Affine alternating sign matrices

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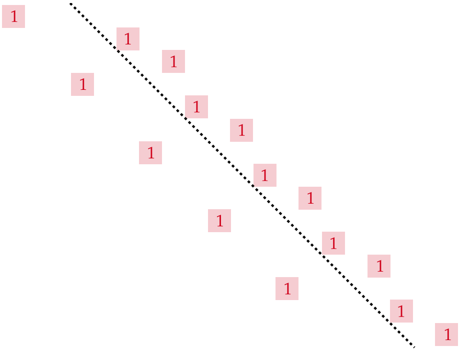
SLC 95

Affine permutations

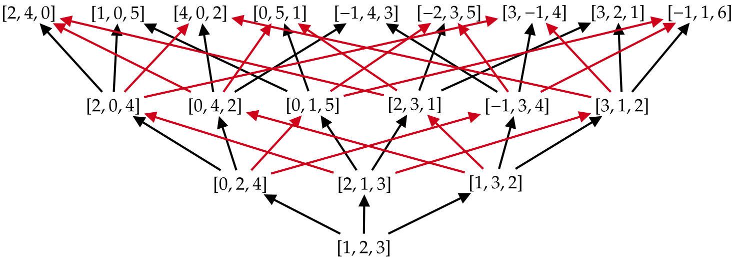
An affine permutation of period n is a bijection $w : \mathbb{Z} \to \mathbb{Z}$ such that:

- $w(i+n) = w(i) + n$;
- $\displaystyle\sum_{i=1}^{n} w(i) = \frac{n(n+1)}{2}$.



Affine permutations of period n form the affine Coxeter group $\tilde{S}_n$.
The weak and Bruhat order on $\tilde{S}_n$ are ranked with generating function

$$\prod_{k=1}^{n-1} \frac{1-x^{k+1}}{(1-x)(1-x^k)}.$$



ASMs are matrices with coefficients in $\{-1, 0, 1\}$ such that in each row and column, nonzero coefficients alternate between 1 and -1 and sum to 1.

$$
\begin{pmatrix}
 & & & 1 & & & & & & \\
 & & & & & & & & 1 & \\
 & & & & 1 & & & & & \\
1 & & & & & & & & & \\
 & & 1 & & & & & & & \\
 & & & & & & 1 & & & \\
 & & & & & & & 1 & & \\
 & & & & & & & & & 1 \\
 & 1 & & & & & & & & \\
 & & & & & 1 & & & &
\end{pmatrix}
\qquad
\begin{pmatrix}
 & & & & 1 & & & & & \\
 & & & & & & 1 & & & \\
 & 1 & & & -1 & & & 1 & & \\
 & & & 1 & & & & -1 & 1 & \\
 & & 1 & -1 & & 1 & & & & \\
 & & & & 1 & & & & -1 & 1 \\
1 & & -1 & 1 & & & -1 & 1 & & \\
 & & 1 & & & & & -1 & 1 & \\
 & & & & & & & 1 & & \\
 & & & & & & 1 & & &
\end{pmatrix}
$$

The number of ASMs of size n is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

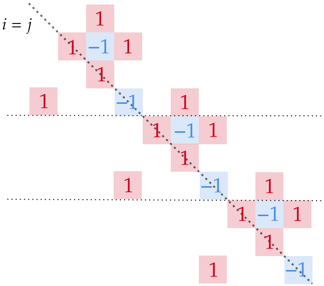
$$
\begin{array}{ccc}
S_n & \longrightarrow & \tilde{S}_n \\
\downarrow & & \downarrow \\
\mathbf{ASM}_n & \longrightarrow & ?
\end{array}
$$

Definition of affine ASMs

Definition

An affine alternating sign matrix of period n is an infinite array $(M_{i,j})_{i,j\in\mathbb{Z}}$ *with coefficients* $M_{i,j} \in \{-1, 0, 1\}$*, such that:*

- *in each row and column there is a finite number of nonzero entries, alternating between* 1 *and* -1 *and with sum equal to* 1*;*
- *(n-periodicity) for all* $i, j \in \mathbb{Z}$ $M_{i+n,j+n} = M_{i,j}$*;*
- *(centering)* $\sum_{i=1}^{n} \sum_{j\in\mathbb{Z}} M_{i,j} \cdot j = \frac{n(n+1)}{2}$



