

ALL DISTRIBUTIVE LATTICES BETWEEN THE WEAK AND BRUHAT ORDERS

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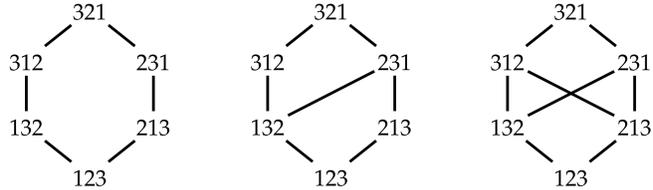


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Weak, middle and Bruhat orders

Consider the (right) weak order, the recently described middle order [1] and the Bruhat order on the symmetric group \mathfrak{S}_n :



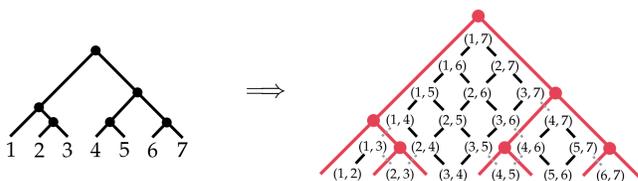
Right weak order: $a < b \implies \dots a b \dots \leq_R \dots b a \dots$

Middle order $a < b < \min(c_1, \dots, c_k) \implies \dots a c_1 \dots c_k b \dots \leq_M \dots b c_1 \dots c_k a \dots$

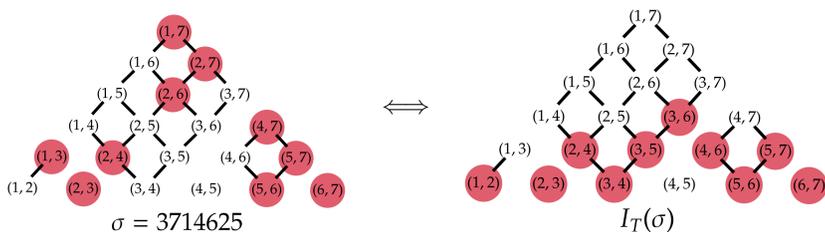
Bruhat order: $a < b \implies \dots a \dots b \dots \leq \dots b \dots a \dots$

From trees towards middle orders

Let \mathcal{T}_n be the set of binary trees with n leaves and let \mathcal{P}_n be the poset of pairs (i, j) ordered by $(i_1, j_1) \leq (i_2, j_2)$ if $i_1 \geq i_2$ and $j_1 \leq j_2$. Each tree $T \in \mathcal{T}_n$ induces a partition of \mathcal{P}_n into rectangular posets, which we write \mathcal{R}_T :



We have a bijection I_T from inversion sets of permutations to lower sets of \mathcal{R}_T :

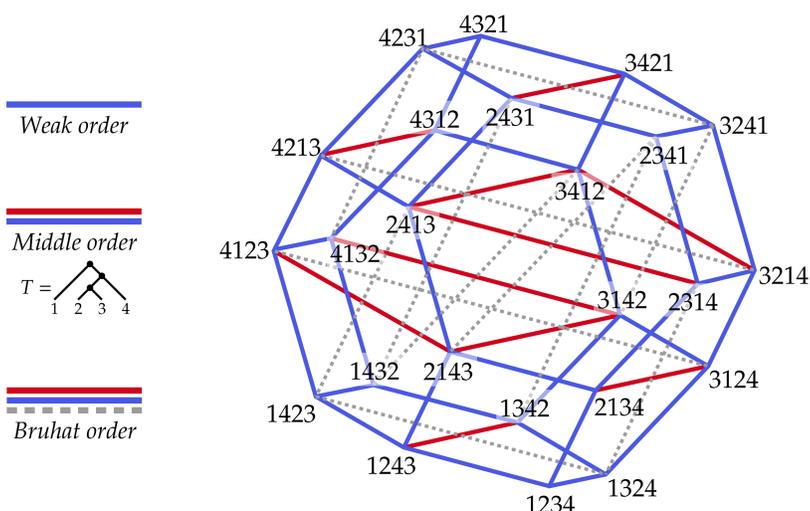


By Birkhoff's representation theorem, ordering lower sets of \mathcal{R}_T by inclusion endows \mathfrak{S}_n with the structure of a distributive lattice.

Main results

Using this bijection, we construct C_{n-1} middle orders on \mathfrak{S}_n :

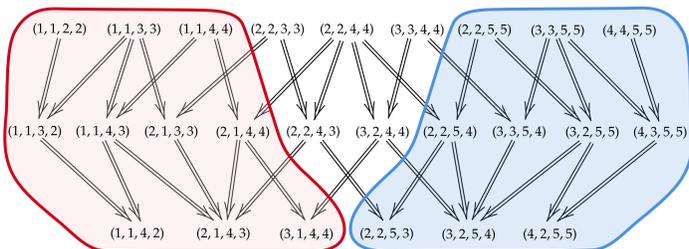
Proposition. Let $\sigma \leq_T \tau$ if and only if $I_T(\sigma) \subset I_T(\tau)$. (\mathfrak{S}_n, \leq_T) is a distributive lattice between the weak and Bruhat orders.



Theorem. All distributive lattices between the weak and Bruhat orders on \mathfrak{S}_n are equal to (\mathfrak{S}_n, \leq_T) for some $T \in \mathcal{T}_n$.

