yield the local rules of Figure 2.11 for bi-orientations.

- every dark face (hyperedge) of degree k has weight $kd - k - d$,
- every light inner face of degree k has weight $-k + d$.

Note that it extends the result for maps of outer degree d and girth d, since any dark inner face of degree 2 has weight $d-2$ (which is consistent with the map context where every inner edge has weight $d - 2$).

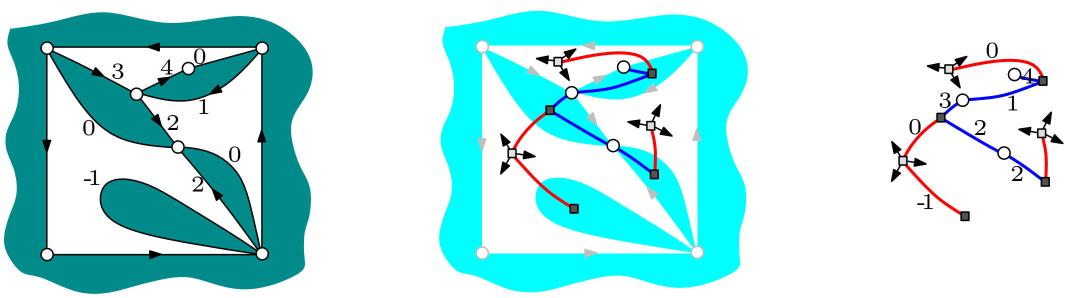

FIGURE 5.8. Left: a plane hypermap with a dark outer face of degree d and ingirth d ($d = 4$ in the example), endowed with its canonical hyperorientation. Right: the associated weighted hypermobile.

One can then use the master bijection for hyperorientations, which yields a bijection (for any $d \geq 1$) between these plane hypermaps and a family of weighted hypermobiles, see Figure 5.8 for an example. For $d = 1$ we recover in a dual setting the bijection in [**21**] for bipartite map with a marked black vertex of degree 1, and for $d \geq 2$ our bijection specialized to constellations (all dark faces have degree d, all light faces have degree divisible by d) coincides with the one introduced in [**20**].

Again the results extend to annular hypermaps (and even more, they extend to a general setting with so-called *charges* at vertices, edges, and faces, where the charge-parameters add constraints on the minimal length of a cycle enclosing them, see [**S1**]). And again the hypermobile generating functions can be specified by a system of equations (which is algebraic when forbidding face-degrees larger than a fixed threshold N), and the generating function of annular hypermaps with fixed non-separating and separating ingirth lower bounds can be determined, see [**S1**] for details and expressions.

5.2. Perspectives on Chapter 3

5.2.1. Irreducible maps. It should be possible to extend our results on bipartite maps (Sections 5.1.2 and 5.1.3) and general maps (Section 5.1.4) under a slightly extended notion of girth that is also considered in [**31**]. Let us just discuss here the extension of Lemma 52 and Proposition 53 (for bipartite maps). For $b \geq 1$ (the extended notion of girth becomes a non-trivial extension only for $b \geq 3$), a bipartite map is said to be $2b$*-irreducible* iff all cycles have length at least $2b$ and the only cycles of length $2b$ are facial (i.e., are contours of faces of degree $2b$). The faces of degree $2b$ are called *small faces*. Note that we recover the classical notion of girth at least $2b+2$ when there are no small faces.

For $b \geq 3$ and M a bipartite plane map of outer degree $2b$ and with all inner faces of degree at least $2b-2$, define a $b/(b-1)$*-orientation* of M as a weighted bi-orientation of M such that:

- every inner (resp. outer) vertex has weight b (resp. weight 1),
- every outer edge has weight 1, and every inner edge has weight $b-1$ or $b-2$; those of weight $b-2$ are 1-way of weights $(0, b-2)$, these inner edges are called *small edges*,
- for $k \geq b$, every inner face of degree $2k$ has weight $-k+b$, and every small face has weight 0,
- every small edge has a small face on its right, and every small face f has clockwise-degree 1, and the unique outgoing half-edge with f on its right belongs to a small edge (thus there is a 1-to-1 correspondence between small edges and small faces).

Note that this extends the definition of $b/(b-1)$-orientations given in Section 5.1.2 (which corresponds to the case without small faces). As an extension of Lemma 52 it should be possible to prove:

LEMMA 59. *For $b \geq 3$, and M a bipartite plane map of outer degree $2b$ and with all inner faces of degree at least $2b-2$, M admits a $b/(b-1)$-orientation iff M is $(2b-2)$-irreducible, in which case M admits a unique $b/(b-1)$-orientation in $\mathcal{O}_{-2b}$.*

Now define an *extended $2b$-branching mobile* as a weighted bi-mobile where every black vertex has even degree at least $2b-2$ (black vertices of degree $2b-2$ are called *small*) such that:

- every white vertex has weight b,
- every edge has weight $b-1$ or $b-2$, edges of weight $b-2$ have half-edge weights $(0, b-2)$, these are called *small edges*,
- for $k \geq b$, every black vertex of degree $2k$ has weight $-k+b$, and every small black vertex has weight 0,
- the black extremity of every small edge is small, and every small black vertex has $2b-3$ buds and the unique incident edge is small (thus there is a 1-to-1 correspondence between small edges and small black vertices).

The master bijection would then give the following extension of Proposition 53:

PROPOSITION 60. *For $b \geq 3$, plane bipartite maps that are $(b-1)$-irreducible and have outer degree $2b$ are in bijection with extended $2b$-branching mobiles. For $k \geq b-1$, every inner face of degree $2k$ in the map corresponds to a black vertex of degree $2k$ in the associated weighted bi-mobile.*

For $b = 3$ and when all inner faces are small (quadrangular), we should recover the bijection in [**A6**] (with unrooted binary trees) for irreducible dissections of the hexagon.

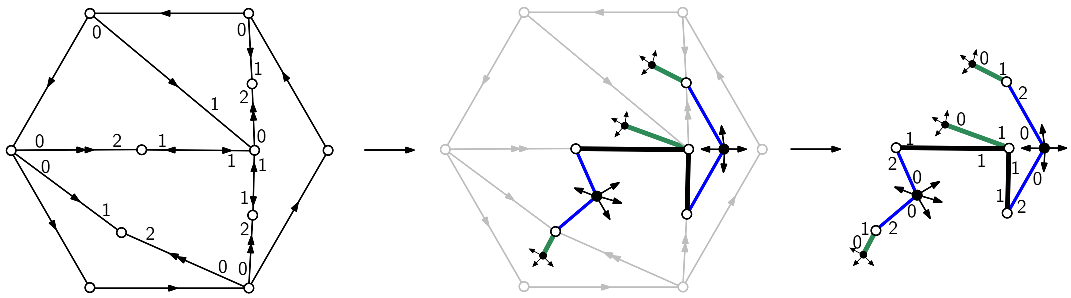

FIGURE 5.9. Left: a plane bipartite map of outer degree $2b = 6$ and 4-irreducible, endowed with its unique $b/(b-1)$-orientation in $\mathcal{O}_{-2b}$ (since all inner faces have degree in $\{2b-2, 2b\}$ there is no half-edge of negative weight). Right: the associated weighted bi-mobile.