The corner sum of an affine ASM M of order n is $(\mathrm{CS}(M)_{i,j})_{i,j\in\mathbb{Z}}$ where

$$\mathrm{CS}(M)_{i,j} = \sum_{k\leq i,\ell\geq j} M_{k,\ell}$$

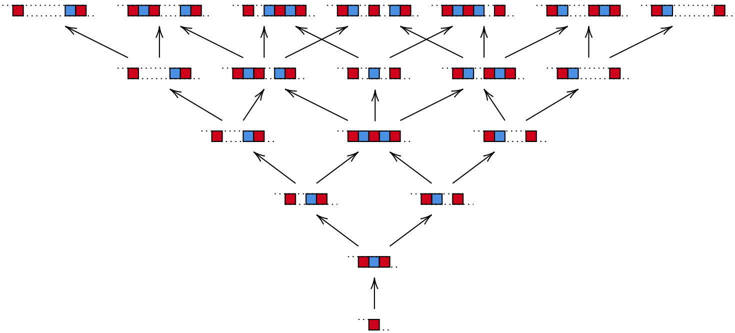
$$\begin{array}{cccccccccc}
 & & 1 & & & & & & & \\
 & 1 & -1 & 1 & & & & & & \\
 & & 1 & & & & & & & \\
1 & & & -1 & & 1 & & & & \\
 & & & & 1 & -1 & 1 & & & \\
 & & & & & 1 & & & & \\
 & & & 1 & & & -1 & & 1 & \\
 & & & & & & & 1 & -1 & 1 \\
 & & & & & & & & 1 & \\
 & & & & & & 1 & & & -1
\end{array}
\qquad
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 2 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 1 & 2 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}$$

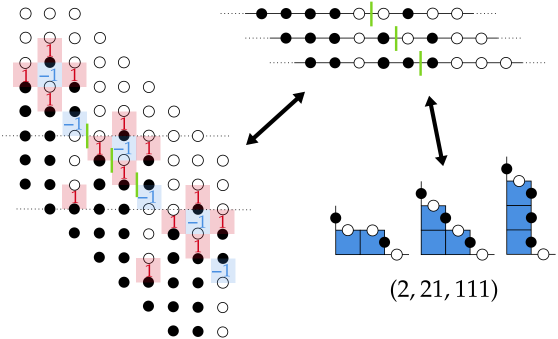
$\widetilde{\mathbf{ASM}}_n$ is the set of affine ASMs of order n ordered by $M_1 \leq M_2$ if and only if $\mathrm{CS}(M_1) \leq \mathrm{CS}(M_2)$ coordinatewise.

$\widetilde{\mathbf{ASM}}_n$ is a distributive lattice!

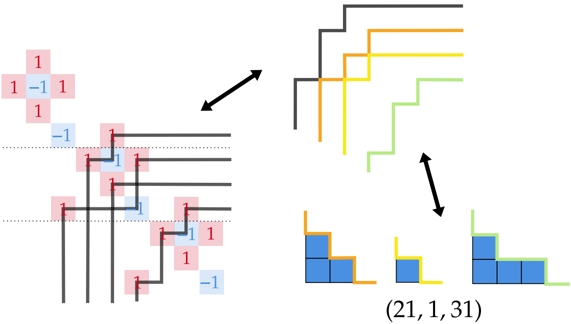
M_2 covers M_1 iff M_2 is the sum of M_1 and $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ periodically.

Affine ASMs of period 1 are in bijection with integer partitions, and $\widetilde{\mathbf{ASM}}_1$ is isomorphic to the Young lattice.



