Quotients of alternating sign matrices and bases of Temperley-Lieb algebra

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42 Years of Alternating Sign Matrices, Ljubljana

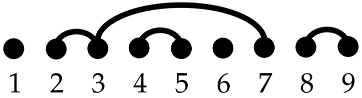
Proposition (Bergeron, Gagnon '24)

Let $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$.

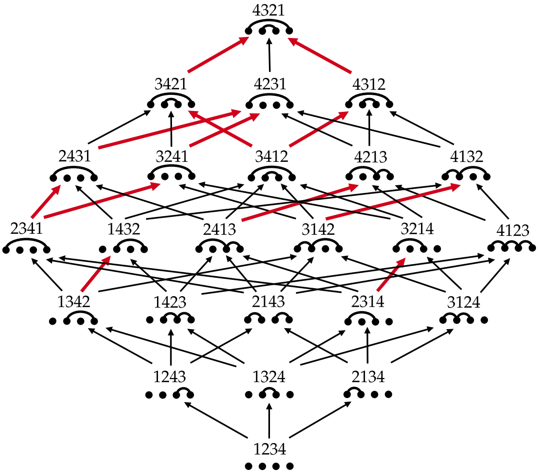
$$E_{pos}(w) := \{i \mid i \leq w_i\} \quad \text{and} \quad E_{val}(w) := \{w_i \mid i \leq w_i\}$$

There is a unique noncrossing partition λ *such that* $E_{val}(w) = [n] - \lambda^+$ *and* $E_{pos}(w) = [n] - \lambda^-$.

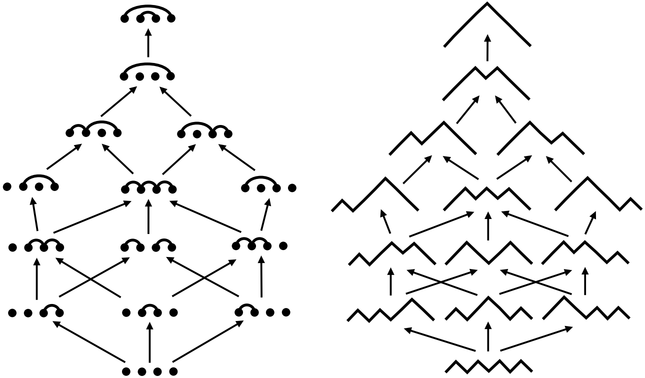
1 5 2 7 4 6 3 9 8



Classes $\mathcal{C}_\lambda$ are intervals of the Bruhat order (Bergeron, Gagnon '24).



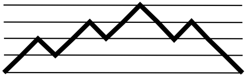
The induced order on classes $\mathcal{C}_\lambda$ is isomorphic to the lattice of Stanley lattice of Dyck paths Dyck_n (Bergeron, Gagnon '24).

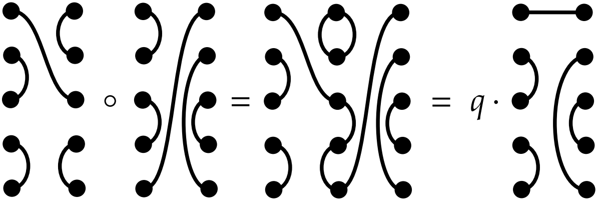


Proposition (Hivert, Pilaud, S. '25)

The excedance quotient is induced by a quotient of the lattice of alternating sign matrices ASM_n *isomorphic to* Dyck_n*. The corresponding Dyck paths can be read on two diagonals of the height function of the ASMs.*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 2 & 3 & 4 & 5 & 4 & 5 \\ 3 & 4 & 3 & 2 & 3 & 4 & 5 & 4 \\ 4 & 5 & 4 & 3 & 4 & 3 & 4 & 3 \\ 5 & 4 & 3 & 4 & 3 & 2 & 3 & 2 \\ 6 & 5 & 4 & 5 & 4 & 3 & 2 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$





The Temperley-Lieb algebra is isomorphic to a quotient of $\mathbb{C}[\mathfrak{S}_n]$:

$$TL_n(2) \cong \mathbb{C}[\mathfrak{S}_n]/\langle e - (12) - (13) - (23) + (123) + (132)\rangle$$

Theorem (Bergeron, Gagnon '24)

Let $n \geq 0$ and for each noncrossing partition $\lambda \in NCP_n$, fix $w_\lambda \in \mathcal{C}_\lambda$. Then the set $\{w_\lambda | \lambda \in NCP_n\}$ is a basis of $TL_n(2)$.

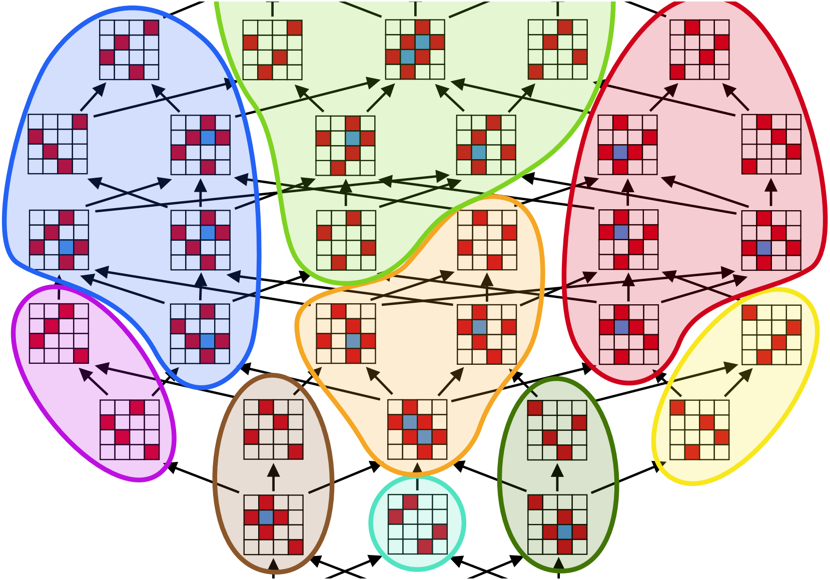
Main results

Theorem (Hivert, Pilaud, S. '25)