Figure 5.9 shows an example that mixes the bijection from **[A6]** with the bijection for hexangulations of girth 6 (given in Section 5.1.1).

Similarly it should be possible to extend Proposition 53 (for annular bipartite maps) allowing for small faces of degree $2b-2$ and where the only allowed non-separating cycle of length smaller than $2b$ have length $2b-2$ and are contours of small faces. And it should be possible to extend the results of Section 5.1.4 (for general annular maps), this time allowing for small faces of degree $d-1$. The generating function expressions obtained by our method should match the expressions obtained recently in [**31**] using a different combinatorial strategy relying on so-called *slices* (certain portions of maps). We expect a bijective link between our mobiles and slices, which would establish a bridge between the two methods.

5.2.2. Mobile simplification when dropping the control on face-degrees. As a special case of the results of Section 5.1.4, for $d \geq 3$, plane maps of outer degree d and girth d are in bijection with weighted bi-mobiles where every white vertex has weight d, every edge has weight $d-2$, and every face of degree k has weight $-k+d$. The correspondence is quite strong, since it keeps track of the face-degree distribution of the map. We can use this correspondence and mobile enumeration to extract the number $a_n^{(d)}$ of plane maps of outer degree d, girth d, with n edges, and a distinguished (root) corner in the outer face. Surprisingly, for $d = 3$, we observe that the coefficient is very simple:

$$a_n^{(3)} = \frac{3 \cdot 2^{n-1}(2n)!}{(n+2)!n!}. \tag{35}$$

The right-hand side is also well-known to be the number of rooted Eulerian triangulations with $2n+2$ faces. And there is a combinatorial proof of the latter, based on a bijective correspondence between plane Eulerian triangulations with $2n+2$ faces, and unrooted binary trees with n nodes (vertices of degree 3), $n+2$ leaves, and $n-1$ inner edges (edges connecting two nodes) each of which is directed.

In **[C21]** we give a bijection between outer-triangular simple plane maps with n inner edges and plane Eulerian triangulations with $2n+2$ faces. The composition of this bijection with the above mentioned correspondence gives us a bijective proof of (35), and even more, the bijection keeps track of the number of inner faces of the outer-triangular simple plane map (it corresponds to the number of nodes with no incoming inner edge in the oriented binary tree). We lose here control on the face-degree distribution, which seems to be the price to pay in order to have a simplified encoding tree-structure [2].

We expect to have a similar procedure for any $d \geq 3$ using Eulerian d-angulations as intermediate structures, which should then yield simplified generating function expressions when keeping track only of the number of edges and the number of faces. We also plan to investigate if similar simplifications occur in the bipartite case.

5.3. Other results on maps with boundaries

5.3.1. Loopless triangulations with boundaries. We may apply the strategy of Section 4.2 to loopless triangulations with boundaries. We would thus consider, for each $a \geq 2$, the family $\mathcal{D}^{(a)}$ of loopless triangulations with boundaries, with a distinguished boundary face f_0 of degree a taken as the outer face; and consider the associated family $\overline{\mathcal{D}}^{(a)}$ of maps from $\mathcal{D}^{(a)}$ with a distinguished edge. For $M \in \overline{\mathcal{D}}^{(a)}$, we can open the marked edge into an internal face f_1 of degree 2 (taken as the marked inner face). Then there is an outermost 2-cycle c among the 2-cycles that enclose f_1 while not touched (at a vertex or an edge) by a boundary face in the enclosed area. Cutting along c yields two maps M_0, M_1, with M_i the part containing f_i for $i \in \{0, 1\}$, and where in M_i the face delimited by c is taken as the outer face, while f_i is taken as the marked inner face.

Then, similarly as in Section 4.2, one can show that M_0 (resp. M_1) can be endowed with a unique weighted bi-orientation in $\widehat{\mathcal{O}}_{-2}$ (resp. $\widehat{\mathcal{O}}_2$) such that every internal vertex has weight 2, every internal edge has weight 0, every internal face (of degree 3) has weight -1, every non-marked boundary of degree k has weight $k+2$, and in M_0 the marked boundary (of length a) has weight $a-2$. Note that the condition at internal faces implies that internal edges are either 0-way of weights $(0, 0)$, or 1-way of weights $(-1, 1)$.

The corresponding weighted bi-mobiles can be enumerated via generating functions. Defining $R \equiv R(t; x_0, x_2, x_3, \ldots)$ by

$$R = 1 + 2x_0 R^3 + 2\sum_{i \geq 2} x_i \binom{2i+1}{i} R^{2i+3},$$

we find that the generating functions for maps from $\overline{\mathcal{D}}^{(a)}$ with a marked corner in the outer face (and where x_0 is conjugate to the number of internal vertices and x_i is conjugate to the number of inner boundaries of length i) should be equal to $\binom{2a-3}{a-1}(R^2)^{a-2} \times R^2 = \binom{2a-3}{a-1} R^{2a-2}$.

However, as already mentioned in Section 4.2 (just after the proof of Lemma 40), a crucial part is missing, namely the necessity of being loopless to admit the above defined weighted bi-orientations. Hence, what we are counting is actually a superfamily of $\overline{\mathcal{D}}^{(a)}$; these maps can have loops but these loops should be quite constrained (it would be interesting to determine exactly how); for instance any loop ℓ in such a map must be touched (at its incident vertex) by a boundary face in the area inside ℓ. In the special case where there is a unique boundary face (the outer face), being loopless should thus be sufficient *and* necessary, and thus in that case we can exactly compute the generating function for maps in $\overline{\mathcal{D}}^{(a)}$ with

[2]Having a simpler encoding tree-structure, we have been able to show in **[C23]** that, for the random rooted simple map M_n with n edges, the random discrete measure given by the distances from the root of the n edges, rescaled by $(2n)^{1/4}$, converges in law to a random measure closely related to ISE, similarly as for random quadrangulations [**38**] and random maps [**69**]. In a work in progress [**2**], we expect to show the stronger result that, as a random discrete metric space rescaled by $(2n)^{1/4}$, the random rooted simple map with n edges converges to the Brownian map.

a distinguished corner in the outer face (and with x_0 conjugate to the number of internal vertices), as

$$\binom{2a-3}{a-1}R^{2a-2}, \quad \text{with } R = 1 + 2x_0R^3,$$

and from the Lagrange inversion formula we find that the number $a_{n,k}$ of loopless triangulations with a single boundary face of degree $k+2$, n internal vertices, and a marked corner in the boundary face satisfies

$$a_{n,k} = \frac{(2k+1)!}{k!^2}\frac{2^{n+1}(2k+3n)!}{n!(2k+2n+2)!},$$

recovering in a bijective way the formula originally found by Mullin [**73**] and mentioned in Section 1.2.1. The first bijective proof of this formula has been given in [**77**] (contrary to our method which produces two mobiles, the construction in [**77**] encodes the map by a single tree-structure).

