Counting Signatures of Monic Polynomials

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following a work of Norbert A'Campo

Contents

Configurations of Monic Polynomials

Consider your favorite monic polynomial P and draw:

1. its real antecedents, i.e. $\{z \in \mathbb{C} : P(z) \in \mathbb{R}\}$
2. its imaginary antecedents, i.e. $\{z \in \mathbb{C} : P(z) \in i\mathbb{R}\}$

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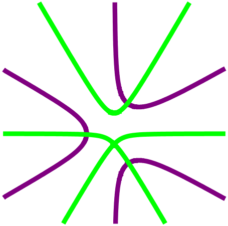
$$P(X) = X^3 + 2iX^2 - X + 1$$



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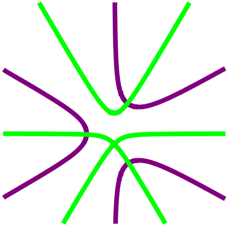

1. Roots of P (with multiplicity)
2. Roots of P' lying on the curves
3. Asymptotic rays

From Configurations to Signatures

Which configurations are isotopic to each other?

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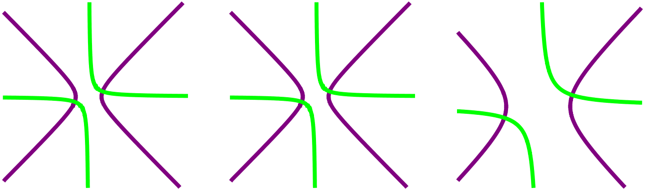

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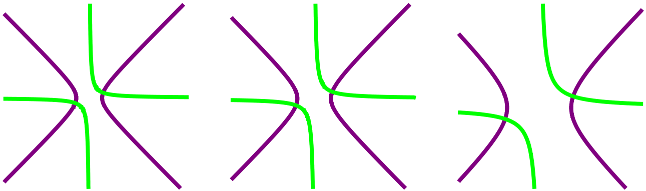

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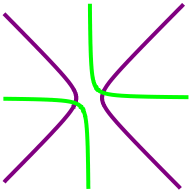

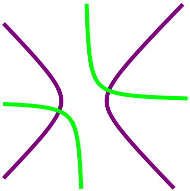

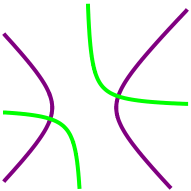

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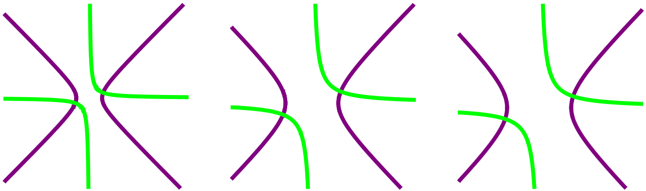

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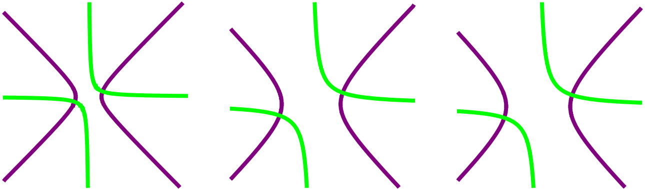

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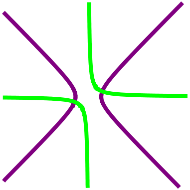

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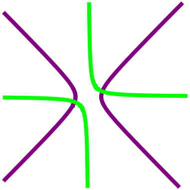

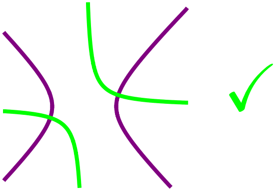

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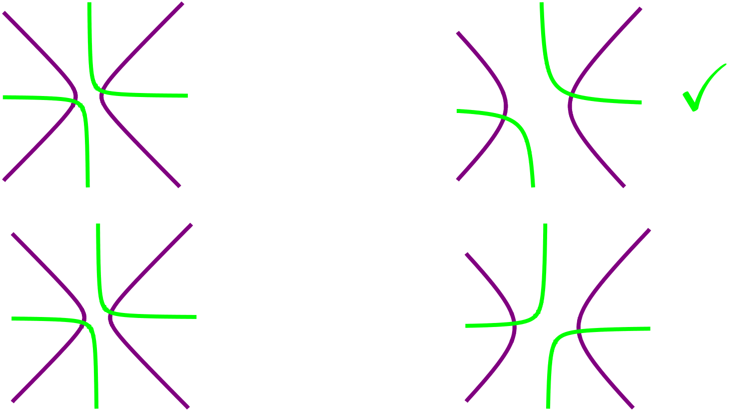

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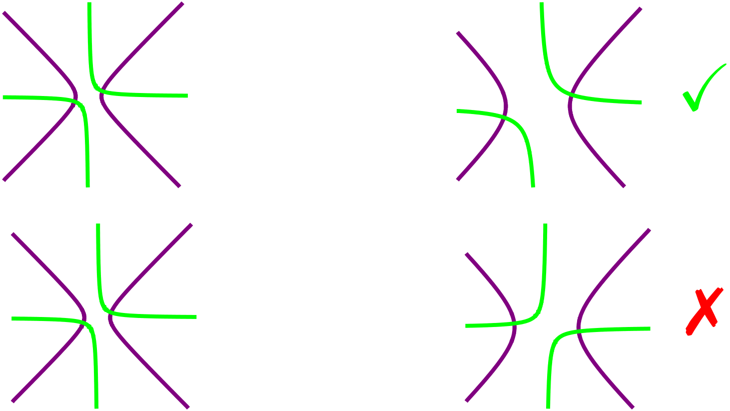