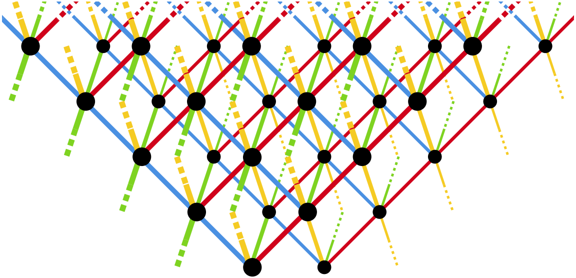


This generalizes the bijection between affine permutations and tuples of abaci by Salim Rostam, who showed that affine permutations correspond to tuples of *n*-cores (Rostam 2025).



inv(M) is the number of crossings in one period of M (here inv(M) $= 3$).
For affine permutations $w \in \tilde{S}_n$, inv(w) $= \ell(w)$.

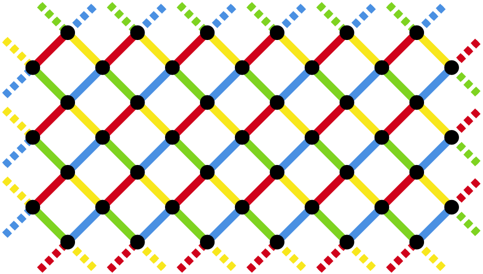
We can describe the poset of join-irreducible elements of $\widetilde{\mathbf{ASM}}_n$:



(Triangular posets correspond to join-irreducible elements of the Young lattice.)

Let G be the graph whose vertices are the pairs $(2i, 2j+1)$ and $(2i+1, 2j)$ for all $i, j \in \mathbb{Z}$, with edges between

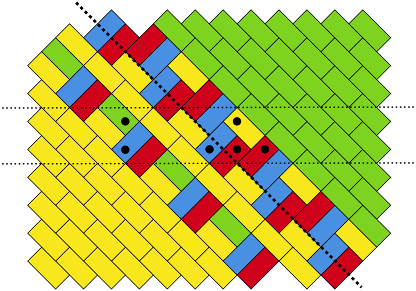
- $(2i-1, 2j)$ and $(2i, 2j+1)$ (green edges);
- $(2i, 2j-1)$ and $(2i+1, 2j)$ (yellow edges);
- $(2i-1, 2j)$ and $(2i, 2j-1)$ (blue edges);
- $(2i, 2j+1)$ and $(2i+1, 2j)$ (red edges).



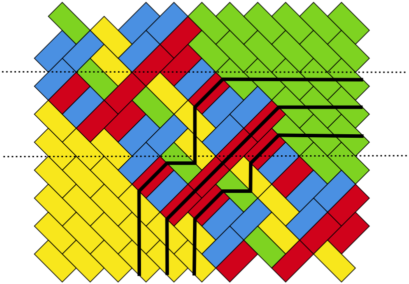
Definition

An affine domino tiling of order n is a perfect matching of G such that

- *for all $i \in \mathbb{Z}$, vertices $(2i, 2k+1)$ only belong to a green edges for k big enough, and to yellow edges for k small enough.*
- *(n-periodicity) invariant by the translation $(i, j) \mapsto (i+n, j+n)$;*
- *(centering) among the edges containing $(2i, 2j+1)$ for $1 \leq i \leq n$, the number of green and blue edges with $i > j$ is equal to the number of yellow and red edges with $i \leq j$.*

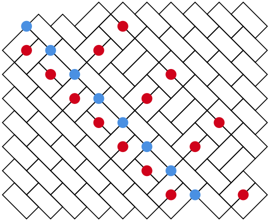


Affine domino tilings of period n can be seen as n-tuples of nonintersecting Schröder paths:

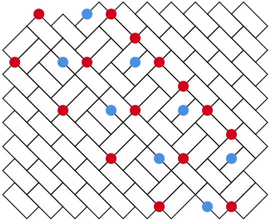


Affine domino tilings

Two morphisms from $\widetilde{\mathbf{TOAD}}_n$ to $\widetilde{\mathbf{ASM}}_n$:

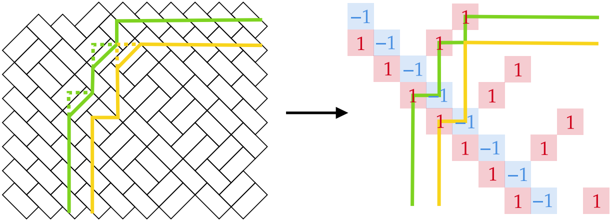


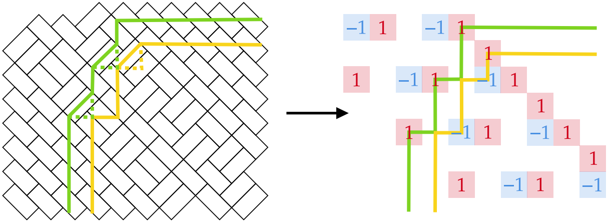
$$\longrightarrow \quad \begin{array}{cccccccccc}
-1 & & & & 1 & & & & & \\
1 & -1 & & 1 & & & & & & \\
 & 1 & -1 & & & & 1 & & & \\
 & & 1 & -1 & & 1 & & & & \\
 & & & 1 & -1 & & & & 1 & \\
 & & & & 1 & -1 & & 1 & & \\
 & & & & & 1 & -1 & & & \\
 & & & & & & 1 & -1 & & 1
\end{array}$$



$$\longrightarrow \quad \begin{array}{cccccccccc}
 & & & & & 1 & & & & \\
1 & & -1 & 1 & & -1 & 1 & & & \\
 & & & & & & & 1 & & \\
 & & 1 & & -1 & 1 & & -1 & 1 & \\
 & & & & & & & & & 1 \\
 & & & & 1 & & -1 & 1 & & -1 \\
 & & & & & & & & & \\
 & & & & & & 1 & & -1 & 1
\end{array}$$

Two morphisms from $\widetilde{\mathbf{TOAD}}_n$ to $\widetilde{\mathbf{ASM}}_n$:





Preimages of an affine ASM M by these morphisms are boolean intervals of dimension $\mathrm{N}_+(M)$ or $\mathrm{N}_-(M)$.

Proposition

$$\sum_{T \in \widetilde{\mathbf{TOAD}}_n} x^{|T|} y^{\mathrm{flip}(T)} = \sum_{M \in \widetilde{\mathbf{ASM}}_n} x^{2|M|} (xy)^{\mathrm{inv}(M)} (1 + xy)^{\mathrm{N}_+(M)}$$

$$= \sum_{M \in \widetilde{\mathbf{ASM}}_n} x^{2|M|} \left(\frac{y}{x}\right)^{\mathrm{inv}(M)} \left(1 + \frac{y}{x}\right)^{\mathrm{N}_-(M)}$$

$\mathrm{flip}(T)$ is the number of red tiles in a period of T, $\mathrm{inv}(M)$ is the number of inversions of M.

Definition

An affine Gelfand-Tsetlin pattern of period n is an array $(X_{i,j})_{i\in\mathbb{Z}, j\in\mathbb{N}}$ s.t.

- $X_{i-1,j} \leq X_{i,j} \leq X_{i-1,j+1}$;
- $X_{i+n,j} = X_{i,j} + n$
- $X_{i,j} \leq i + j$, *and* $X_{i,j} = i + j$ *for* j *sufficiently big.*

$j \in \mathbb{N}$

0 1 3 4 5 6 7 8

1 2 4 5 6 7 8

1 4 4 6 7 8 9

2 4 6 7 8 9

$i \in \mathbb{Z}$ 4 5 7 8 9 10

4 7 7 9 10

5 7 9 10 11

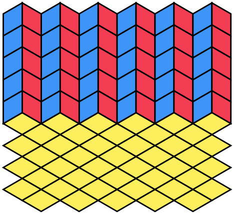
7 8 10 11

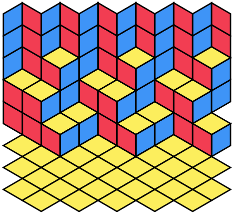
Proposition

Affine ASMs $\iff$ *affine Gelfand-Tsetlin patterns with strictly increasing rows.*

We call these "affine Gog patterns".

Affine Gelfand-Tsetlin patterns can be seen as periodical stackings of cubes:





Affine Gelfand-Tsetlin patterns

Definition

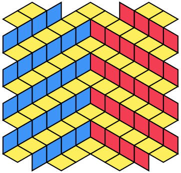
An affine Magog pattern of period n is an array of integers $(X_{i,j})_{i\in\mathbb{N},j\in\mathbb{Z}}$ s.t.