5.3.2. Other results and obstacles for counting maps with boundaries. Ideally, similarly as in Section 5.1, we would like to extend the results obtained in Chapter 4 in order to have a bijection in each $d \geq 1$ for maps with boundaries, of girth at least d, the bijection keeping track of the degrees of the internal faces and the degrees of the boundary faces. However we see at the moment two main obstacles to achieve this goal:

- for M such a map, we can compute a certain (almost-) minimal regular boundary-α-orientation on a map $\sigma(M)$ derived from M, but then the nice local properties (Property 26 in Chapter 3, Properties 41 and 48 in Chapter 4) that made it possible to apply transfer rules in order to get a weighted bi-orientation in $\mathcal{O}$ might fail to hold when some internal faces have degree larger than $d+2$,
- even in the favorable case when the internal face degrees are in $\{d, d+1, d+2\}$, where we obtain from the transfer rules a weighted bi-orientation in $\mathcal{O}$, we have the problem (as already seen in Chapter 4 after proving Lemma 40, and above in Section 5.3.1) that, when there is more than one boundary and the girth constraint is non-trival (it is trivial for $d=1$, and for $d=2$ in the bipartite case), having girth at least d does not seem to be necessary to admit such a weighted bi-orientation, and we are actually counting a superfamily of the one we are interested in.

Given these obstacles, the enumeration of maps with more than one boundary is thus limited to maps with internal faces of degree at most 3, and to bipartite maps with internal faces of degree at most 4, which is essentially what we have done respectively in Section 4.3 and Section 4.2 (we did 'exactly' instead of 'at most', but the two problems are essentially the same, since for instance allowing for internal faces of degree 2 in the bipartite case just comes down to opening some of the edges into such faces).

If there is a unique boundary, the enumeration should be limited, in each $d \geq 1$, to the case where the internal faces have degree in $\{d, d+1, d+2\}$. For $d=2$ (loopless maps) this would thus slightly extend the result at the end of Section 5.3.1, by allowing for internal faces that are either triangular or quadrangular. For $d \geq 3$, we have already given in [**A21**] a bijection when all internal faces have degree d (d-angulations with a boundary), and thus it should be possible to extend it by allowing for internal faces of degrees in $\{d, d+1, d+2\}$.

Finally, in the case of one boundary, it should also be possible to deal with the irreducibility condition (as seen in Section 5.2.1), that is, for each $d \geq 3$, to have a bijective encoding of maps with one boundary, that are d-irreducible, and with internal face degrees in $\{d, d+1, d+2\}$. In particular we should be able to obtain bijective proofs of the counting formulas (respectively obtained in [**87**] and [**74**]) for irreducible triangulations with a boundary and irreducible quadrangulations with a boundary.

5.4. Distance properties of the master bijection

We now give a reformulation of the master bijection Ψ (from mobiles to oriented maps, as presented in Section 2.2), which relies on a canonical labelling of the corners at the white vertices of the mobile. The advantage of this alternative formulation is that it is well-suited to a control on the distances of the associated (oriented) map. For the sake of conciseness, we only discuss here the case $\delta < 0$ (i.e., the mobile has more buds than edges), the other cases ($\delta = 0$ and $\delta > 0$) having similar reformulations. In a collaboration with M. Albenque, G. Collet and O. Bernardi [**2**] (article in preparation) we give a similar labelled reformulation for the bijection mentioned in Section 5.2.2 (between oriented binary trees and outer-triangular simple maps) and exploit it to show the convergence of the random rooted simple maps with n edges to the Brownian map.

5.4.1. A labelled reformulation of Ψ. Let $\delta < 0$ and let T be a mobile of excess δ. Recall from Section 2.2 that in its first formulation, the mapping Ψ proceeds with the following steps:

(1) a bid (ingoing half-edge) is inserted at each 'down-corner' of T (corner just after an edge in ccw order around a black vertex),
(2) the buds are matched with the bids according to a ccw-walk around T where buds are considered as opening parentheses and bids as closing parentheses, and a directed edge e, called a *closure-edge*, is created out of each matched pair,
(3) the $|\delta|$ unmatched buds are extended into edges reaching to a new vertex v_∞ inserted in the outer face.

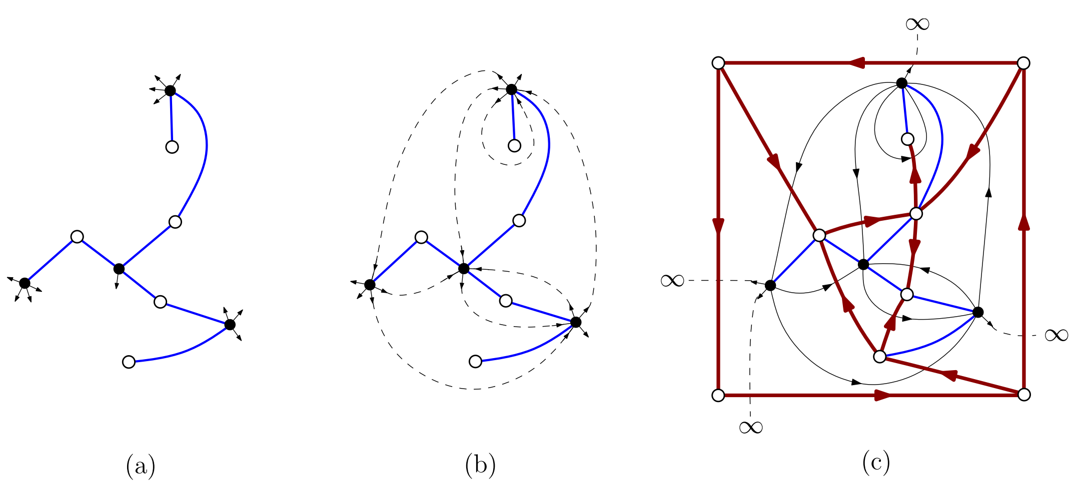

FIGURE 5.10. (a) A mobile T of excess $\delta = -4$ (b) The matching of bids with buds (following a ccw-walk around T with buds as opening parentheses and bids as closing parentheses). (c) The map $\Lambda(T)$, superimposed with the oriented map $O = \Psi(T)$ (which is also the dual of the oriented map obtained from $\Lambda(T)$ by erasing the white vertices and edges of T).