From Configurations to Signatures (a.k.a. Isotopy Classes)

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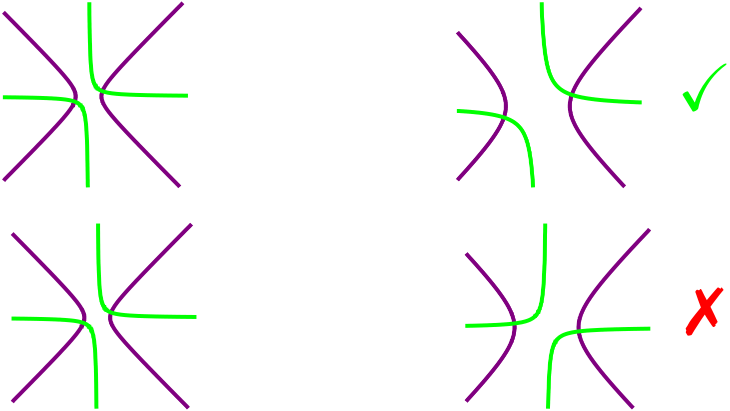

Signatures are used to compute cohomologies of braid groups

An Abstract View of Signatures

Four necessary criteria for being a signature of degree d...

1. No cycle — complex analysis and meromorphic functions
2. $2d$ bicolored edges — alternating and starting from **even** edges
3. 1-colored contact points — with even valency $\equiv 0 \pmod 2$
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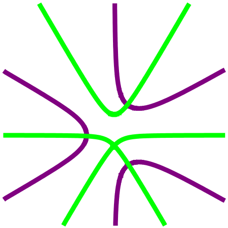

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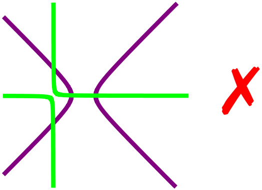

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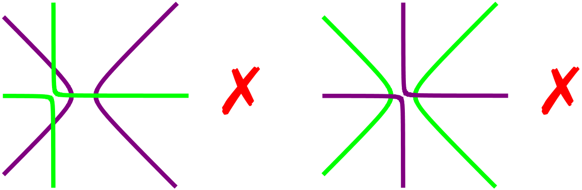

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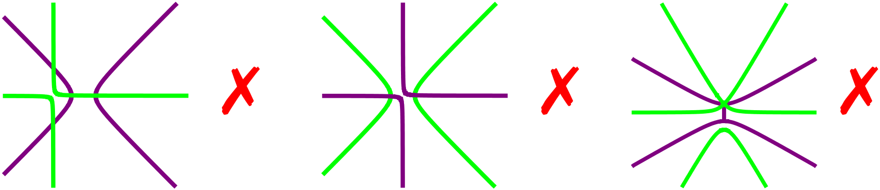

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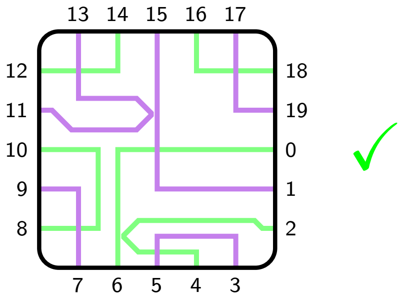

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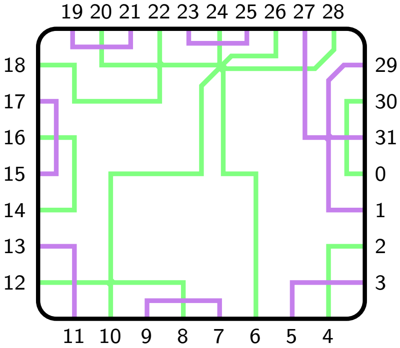
How many faces does the complex have?

Contents

Counting Which Signatures?

Three parameters of interest



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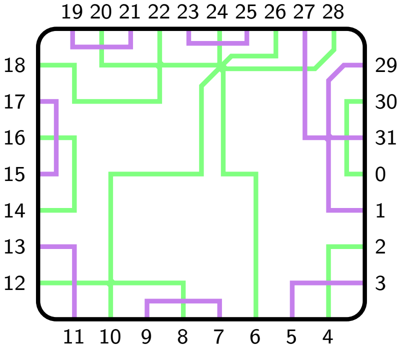
1. Degree of the polynomial $d = \frac{1}{2}\#\text{edges}$



$d = 8$

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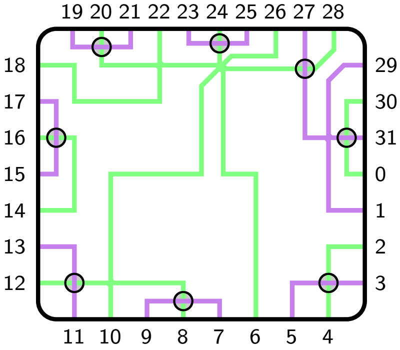

$d = 8$
$r = 0$

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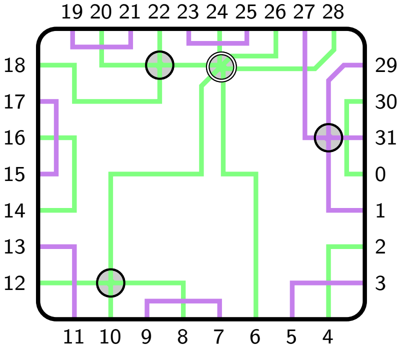

$d = 8$
$r = 0$
$c = 6$

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How many signatures with parameters (c, d, r) are there?