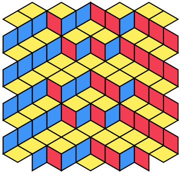
- $X_{i-1,j} \leq X_{i,j} \leq X_{i-1,j+1}$ *for* $i > 0$;
- $X_{i,j} \leq j$ *and* $X_{k,j} = j$ *for* i *sufficiently big;*
- *(periodicity)* $X_{i,j+n} = X_{i,j} + n$.

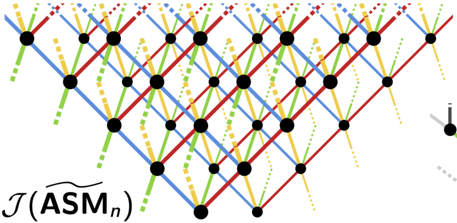
$j \in \mathbb{Z}$

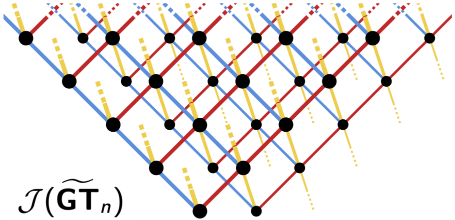
$i \in \mathbb{N}$

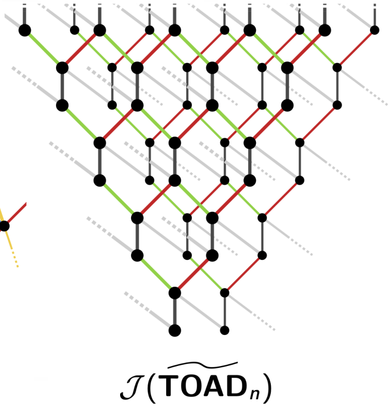
$$
\begin{array}{ccccccccccccccc}
 & 2 & & 4 & & 4 & & 5 & & 7 & & 7 & & 8 & \\
1 & & 3 & & 4 & & 4 & & 6 & & 7 & & 7 & & 9 \\
 & 3 & & 3 & & 4 & & 6 & & 6 & & 7 & & 9 & \\
1 & & 3 & & 4 & & 4 & & 6 & & 7 & & 7 & & 9 \\
 & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\
 & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 &
\end{array}
$$