Let $\Lambda(T)$ be the (vertex-pointed oriented) map obtained after these steps, and let $\lambda(T)$ be the map obtained from $\Lambda(T)$ after erasing the white vertices and the edges of T; recall that $O = \Psi(T)$ is obtained as the dual of $\lambda(T)$. In $\Lambda(T)$, the $|\delta|$ faces incident to v_∞ are called *exterior faces* and the other faces are called *interior faces*. It is easy to check (for instance performing the matchings bud/bid step by step, from innermost to outermost) that for every interior face f there is a unique closure-edge e_f with f on its left and there is a unique white corner (corner at a white vertex) incident to f; this corner denoted by c_f is just after the corner at the end-vertex of e_f in ccw order around f; the white vertex at c_f is denoted by w_f. The *parent-face* of f is the face on the right of e_f.

Let $\Lambda'(T)$ be obtained from $\Lambda(T)$ by inserting in each exterior face f a white vertex denoted by w_f. Then it is easy to see that the edges of $O = \Psi(T)$ are recovered from $\Lambda'(T)$ as follows (see Figure 5.10): each closure-edge e, with f (resp. f') the face on the left (resp. right) of e, yields an inner edge e^* of O, which goes from w_f to $w_{f'}$. And moreover the $|\delta|$ white vertices in the respective exterior faces have to be connected together by an outer $|\delta|$-gon, which forms the outer face of O.

Define now the *depth-labelling* of $\Lambda(T)$ as the unique labelling of the faces of $\Lambda(T)$ by labels in $\mathbb{N}$ such that the $|\delta|$ outer faces have label 0, and the label of every inner face is one more than the label of its parent-face, see Figure 5.11(a) for an example. The *canonical labelling* of T is then defined as the induced labelling of the white corners (corners at white vertices) of T. Alternatively this labelling can be computed directly on T as follows (see Figure 5.11(b)): starting from an exposed bud, initialize the current label value s to be 0, then walking counter-clockwise around T:

- each time a non-exposed bud is crossed, s is increased by 1,
- each time an edge is traversed from the white to the black extremity, s is decreased by 1,
- each white corner is assigned the current value of s at the time it is visited.

Let e be a closure-edge of $\Lambda(T)$, call f the face on the left (always an interior face) and f' the face on the right, and let ℓ be the label of c_f. Assume that f' is also an interior face (which is equivalent to c_f having label greater than 1). Then it is not difficult to see that $c_{f'}$ is characterized as the *first corner* of label $\ell - 1$ after c_f in a counter-clockwise walk around T; $c_{f'}$ is called the *successor* of c_f. From this observation we get the following labelled reformulation of the master bijection Ψ (case $\delta < 0$):

(1) determine the $|\delta|$ exposed buds, and draw an infinite ray starting from each of these buds so as to split the outer face f_0 into $|\delta|$ outer faces $f_1, \ldots, f_{|\delta|}$; then insert an isolated white vertex v_i in each outer face f_i for $i \in [1..|\delta|]$,
(2) endow T with its canonical labelling of the white corners, as specified above and shown in Figure 5.11(b),
(3) for each white corner $c \in T$ of label $\ell \geq 2$, connect c to its successor c' (the next white corner of label $\ell - 1$ after c in a counter-clockwise walk around T), the new edge being directed from c' to c,
(4) for each $i \in [1..|\delta|]$ and for each white corner $c \in T$ of label 1 and incident to the outer face f_i, connect c to the white vertex v_i, the new edge being directed from v_i to c,
(5) connect the $|\delta|$ outer vertices $v_1, \ldots, v_{|\delta|}$ by an outer $|\delta|$-gon that is oriented ccw.

This reformulation has a similar flavor as the Schaeffer bijection [**82**, Ch.6] from well-labelled trees to vertex-pointed quadrangulations, and thus we can expect to obtain useful estimates/bounds for the distances between vertices in terms of the labels of the corresponding mobile.

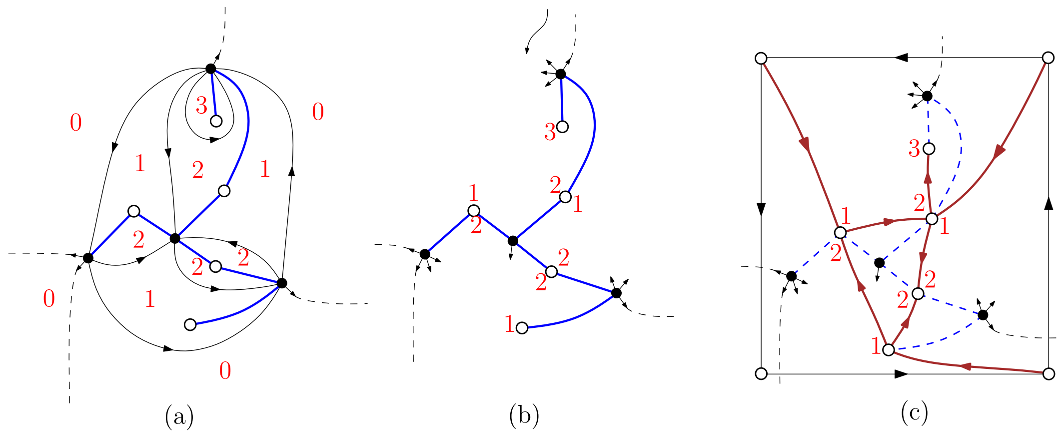

FIGURE 5.11. (a) The depth-labelling of the map $\Lambda(T)$ obtained in Figure 5.10. (b) The induced labelling of the white corners of T, called the canonical labelling of T, which can be alternatively obtained from a ccw walk around T starting at an exposed bud. (c) The closure $O = \Psi(T)$ can be obtained from T endowed with its canonical labelling.

5.4.2. Interpretation of the labels and bounds on the distances. Let $\delta < 0$ and let $O \in \mathcal{O}_\delta$. As shown in [**11**], for each inner edge $e \in O$, there is a unique directed path P_e of inner edges that starts at some outer vertex, ends at e, and such that no edge arrives to P_e from the left-side, see Figure 5.12. The path P_e is called the *rightmost path* of e, and its length is denoted by $L(e)$.