Counting Signatures: First Steps

Evaluating $s_{c,d,r} = \#\{$signatures with parameters $(c, d, r)\}$

Counting Signatures: First Steps

Evaluating $\mathsf{s}_{c,d,r} = \#\{$signatures with parameters $(c, d, r)\}$

Recursion Formula for Facets

$$\mathsf{s}_{0,d+1,0} = \sum_{d_1+d_2+d_3+d_4=d} \mathsf{s}_{0,d_1,0} \times \mathsf{s}_{0,d_2,0} \times \mathsf{s}_{0,d_3,0} \times \mathsf{s}_{0,d_4,0}$$

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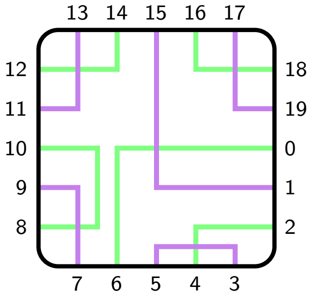
Proof:



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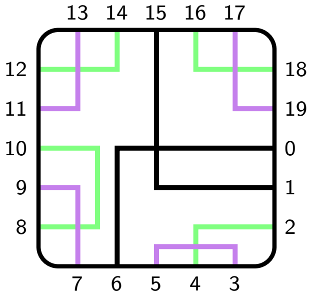

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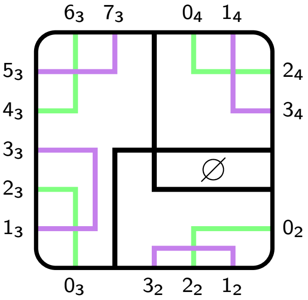

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Counting Facets with Fuss-Catalan Numbers (A'Campo 17)

$$\mathsf{s}_{0,d,0} = \frac{1}{3d+1}\binom{4d}{d}$$

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$\Rightarrow$ **What next?**

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Strategy: Use recursion formulæ and generating functions

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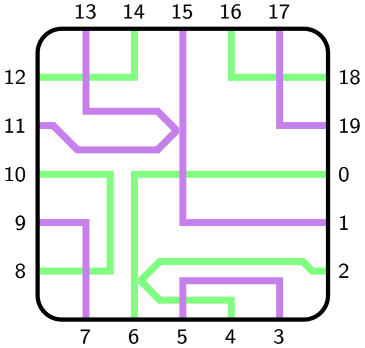

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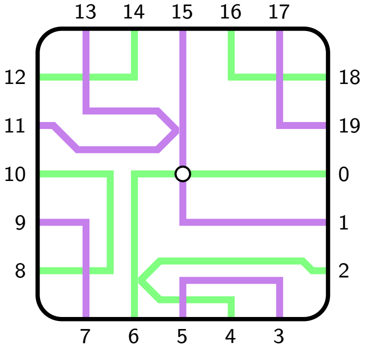

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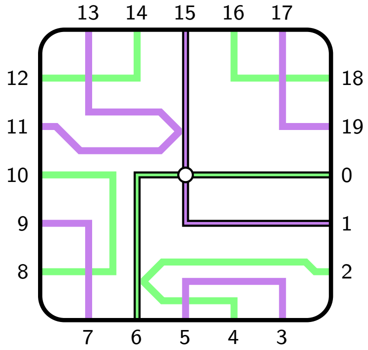

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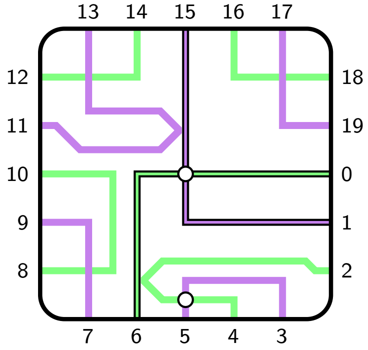

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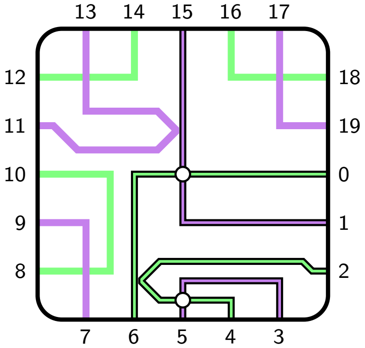

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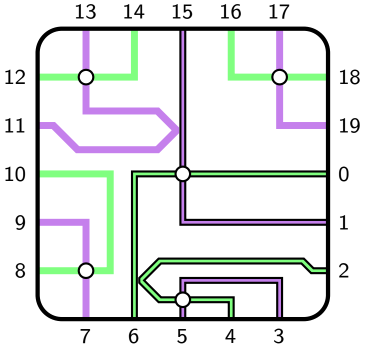

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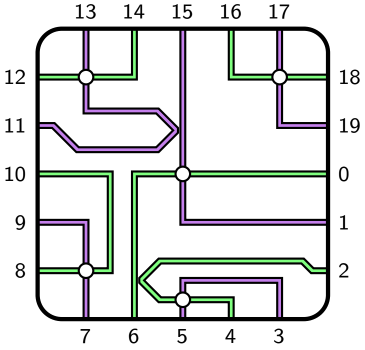

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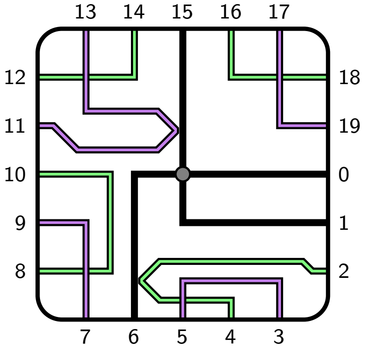
3. Recursive decomposition