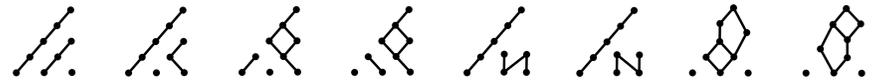
Some ingredients of the proof: Let $1 \leq i \leq a < b \leq j \leq n$, $\overline{(a, i, j, b)}$ is the set of edges (v, w) of (\mathfrak{S}_n, \leq) such that $w = v \circ (a, b)$ and $\{i, \dots, j\}$ is the largest interval containing $\{a, \dots, b\}$ and no values which are between a and b in v .

Middle orders are unions of such equivalence classes. Between these classes there are implication and exclusion relations (left edges forbid right edges and reciprocally):



Middle orders on other Coxeter groups

On Coxeter groups, a **middle order** is any distributive lattice between the weak and Bruhat orders. The 8 middle orders on B_3 are isomorphic to the lattices of lower sets of these posets:



It seems difficult to classify these lattices in other types than type A . Let (W, S) be a Coxeter group, $J \subset S$ and W_J the parabolic subgroup generated by J . Let W^J be the set of elements of minimal length of cosets in W/W_J . We have the inclusion

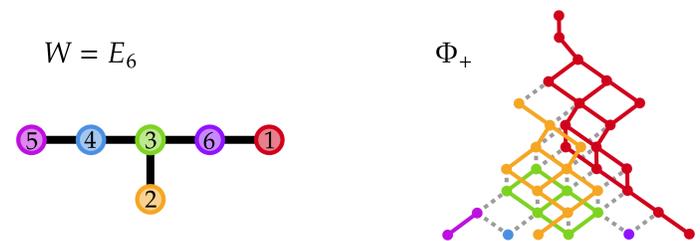
$$(W, \leq_R) \subset (W_J, \leq_R) \times (W^J, \leq_L) \subset (W_J, \leq) \times (W^J, \leq) \subset (W, \leq).$$

Hence if we have middle orders on W_J and W^J , their product is a middle order on W . Using this property, we shall describe an interesting subset of middle orders.

Minuscule middle orders

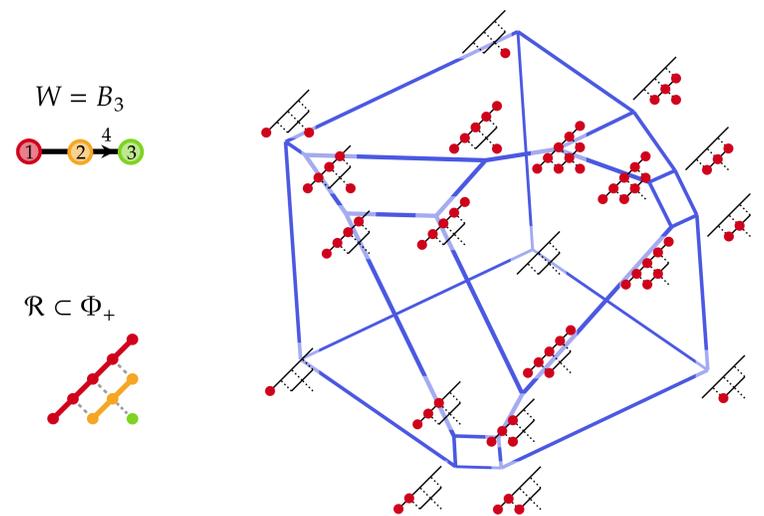
Let (W, S) be a Weyl group with root poset Φ_+ , let $\alpha_s \in \Phi_+$ be a minuscule root corresponding to $s \in S$, and $J = S \setminus \{s\}$. The weak and Bruhat orders on W^J are both isomorphic to $\mathbf{Low}(\mathcal{P}_s)$ where \mathcal{P}_s is the upper set of Φ_+ generated by α_s [2].

Let s_1, \dots, s_r be a permutation of S such that all α_{s_i} are minuscule roots of $W_{\{s_1, \dots, s_r\}}$. Using this we construct a partition of Φ_+ into a disjoint union of minuscule posets, whose lower sets form a distributive lattice between the weak and Bruhat orders on W . Distributive lattices constructed this way are called **minuscule middle orders**.



Jurassian lattices

A minuscule middle order is induced by a bijection from inversion sets of Φ_+ to lower sets of \mathcal{R} , where \mathcal{R} is a partition of Φ_+ into minuscule posets. A subset of Φ_+ is **\mathcal{R} -packed** if it is both an inversion set and a lower set of \mathcal{R} . A **Jurassian lattice** is the set of \mathcal{R} -packed subsets of Φ_+ ordered by inclusion.



Conjecture. The Jurassian lattices are semidistributive and extremal lattices of size $\text{Cat}(W)$, whose Hasse diagram is regular and is the 1-skeleton of a polytope.

Remark. Polytopes obtained from Jurassian lattices are not always associahedra.

Open problems

- Enumerate middle orders up to isomorphism (cf. sequence A247139 of the OEIS).
- Study regular subdivisions of the permutahedron into Bruhat interval polytopes compatible with middle orders.

References

- [1] M. Bouvel, L. Ferrari and B. E. Tenner. Between weak and Bruhat: the middle order on permutations. *Graphs Comb.*, 41(2), April 2025.
- [2] R. A. Proctor. Bruhat lattices, plane partition generating functions, and minuscule representations. *Eur. J. Comb.*, 5(4):331–350, 1984.