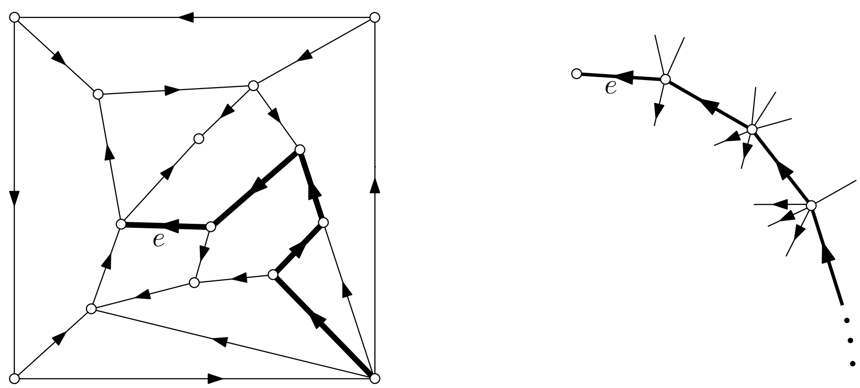

FIGURE 5.12. Left: an orientation $O \in \mathcal{O}_{-4}$, with a distinguished inner edge e, the rightmost path of e being shown bolder. Right: generic situation for the rightmost path of an inner edge e.

In the labelled reformulation of Ψ given above, it is easy to see that each white corner c of T corresponds to an inner edge e_c of $O = \Psi(T)$, and moreover, with $k \geq 1$ the label of c, the backward path in O starting from c and jumping at each step to the next successor until reaching an outer vertex is the rightmost path of e. Since this path decreases by 1 in label at each step until reaching an outer vertex (of label 0), we conclude that the label $\ell(c)$

of c is equal to $L(e_c)$. In particular, if we denote by $d(v)$ the distance of any inner vertex v from the outer contour of O (length of a shortest path, not necessarily directed, connecting v to some outer vertex), and denote by v_c the inner vertex of O corresponding to the white vertex of T at c (v_c is also the end-vertex of e_c) then $d(v_c) \leq \ell(c)$. For each inner edge $e \in O$, denote by $d(e)$ the distance of the end-vertex of e to the outer contour of O, and denote by $K(O)$ the maximum of $L(e) - d(e)$ over all inner edges e of O. Then we have, for any white corner c of T,

$$|d(v_c) - \ell(c)| \leq K(O), \tag{36}$$

which is to be read as $\ell(c)$ providing an estimate (up to $K(O)$) of the distance from v_c to the outer face contour.

FIGURE 5.13. Illustration that the rightmost paths starting at respective white corners c, c' meet at label $m(c, c') - 1$, when $m(c, c') \geq 2$ (i.e., when c, c' are incident to the same outer face f_i, for some $i \in [1..|d|]$).

Next we consider the distance in O, denoted $d(v, v')$, between two arbitrary (inner) vertices v, v', i.e., the length of a shortest (not necessarily directed) path connecting v to v'. For c, c' two white corners of T, we have seen that there is (in O) a path of length $\ell(c)$ (resp. $\ell(c')$) from v_c (resp. $v_{c'}$) to the outer contour. Since any two outer vertices are at distance at most $\lfloor |\delta|/2 \rfloor$ we have the easy bound

$$d(v_c, v_{c'}) \leq \ell(c) + \ell(c') + \lfloor |\delta|/2 \rfloor.$$

Similarly as in the Schaeffer bijection [**82**, Ch.6] for quadrangulations (see e.g. [**53**]), we can get a better bound, expressed in terms of the quantity $m(c, c')$ defined as the minimal label of all white corners that are between c and c' in a ccw walk around T. If $m(c, c') \geq 2$, then as shown in Figure 5.13 the rightmost path of e_c meets the rightmost path of $e_{c'}$ at the corner of label $m(c, c') - 1$, so that there is a path in O connecting v_c to $v_{c'}$ and of length $\ell(c) - (m(c, c') - 1) + \ell(c') - (m(c, c') - 1)$. Hence, for $m(c, c') \geq 2$ we have

$$d(v_c, v_{c'}) \leq \ell(c) + \ell(c') - 2\, m(c, c') + 2,$$

while for $m(c, c') = 1$ we can use the upper bound given above. We can combine both upper bounds, and symmetrize in c, c', to get

$$d(v_c, v_{c'}) \leq \ell(c) + \ell(c') - 2 \max(m(c, c'), m(c', c)) + \lfloor |\delta|/2 \rfloor + 2. \tag{37}$$

Let us finally mention that the extended master bijection for (possibly weighted) bi-orientations presented in Section 2.4 can also be given a labelled reformulation, since this extension can be seen as a mere adaptation of the master bijection for orientations (as shown in Figure 2.9).

5.4.3. Possible applications. Let Q_n denote a uniformly random rooted quadrangulation, and denote by $d(.)$ the distance in Q_n, and for $c > 0$ denote by $d_c(.)$ the distance rescaled by c, i.e., $d_c(v, v') := d(v, v')/c$. Note that for any $u_n > 0$, $(Q_n, d_{u_n}(.))$ forms a sequence of random discrete metric spaces. A seminal result obtained recently independently in [**63**] and [**67**] is that for $u_n = (8n/9)^{1/4}$, $(Q_n, d_{u_n}(.))$ converges in law (for the so-called *Gromov-Hausdorff* topology) as $n \to \infty$ to a random metric space called the *Brownian map*, which has almost surely the topology of the sphere [**53, 66, 54**] (it was already known [**38, 62**] that the random discrete *measure* given by the distances from the root-vertex, renormalized by $(8n/9)^{1/4}$, converges to an explicit random discrete measure on $\mathbb{R}_+$ that is closely related to the ISE law [**4**]).