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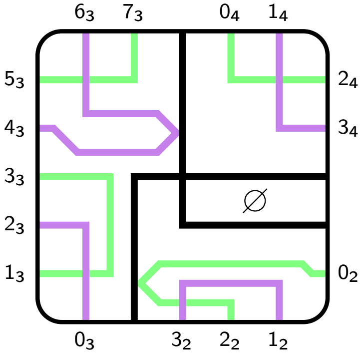

Counting Signatures: Some More Tools

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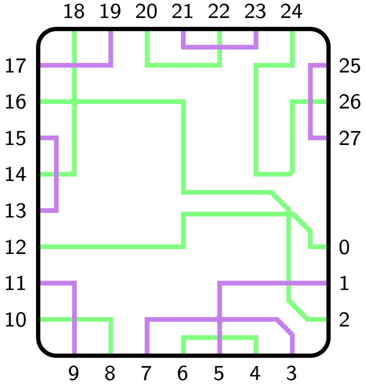

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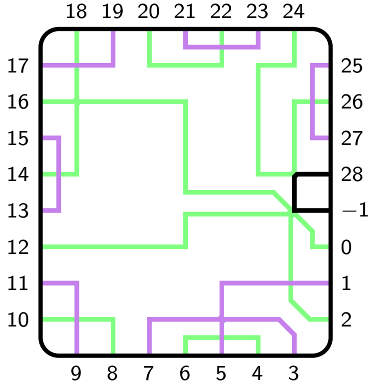

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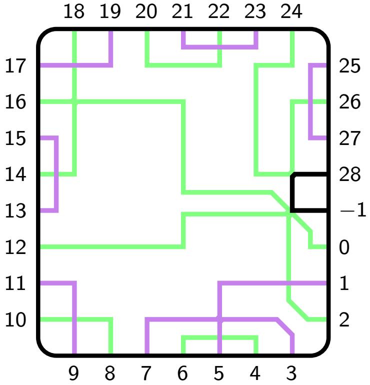

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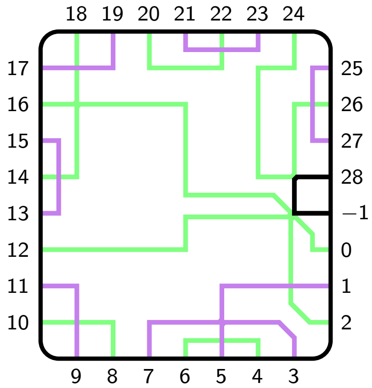
2. Another variant of signatures with generating series $\mathcal{D}(x, y, z)$



Counting Signatures: The End is Near

Three algebraic equations (using bijective proofs)

$$\begin{aligned}
\mathcal{S} &= 1 + y\mathcal{C}^4/(1 - x^2 yz\mathcal{C}^4) \\
\mathcal{C} &= \mathcal{D}\mathcal{S} \\
\mathcal{D} &= 1 + xy\mathcal{C}^4\mathcal{D}^2/(1 - x^2 y\mathcal{C}^4\mathcal{D})
\end{aligned}$$

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Theorem

The generating function $\mathcal{S}(x, y, z)$ is algebraic!

Counting Signatures: The End is Near

Three algebraic equations (using bijective proofs)

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Theorem

The generating function $\mathcal{S}(x, y, z)$ is algebraic!

and its minimal polynomial is not so nice...

$$x^4(x+1)^4 + \mathcal{S}^4yv\left(x^4 - x^8 + 4vx(x^4 + x^2v + v^2) - (x^2 + v^2)^2 + \mathcal{S}^4yv^5\right) = 0$$

with $v = x^2z + 1/(\mathcal{S} - 1)$.

Counting Signatures Efficiently

Three ideas for computing $\mathsf{s}_{c,d,r}$

1. Using directly the minimal polynomial of $\mathcal{S}$ Did not work ☹

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Corollary

The family of coefficients $(\mathbf{s}_{c,d,r})_{c \leqslant \mathbf{C}, d \leqslant \mathbf{D}, r \leqslant \mathbf{R}}$ can be computed in time

$$\mathcal{O}(\min\{\mathbf{C},\mathbf{D},\mathbf{R}\}^2 \min\{\mathbf{C},\mathbf{D}\}^2 \mathbf{D}^4).$$

Contents

What if $d \to +\infty$?

Problem: Fix c and r and evaluate $\lim \mathsf{s}_{c,d,r}$ when $d \to +\infty$

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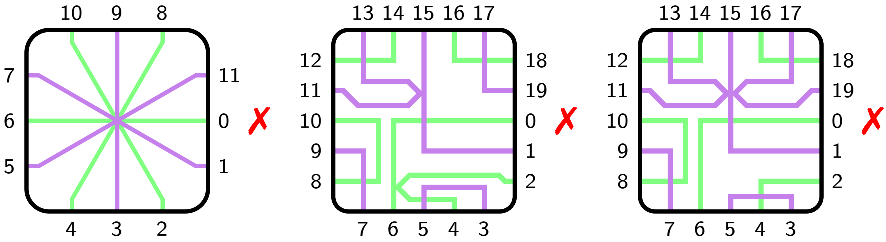

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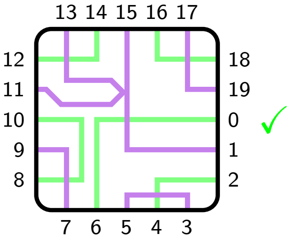

Counting Typical Signatures (1/2)

Another generating function: $\mathcal{T}(x, y, z) = \sum \mathbf{t}_{c,d,r} x^c y^d z^r$

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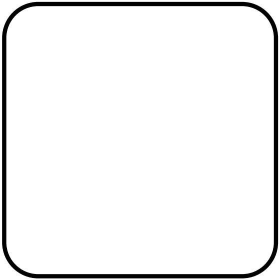