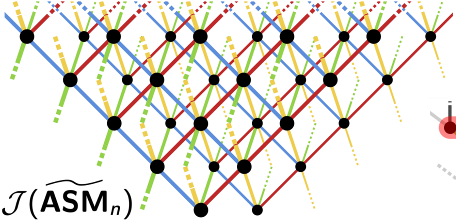



$\mathcal{J}(\widetilde{\mathbf{ASM}_n})$

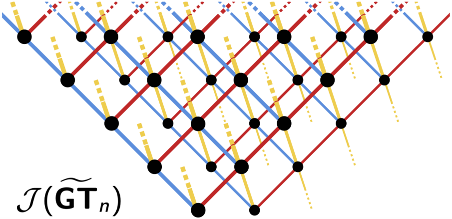

$\mathcal{J}(\widetilde{\mathbf{GT}_n})$


$\mathcal{J}(\widetilde{\mathbf{TOAD}_n})$

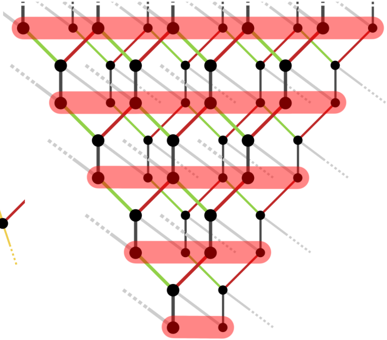
Lattice structures



$\mathcal{J}(\widetilde{\mathbf{ASM}_n})$

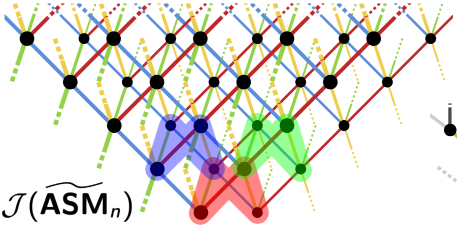


$\mathcal{J}(\widetilde{\mathbf{GT}_n})$

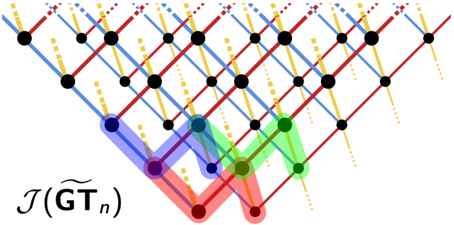


$\mathcal{J}(\widetilde{\mathbf{TOAD}_n})$

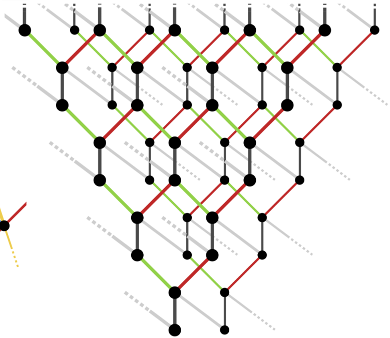
Lattice structures



$\mathcal{J}(\widetilde{\mathbf{ASM}}_n)$



$\mathcal{J}(\widetilde{\mathbf{GT}}_n)$



$\mathcal{J}(\widetilde{\mathbf{TOAD}}_n)$

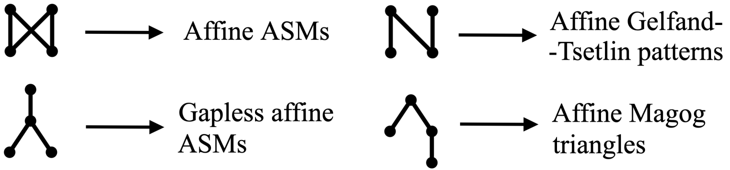
Lattice structures

$\mathbf{ASM}_{n+2}, \mathbf{GT}_{n+2}$ are isomorphic to $\mathbf{Low}(\mathcal{P}^n/\mathfrak{S}_n)$ for some poset $\mathcal{P}$ of size 4.

Definition

Let $\mathcal{P}$ be a finite poset with two minimal elements a and b. $\mathcal{A}^\infty(\mathcal{P})$ is the set of tuples $\mathbf{x} \in \mathbb{Z}^{\mathcal{P}}$ with $\mathbf{x}_a = -\mathbf{x}_b$ and $\mathbf{x}_c \geq 0$ for $c \notin \{a, b\}$, ordered by $\mathbf{x} \leq \mathbf{y}$ iff $\sum_{i \in F} \mathbf{x}_i \leq \sum_{i \in F} \mathbf{y}_i$ for all $F \in \mathbf{Upp}(\mathcal{P})$. $\mathcal{A}_n^\infty(\mathcal{P})$ is the quotient of $\mathcal{A}^\infty(\mathcal{P})$ by the equivalence relation $\mathbf{x} \sim \mathbf{y}$ iff $\mathbf{x}_a \equiv \mathbf{y}_a \mod n$ and $\mathbf{x}_c = \mathbf{y}_c$ for $c \notin \{a, b\}$.

Then $\widetilde{\mathbf{ASM}}_n$ and $\widetilde{\mathbf{GT}}_n$ are isomorphic to $\mathbf{Low}(\mathcal{A}_n^\infty(\mathcal{P}))$.



Conjecture

$$\sum_{X\in\widetilde{\mathbf{GT}}_n} x^{|X|}y^{\mathrm{odd}(X)} = \prod_{k\geq 1}\frac{1}{(1-x^{2k-1}y)^n(1-x^{2kn})}.$$

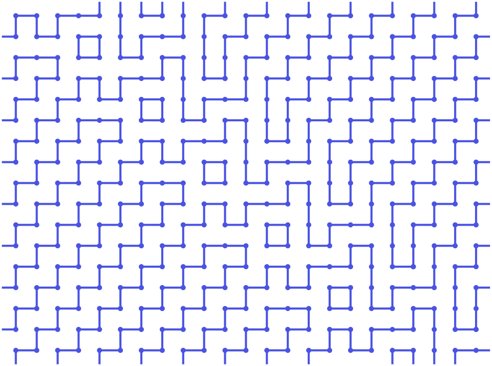
where $\mathrm{odd}(X)$ *is the number of entries s.t.* $X_{i,j}\neq i+j \mod 2$ *in a period.*