A key ingredient to prove this result is the Schaeffer bijection [**82**, Ch.6] (reformulating a construction of Cori and Vauquelin [**42**], and generalized to higher genus in [**36**]), which also corresponds to the BDG bijection [**26**] (reviewed in Section 2.3) specialized to quadrangulations. This bijection maps a so-called *well-labelled tree* T (i.e., an unrooted plane tree with a positive label $\ell(v)$ at each vertex v such that the minimum label over all vertices is 1 and $|\ell(v) - \ell(u)| \leq 1$ for any two adjacent vertices u, v) with n edges to a vertex-pointed quadrangulation Q with n faces. And a crucial point is that, with $d(.)$ the distance in Q and v_0 the pointed vertex of Q, one has

$$d(v, v_0) = \ell(v). \tag{38}$$

Moreover, for two vertices $v, v' \in T$, denote by $m(v, v')$ the smallest label seen in a ccw walk around T starting at v, ending at v', and never seeing v or v' in-between. Then

$$d(v, v') \leq \ell(v) + \ell(v') - 2\max(m(v, v'), m(v', v)) + 2. \tag{39}$$

The two bounds (36) and (37) we have obtained in Section 5.4.2 can thus be seen as respective analogues of (38) and (39), up to the additional term $K(O)$ in (36) and the additional term $\lfloor|\delta|/2\rfloor$ in (37). We thus expect that the master bijection could make it possible to prove convergence to the Brownian map for random maps in families specified by girth and face-degree constraints (bijections in Chapter 3 and its generalizations in Section 5.1), as in the following conjecture:

Conjecture 61. *Let $d \geq 1$, let Δ be a finite set of integers in $\{d, d+1, \ldots\}$, and let $\mathcal{M}_{d,\Delta}(n)$ be the set of rooted maps of girth at least d, with n faces all of degree in Δ. And let $M_{d,\Delta}(n)$ be a uniformly random map in $\mathcal{M}_{d,\Delta}(n)$. Then there is a computable positive constant $c \equiv c_{d,\Delta}$ such that $(M_{d,\Delta}(n), d_{cn^{1/4}}(.))$ converges in law to the Brownian map.*

Let us briefly discuss here the case of rooted simple quadrangulations, i.e., the case $\Delta = \{4\}$ and $d = 4$. We can first rely on a crucial lemma in the recent article [**1**] (which shows Conjecture 61 for the the case of simple triangulations and simple quadrangulations), where it is proved that, if Q_n denotes a uniformly random simple quadrangulation with n faces endowed with its unique minimal 2-orientation, then $K(Q_n)/n^{1/4}$ converges in law to 0. Proving this relies on a key deterministic result established in [**1**, Prop.7.12] (which hopefully could be extended to the general setting of Conjecture 61): "for Q a rooted simple quadrangulation with n faces, $K(Q) \geq \epsilon n^{1/4}$ implies that Q has a cycle Γ of length $O(\mathrm{Diam}(Q)/(\epsilon n^{1/4}))$ such that both components obtained after cutting along Γ are 'large' (in the sense that they have diameter $\Omega(\epsilon n^{1/4})$)", and as shown in [**1**, Theo.8.1] (based on contour processes, see also Lemma 15 in **[C23]** for a line of proof based on analytic arguments), the latter event is typically unlikely for random maps.

Since $K(Q_n)/n^{1/4}$ converges in law to 0, (36) can be seen as analogous to (38) after renormalizing distances by $n^{1/4}$. More easily (37) can be seen as analogous to (38), again after renormalizing by $n^{1/4}$ (which makes the contribution $\lfloor |\delta|/2 \rfloor$ negligible). The recent article [**63**] has extracted a list of conditions to be checked that guarantee convergence to the Brownian map (after rescaling distances by $cn^{1/4}$ for a certain $c > 0$):

- (i): having analogous of (38) and (39),
- (ii): the pair of contour-processes (around the labelled tree) giving respectively the depth and the label converges in law to the so-called *Brownian snake* [**61**],
- (iii): a certain condition of invariance under re-rooting (essentially, the distance from the root to a random vertex has to have the same limit as the distance between two random vertices).

For simple quadrangulations, we have seen above that (i) holds, and (iii) holds (essentially due to the fact that the root can be placed anywhere in the quadrangulation). And since the encoding mobiles (ternary trees as we have seen in Section 3.1) have bounded arity, one could use the results in [**65**] to guarantee that (ii) holds. Therefore it should be possible to use mobiles in order to show convergence of rooted simple quadrangulations to the Brownian map. The convergence result is shown in [**1**] using a bijection with blossoming trees given in **[T1]** (itself closely related to the bijection by Poulalhon and Schaeffer [**78**] for simple triangulations). The advantage we see in using mobiles instead of blossoming trees in order to attack Conjecture 61 is to have the general bound (37) and also to enable the bijective encoding of maps of girth at least $d \geq 1$, with faces of arbitrary degree (whereas for $d \geq 3$, bijections with blossoming trees are up to now limited to the case of d-angulations of girth d, possibly with a boundary [**3**]).

The main obstacle we see to show Conjecture 61 is the lack (up to now) of a general result that would guarantee (ii) for families of (canonically labelled) trees of bounded arity and that are not simply generated (i.e., require a generation grammar of more than line, such as (32) when $b \geq 3$).

Let us finally mention another possible line of research, namely the convergence of (suitably rescaled) random rooted d-angulations of girth at least e with n faces, where d, e depend on n, say for instance $d = \lfloor n^{\beta} \rfloor$ and $e = \lfloor n^{\alpha} \rfloor$ with $0 < \alpha \leq \beta$. Limiting behaviours different from the Brownian map might be obtained depending on α, β, possibly related to those discovered in [**64**] (as a first step it would already be interesting to determine the asymptotic order of typical distances).

5.5. Other results

In this section I describe (without details) some other research collaborations I have had in recent years.

5.5.1. The 2-point and 3-point function of planar maps. To study the typical distances in random maps, an alternative to the probabilistic method mentioned in Section 5.4 is to try to solve for the so-called 2-*point function* of a family $\mathcal{M} = \bigcup_n \mathcal{M}_n$ of maps. The 2-point function $G_i(x)$ of $\mathcal{M}$ is defined as the generating function of maps in $\mathcal{M}$ with two marked vertices at mutual distance i (depending on the context, $G_i(x)$ can also be defined as the generating function of rooted maps in $\mathcal{M}$ with a marked vertex at distance i from the root-edge). The 2-point function thus gives access to the distribution of the mutual distance X_n between two vertices in a random bi-pointed map from $\mathcal{M}_n$. Remarkably, the 2-point function of several families of rooted maps admits an exact expression, making it possible to show that $X_n/n^{1/4}$ admits an explicit limit law. To prove these exact expressions a variety of insightful tools have been developed, based on guessing/checking [**25, 52**], conserved quantities [**29**], and continued fraction theory [**30**].

The starting point is to use the Cori-Vauquelin Schaeffer bijection for quadrangulations [**42, 82**] and more generally the BDG bijection for maps [**26**] with control on the face-degrees. Indeed in these bijections (reviewed in Section 2.3 for the bipartite case) the map is naturally vertex-pointed, and in the encoding labelled mobile each vertex is labelled by the distance from the pointed vertex. Hence the bijection ensures that the *cumulative* 2-point function $R_i(x) := \sum_{j\leq i} G_j(x)$ can be seen as a generating function of rooted labelled mobiles with positive labels and label i at the root-vertex, and based on this one can (using a decomposition at the root) specify a recurrence for the $R_i(x)$ (for quadrangulations it reads $R_i(x) = 1+xR_i(x)\cdot(R_{i-1}(x)+R_i(x)+R_{i+1}(x)))$ to be solved; or one can express the $R_i(x)$ as the coefficients of the continuous fraction expansion of known map generating functions [**30**]. This latter method has made it possible to obtain very precise results, namely to express the 2-point function of bipartite maps (and more generally, maps) with bounded face-degrees and a weight g_k for each face of degree $2k$. In a recent joint work with E. Guitter [**S2**], we have extended the expressions for bipartite maps to incorporate additionally a weight $t_\bullet$ for each black vertex and a weight $t_\circ$ for each white vertex in the bipartite maps.