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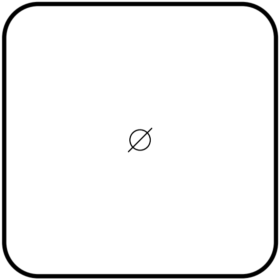

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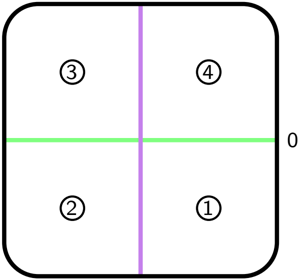

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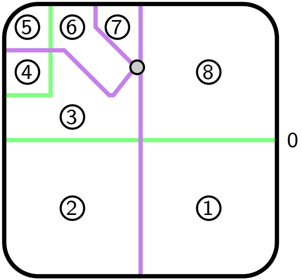

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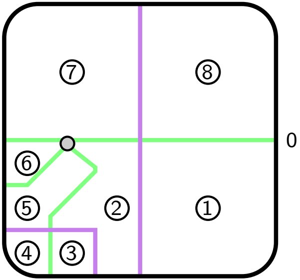

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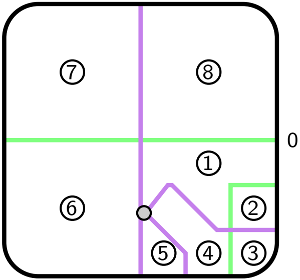

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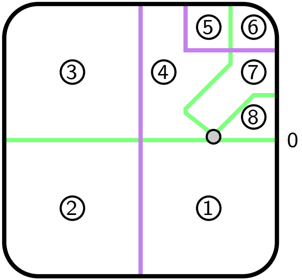

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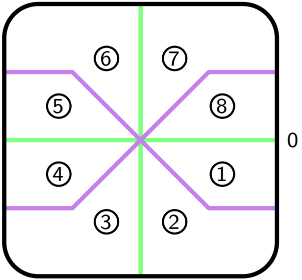

Counting Typical Signatures (2/2)

Algebraic equation with triangular system of variables

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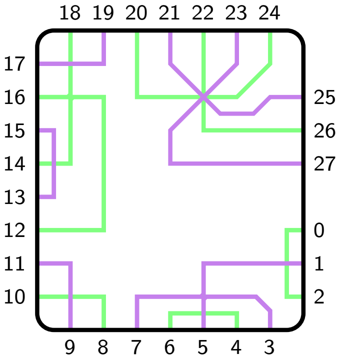
Exact and asymptotic evaluations

$$\mathbf{t}_{c,d,r} = \frac{\mathbf{1}_{c \geqslant 2r} \cdot \mathbf{1}_{d \geqslant 2c-2r} \cdot 4^{c-2r}}{c+3d-r+1} \binom{4d}{c-2r, d-2c-2r, r, c+3d-r}$$

$$\mathbf{t}_{c,d,r} \sim \frac{\mathbf{1}_{c \geqslant 2r}}{r!(c-2r)!} \cdot \sqrt{\frac{2}{27\pi}} \cdot \frac{4^c}{3^c} \cdot \frac{3^r}{16^r} \cdot \frac{4^{4d}}{3^{3d}} \cdot d^{c-r-3/2}$$

Typical Signatures are Typical

Main tool: Reducing a signature



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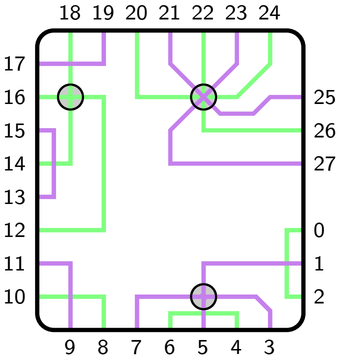

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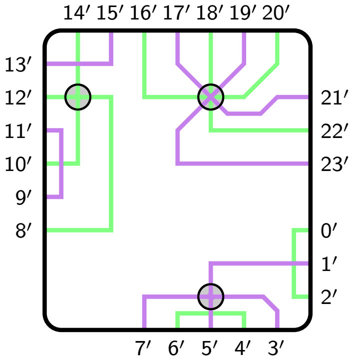

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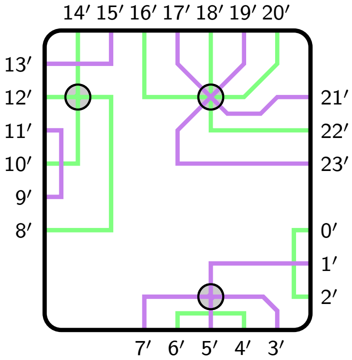

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At most $\mathcal{O}\left(d^{c-r-5/2} \cdot \frac{4^{4d}}{3^{3d}}\right)$ signatures reduce to a non-typical signature σ.

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Proof: with **C** components:

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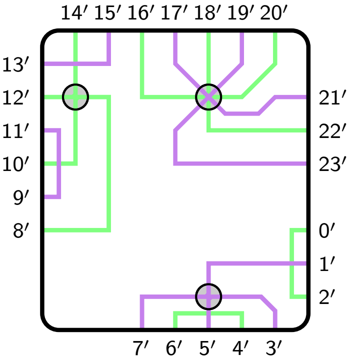

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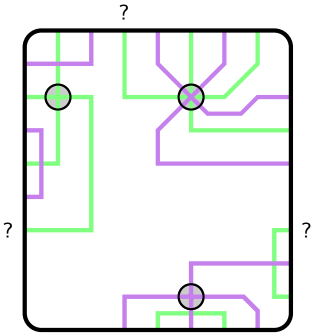

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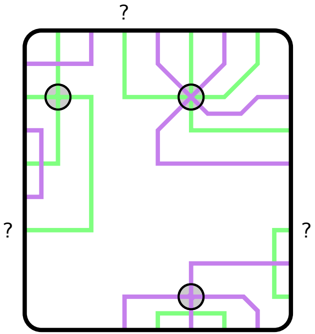

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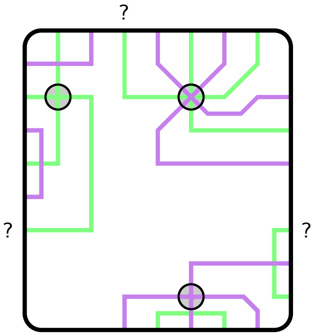
Fixing c $\Rightarrow$ finite number of reductions



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Thank you!