Conjecture

$$\sum_{T\in\widetilde{\mathbf{TOAD}}_n} x^{|T|}y^{\mathrm{flip}(T)} = \prod_{k\geq 1}\frac{(1+x^{2k-1}y)^n}{1-x^{2kn}}.$$

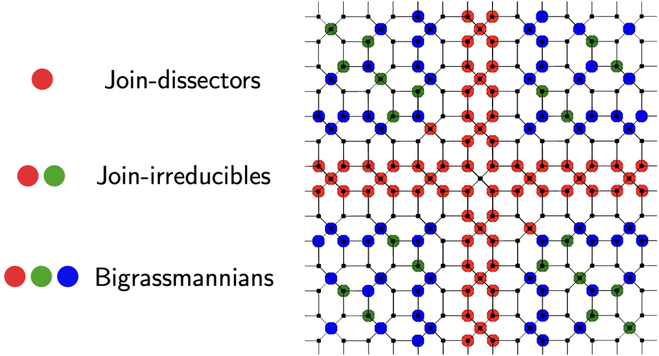
In the finite case, $|\mathbf{GT}_n| = |\mathbf{TOAD}_n| = 2^{\binom{n}{2}}$, with respective rankings $\prod_{k=1}^{n-1}(1+x^k)^{n-k}$ and $\prod_{k=1}^{n-1}(1+x^{2k-1})^{n-k}$.

Study affine generalizations of other ASM related objects (FPLs...)

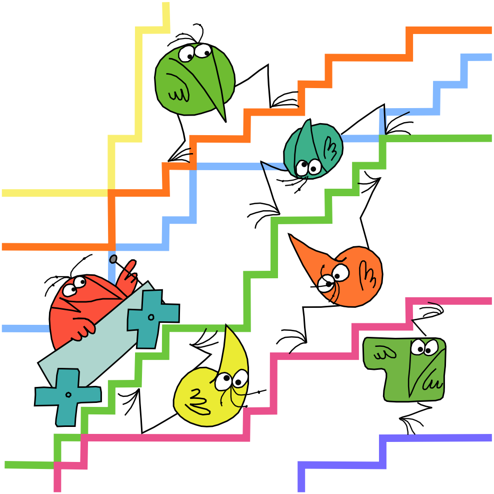


Perspectives

Study the link between $\widetilde{\mathbf{ASM}}_n$ and the completion of the Bruhat order on $\tilde{S}_n$.



The end



THANKS FOR YOUR ATTENTION !

$\widetilde{\mathrm{ASM}}_n$	0	1	2	3	4	5	6	7	8	9	10	11
$n=1$	1	1	2	3	5	7	11	15	22	30	42	56
$n=2$	1	2	1	4	6	8	11	14	25	32	43	54
$n=3$	1	3	3	7	9	15	26	39	51	72	102	138
$n=4$	1	4	6	12	21	32	52	84	130	200	284	412
$n=5$	1	5	10	20	40	66	115	190	300	475	737	1125

$\widetilde{\mathrm{TOAD}}_n$	0	1	2	3	4	5	6	7	8	9	10	11
$n=1$	1	1	1	2	3	4	5	7	10	13	16	21
$n=2$	1	2	1	2	5	6	6	8	15	20	20	26
$n=3$	1	3	3	4	9	12	16	24	33	47	63	81
$n=4$	1	4	6	8	17	28	38	56	85	128	178	240
$n=5$	1	5	10	15	30	56	85	130	205	315	466	670

$\widetilde{\mathrm{GT}}_n$	0	1	2	3	4	5	6	7	8	9	10
$n=1$	1	1	2	3	5	7	11	15	22	30	42
$n=2$	1	2	3	6	10	16	25	38	57	84	121
$n=3$	1	3	6	13	24	42	74	123	198	315	489
$n=4$	1	4	10	24	51	100	190	344	602	1028	1712
$n=5$	1	5	15	40	95	206	425	835	1575	2880	5122