An important recent result regarding the 2-point function is the new bijection by Ambjørn and Budd [**5**] between vertex-pointed quadrangulations and vertex-pointed arbitrary maps that preserves the profile of distances from the pointed vertex (there already existed a classical bijection due to Tutte between both families, but it did not preserve the distance-profile). This makes it possible to solve for the 2-point function of arbitrary maps (with no bound on the face-degrees), which is very closely related to the 2-point function of quadrangulations since they are both encoded by well-labelled trees; moreover assigning a fixed Boltzmann weight $w > 0$ to each face amounts to assigning a weight w to each local max in the well-labelled trees, and remarkably an exact (bivariate) expression also holds, as proved in [**5**] using a guessing/checking approach.

In a recent work with J. Bouttier and E. Guitter [**A30**], we have extended this correspondence to a bijection between vertex-pointed bipartite maps and vertex-pointed arbitrary hypermaps, so as to preserve the profile of distances from the marked vertex, and such that each hyperedge of degree k corresponds to a face of degree $2k$ in the associated bipartite map. Similarly, this ensures that the 2-point function of hypermaps and the 2-point function of bipartite maps are very closely related, being both encoded by labelled mobiles. And assigning a Boltzmann weight w to each face of the hypermap amounts to assigning a weight w to each local max (under a certain sense) in the encoding labelled mobile. With E. Guitter we have recently obtained partial results to compute the 2-point function of hypermaps with the additional weight w at faces [**S3**]; the cumulative 2-point functions $R_i(x, w)$ can be seen as the coefficients of a certain continuous fraction decomposition of a known map generating function. However this continuous fraction is of a different type and the coefficients can not be extracted in a determined way as in [**30**]; with some additional natural assumptions we have been able to carry out the calculations in a constructive way and recover in the case of maps (related to labelled well-labelled trees) the bivariate expressions obtained in [**5**]. It would be interesting to extend our approach to general hypermaps and prove (possibly under mild assumptions) an explicit expression for the 2-point function of hypermaps with a weight w at each face and a weight g_k for each hyperedge of degree k.

More generally, for any $k \geq 2$ one can consider the k-point function of a family $\mathcal{M} = \bigcup_n \mathcal{M}_n$ of maps, that is, the generating functions for maps in $\mathcal{M}$ with k marked vertices and prescribed mutual distances (there are $\binom{k}{2}$ distances to be prescribed, one for each pair of marked vertices). By a generalization of the Cori-Vauquelin-Schaeffer bijection due to Miermont [**70**], labelled quadrangulations with k marked vertices, each having a certain delay assignment, are in bijection with well-labelled maps with k faces. Building on this bijection, Bouttier and Guitter have derived an explicit expression for the 3-point function of quadrangulations [**28**]. Similarly the Ambjørn-Budd bijection yields a correspondence between well-labelled arbitrary maps with k local min (to be taken as marked vertices) and well-labelled maps with k faces. In a collaboration with E. Guitter [**A31**], we derive from it the 3-point function of arbitrary maps, both in the univariate case (with respect to the number of edges) and in the bivariate case (with a Boltzmann weight w at each face). This allows us to compute the limit joint-law of the 3 pairwise distances in random tri-pointed maps with n edges and a fixed Boltzmann weight $w > 0$ at faces, and we show that the limit law (when rescaling by $n^{1/4}$) is the same as for quadrangulations, up to a further rescaling by an explicit constant factor depending on w. By a correspondence between the face-weight w and the edge-density of the random maps, this also yields, for each $\lambda \in (0, 1)$ the limit law for tri-pointed maps with n edges and $\lambda \cdot n$ faces. We also conjecture (without a rigorous proof, which would possibly require a technical saddle-point calculation) the scaling order when λ is close to 0 (say $\lambda \cdot n \simeq n^\alpha$ for $\alpha \in (0, 1)$), corresponding to maps with few faces, or close to 1 (say $n \cdot (1 - \lambda) \simeq n^\alpha$ for $\alpha \in (0, 1)$), corresponding to maps with few vertices.

5.5.2. Bijections for unicellular maps in higher genus. A unicellular map of genus g is a (rooted, i.e., with a marked corner) map with a unique face on the orientable surface of genus g (in genus 0 it corresponds to a plane tree). Equivalently, a unicellular map with n edges is obtained by gluing pairwise the sides of a $2n$-gon, and the genus of the unicellular map is the genus of the obtained surface. Unicellular maps have connections with algebra (they can be seen as factorizations of a long cycle in the symmetric group), probabilities (they encode the nth moments of gaussian ensembles) and topology (seen as gluings of a polygon). As discovered and proved by Harer and Zagier [**58**], the coefficients $\epsilon_g(n)$ giving the number of unicellular maps of genus g with n edges satisfy the following summation formula (proved in [**58**] using matrix integral techniques and a clever polynomiality argument):

$$\sum \epsilon_g(n) N^{n+1-2g} = (2n-1)!! \sum_{r \geq 1} 2^{r-1} \binom{n}{r-1} \binom{N}{r},$$

which yields, using some algebraic manipulations, the very simple recurrence

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (n-1)(2n-1)(2n-3)\epsilon_{g-1}(n-2).$$

Bijective proofs (relying on the BEST theorem) of the summation formula have been given subsequently [**56, 12**], but without a simple combinatorial interpretation of the recurrence.

In a collaboration with G. Chapuy and V. Feray [**A25**] (building on a recursive decomposition of unicellular maps discovered by G. Chapuy [**37**]), we have introduced a new bijection for unicellular maps where for the first time the genus is readily read on the encoding structure, yielding direct combinatorial proofs of all known formulas for unicellular maps (including the Harer-Zagier recurrence, and the very precise Goupil-Schaeffer formula [**57**] that keeps track of the vertex degree distribution). In our bijection a unicellular map M with n edges is simply encoded by a pair made of a (rooted) plane tree T with n edges and a permutation σ of size $n + 1$ with all cycles of odd length, and the genus corresponds to

$\frac{1}{2}(n+1-\#\text{cycles}(\sigma))$. Our bijection has for instance been recently applied to characterize the local limit of random unicellular maps of linear genus [**6**].