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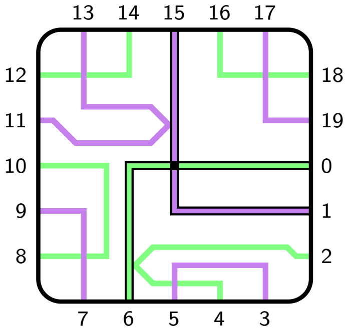

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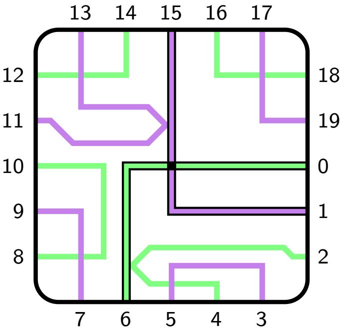

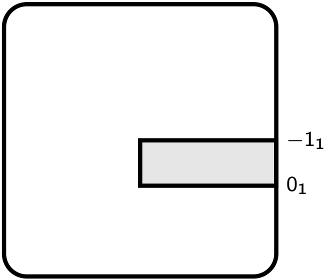

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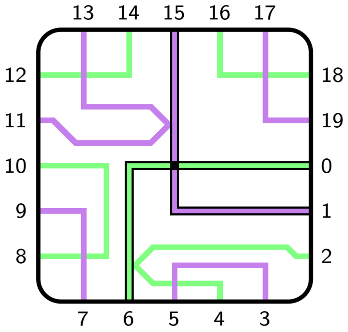

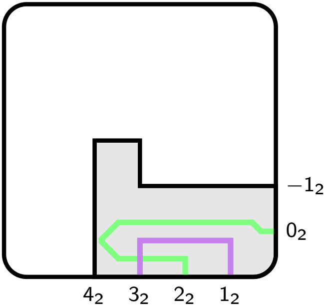

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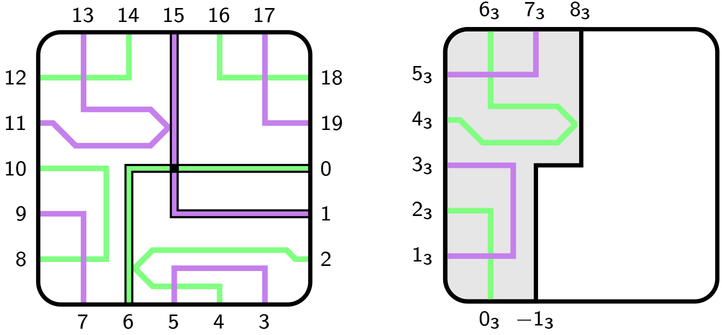

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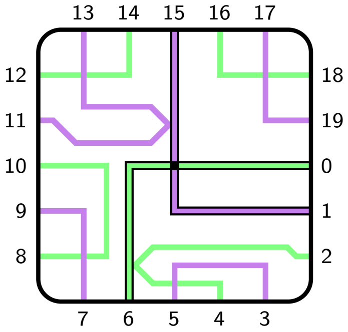

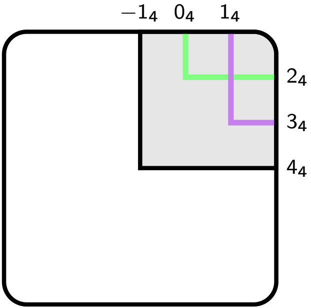

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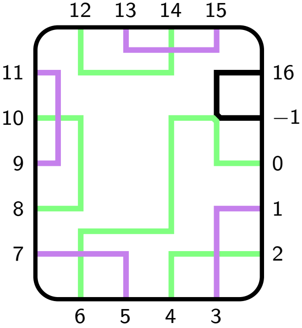

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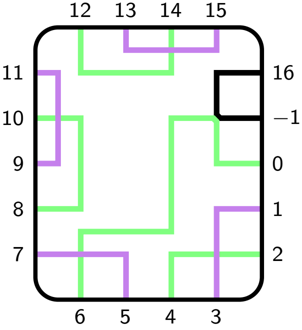

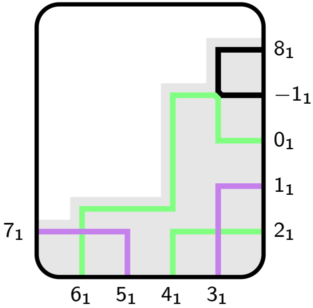

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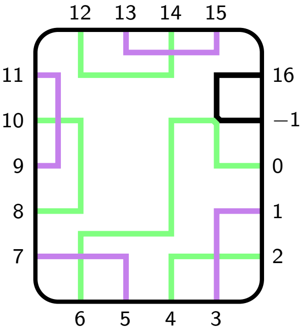