In this topic several interesting questions are open: for example up to now we have only partially succeeded to generalize our bijection to so-called unicellular constellations (which generalize bipartite unicellular maps) and we have not succeeded to find a bijective proof of the Poulalhon-Schaeffer formula [**79**] (which extends the Goupil-Schaeffer formula). We would also like to find a similar bijection for locally orientable unicellular maps, for which (more involved) counting formulas exist; up to now a bijection giving a new summation formula has been introduced in [**12**], and the recursive decomposition in [**15**] has been adapted to the locally orientable case when vertices have degree in $\{1,3\}$.

5.5.3. Baxter families and pattern-avoiding permutations. Baxter permutations are permutations avoiding the patterns $3-14-2$ and $2-41-3$. They can also be defined as the permutations obtained by iteratively inserting the highest entry just before a left-to-right maximum or just after a right-to-left maximum in the permutation already built, and as such they admit a simple generating tree that can be seen as a natural bivariate analogue of the generating tree for Catalan structures (where the parent of a parenthesis word on $\{a,b\}$ is obtained by deleting the last factor ab). As for Catalan numbers, the so-called Baxter numbers, defined as

$$B_n = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{r=0}^{n-1} \binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2},$$

occur recurrently in combinatorics, counting for instance certain 3-line Young tableaux [**39**], plane bipolar orientations, 2-orientations of quadrangulations, twin pairs of binary trees [**48**], and non-intersecting triples of directed paths [**A10,A11,A16**]. Baxter permutations show a strong link with planar maps: as shown in a collaboration with N. Bonichon and M. Bousquet-Mélou [**A11**], there is a direct (of a much geometric flavour) bijection between plane bipolar orientations and Baxter permutations; when specializing this correspondence to Baxter permutations avoiding $2-4-1-3$ we recover (under a more geometric formulation) a bijection due to Dulucq, Gire, and West [**47**] with rooted non-separable planar maps.

I have again encountered Baxter structures in a recent collaboration [**S4**] with S. Burrill, J. Courtiel, S. Melczer, and M. Mishna, in the context of certain walks on the Young lattice ending in a row-shape (these can also be seen as certain walks in the 2D Weyl chamber $\{0 \leq x \leq y\}$, ending on the x-axis). Intriguingly these walks are counted by the Baxter numbers but the natural generating tree they have is different from the usual one for Baxter structures; we have not yet managed to find a bijective proof for their enumeration, even if a bijective connection is suggested by the equidistribution (for which we have up to now only strong computational evidence) between the ending abscissa and a natural "switch-parameter" on the classical Baxter family of non-intersecting triples of lattice paths.

In a more probabilistic direction, an interesting research problem is to determine a convergence in law of pattern-avoiding permutations from a given class (represented as a set of points in the square $[0,1]\times[0,1]$), such as Baxter permutations (where one can exploit the rich combinatorics of this specific class). Such convergence results have up to now been established for simpler families of permutations (avoided patterns of length 3) in [**76**], where the points tend to accumulate in the vicinity of a curve, which shouldn't be the case for families such as Baxter permutations, where the permutations points distribution seems to obey a more chaotic behaviour, similarly as for planar triangulations embedded using standard graph drawing algorithms.

5.5.4. Intervals in Tamari lattices. In recent years there has been a growing interest in the combinatorics of intervals of the Tamari lattice (lattice formed by binary trees with n nodes, whose covering relation is given by the elementary rotation operation on binary trees), starting with the work of Chapoton [**33**] who showed that the number of intervals of the Tamari lattice (of size n) is given by:

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1}.$$

The notion of Tamari lattice can be generalized to any $m \geq 1$ as a lattice on $(m+1)$-ary trees with n nodes (for $m > 1$ the covering relation is better seen on the encoding m-Dyck paths). In a recent joint work with M. Bousquet-Mélou and L.-F. Préville-Ratelle [**A22**], we have shown that the number of intervals (conjecturally giving the dimension of certain coinvariant spaces [**10, 33**]) in the so-called m-Tamari lattice is more generally given by the formula:

$$\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}.$$

The first step of the proof is to show that the associated generating function (with a secondary catalytic variable) satisfies an explicit equation. This equation can be solved for any fixed (small) value of m using a generalization [**19**] of Tutte's quadratic method to general polynomial equations with a catalytic variable, giving the above formula. However to have a proof that works uniformly over all m, the only way we have found is by a guessing/checking approach, where the guessed expression of the (bivariate) generating function is an explicit rational expression after a suitable change of variable (both for the main variable and for the catalytic variable).

It is tempting to search for bijective proofs of these formulas. Up to now there is a bijective proof, by Bonichon and Bernardi [**13**], only in the case $m = 1$, and it relies on simple triangulations. Models of planar maps that would give fruitful intermediate objects for $m \geq 2$ are still to be found. Let us finally mention that there are also beautiful factorized counting formulas for Tamari intervals endowed with certain labellings [**18**]: again it would be interesting to find bijective proofs of such formulas (here it is open in all cases, including $m = 1$), possibly with models of (labelled) maps as intermediate objects.

5.6. Publication list

The pdf files are available at `http://www.lix.polytechnique.fr/Labo/Eric.Fusy/`

Submitted

[S5]. "Bijections for maps with boundaries: Krikun's formula for triangulations, and a quadrangulation analogue" (with Olivier Bernardi), arXiv:1510.05194.

[S4]. "Tableau sequences, open diagrams, and Baxter families" (with Sophie Burrill, Julien Courtiel, Stephen Melczer, and Marni Mishna), arXiv:1506.03544.

[S3]. "Comparing two statistical ensembles of quadrangulations: a continued fraction approach" with Emmanuel Guitter), arXiv:1507.04538.

[S2]. "The two-point function of bicolored planar maps" (with Emmanuel Guitter), arXiv:1411.4406.

[S1]. "Unified bijections for planar hypermaps with general cycle-length constraints" (with Olivier Bernardi), arXiv:1403.5371.

<u>*Published in journals*</u>

[A32]. "Asymptotic expansion of the multi-orientable random tensor model" (with Adrian Tanasa). Electr. J. Comb. 22(1): P1.52, 2015.

[A31]. "The three-point function of general planar maps" (with Emmanuel Guitter). *J. Stat. Mech.* P09012, 2014.

[A30]. "On the two-point function of general planar maps and hypermaps" (with Jérémie Bouttier and Emmanuel Guitter). *Annales de l'institut Henri Poincaré D*, Volume 1, Issue 3, pages 265-306, 2014.

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