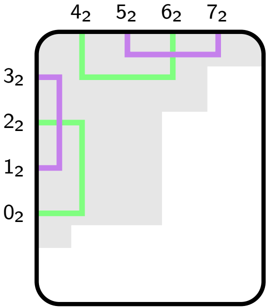

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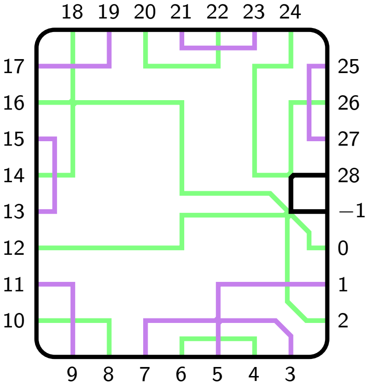

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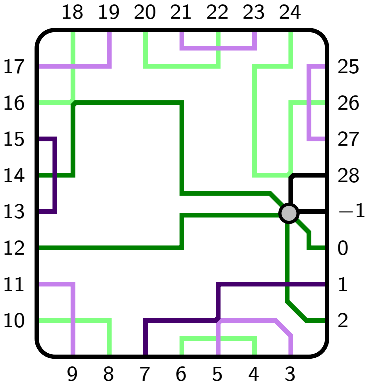

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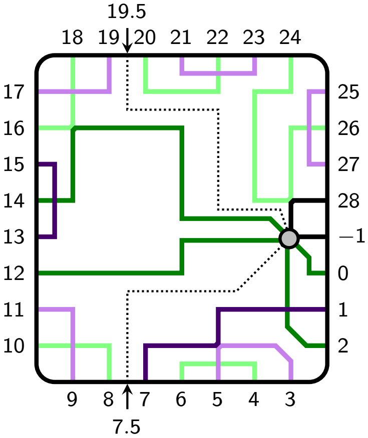

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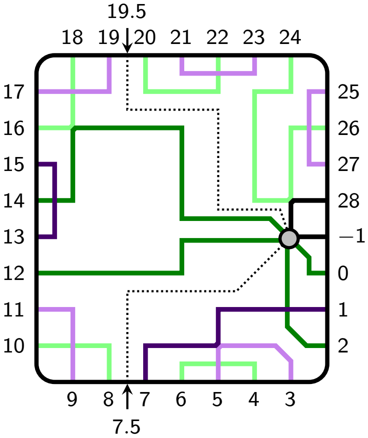

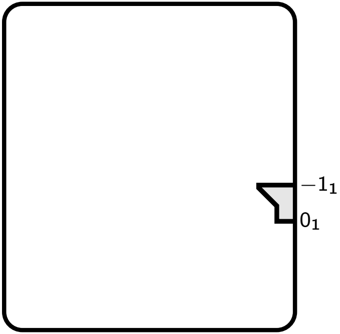

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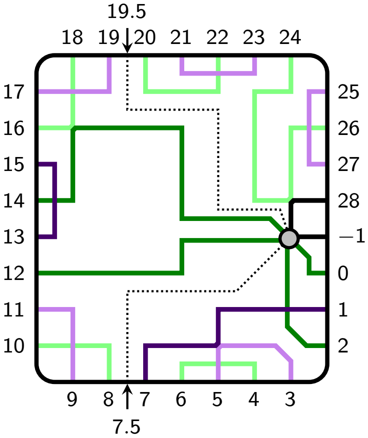

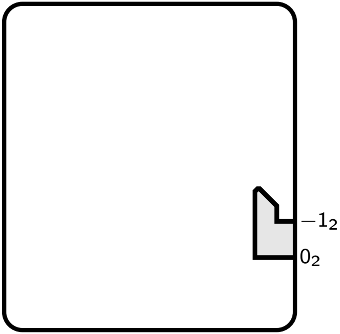

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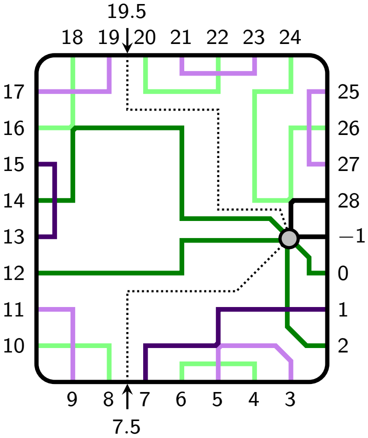

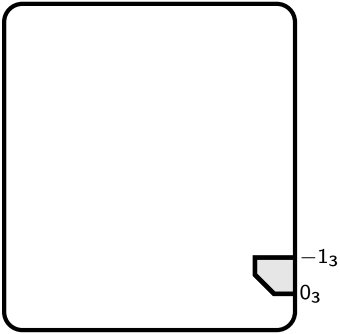

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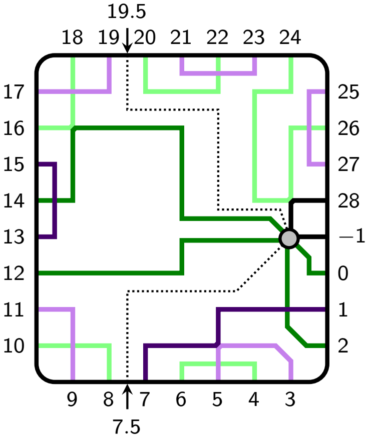

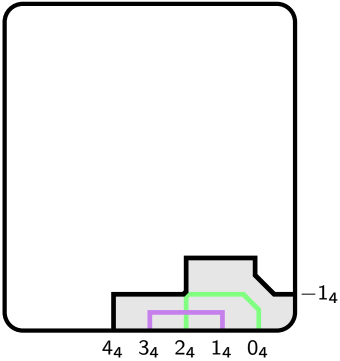

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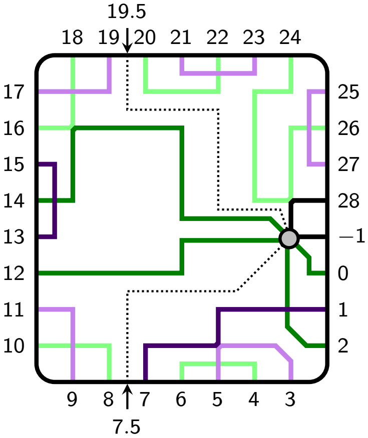

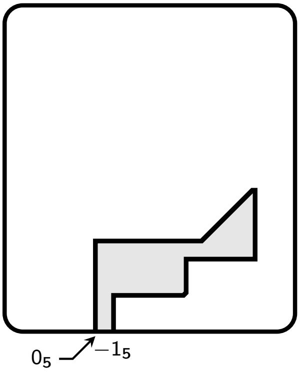

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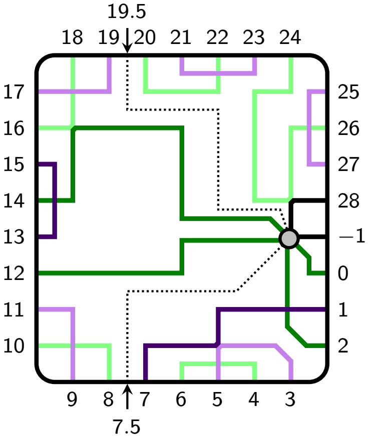

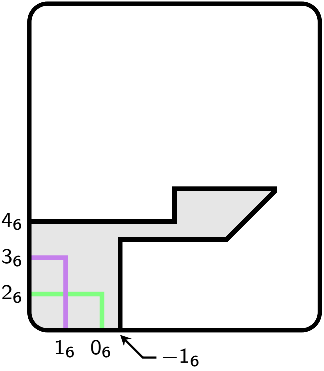

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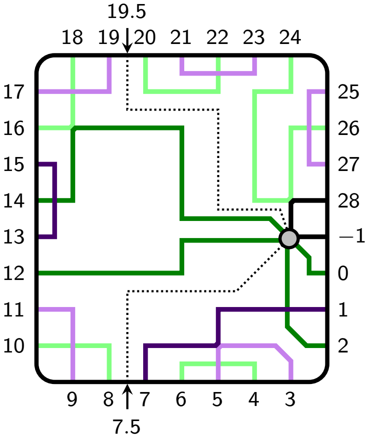

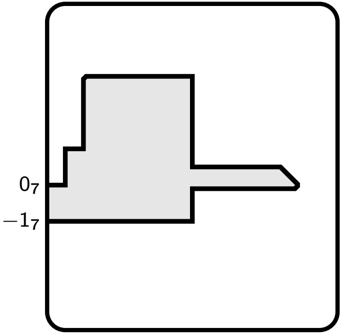

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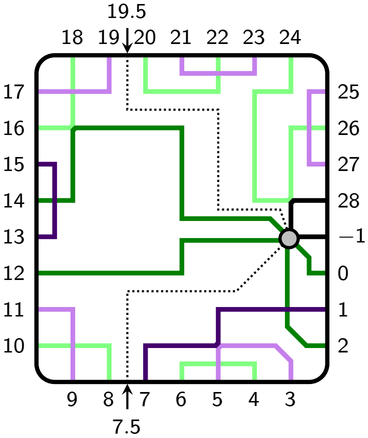

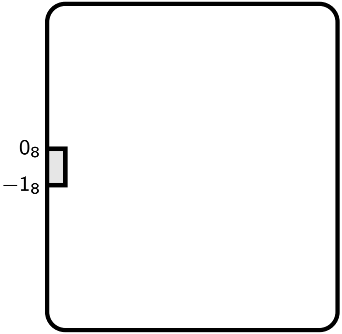

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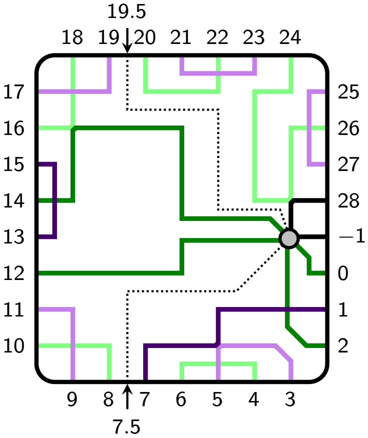

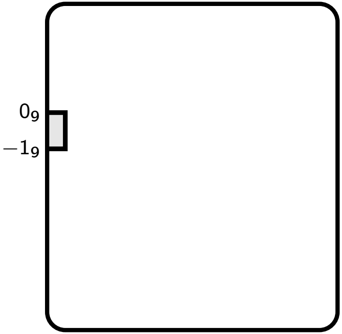

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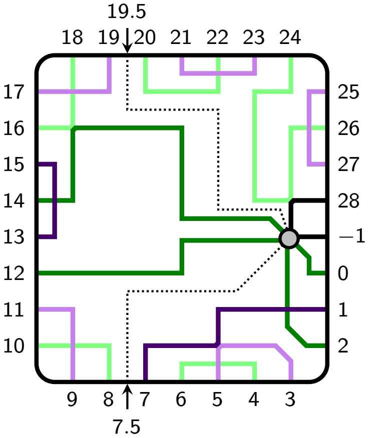

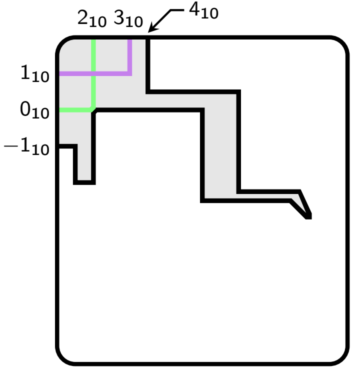

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