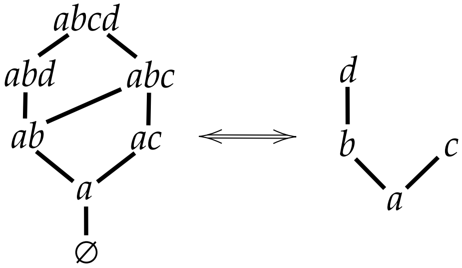
Let $\equiv$ be a lattice congruence of ASM_n whose quotient is isomorphic to Dyck_n. Then:

- *Let $\mathcal{B}$ a set containing exactly one permutation in each congruence class of $\equiv$. Then $\mathcal{B}$ is a basis of $\mathrm{TL}_n(2)$.*
- *Each congruence class of $\equiv$ contains a unique minimal permutation, which avoids the pattern 321;*
- *Maximal elements of congruence classes of $\equiv$ are covexillary permutations, i.e. they avoid the pattern 3412;*

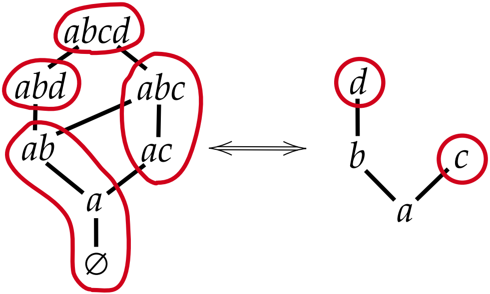
Main results



Let $\mathcal{P}$ be the poset of join-irreducible elements of $\mathbb{L}$. $\mathbb{L}$ is isomorphic to the set lower sets of $\mathcal{P}$ ordered by inclusion $\text{Low}(\mathcal{P})$.

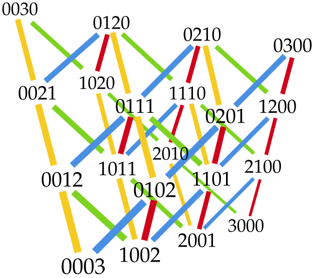


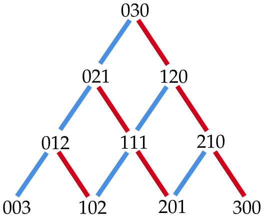
Lattice congruences $\equiv \in \mathrm{Cong}(\mathbb{L})$ are in bijection with subposets $\mathcal{Q} \subset \mathcal{P}$, and $\mathbb{L}/\equiv$ is isomorphic to the lattice of ideals of $\mathrm{Low}(\mathcal{Q})$.

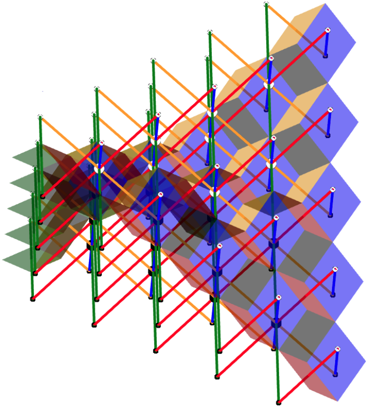


Geometric structure of $\mathcal{J}(\mathrm{ASM}_n)$

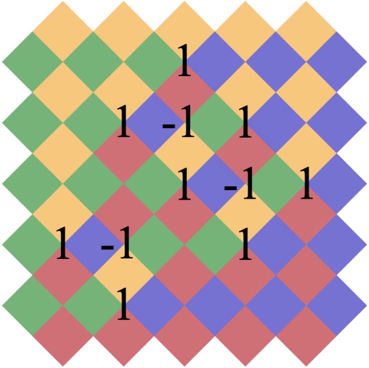
Irreducibles posets of ASM_n and Dyck_n have simple geometric structures (*cf.* EKLP '92, Propp '02, Striker '14 in the case of ASMs)



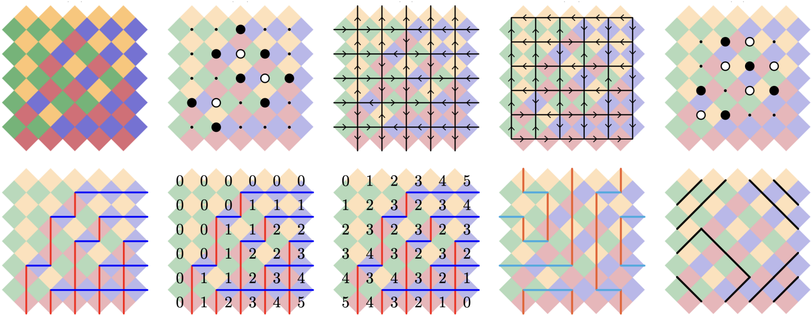




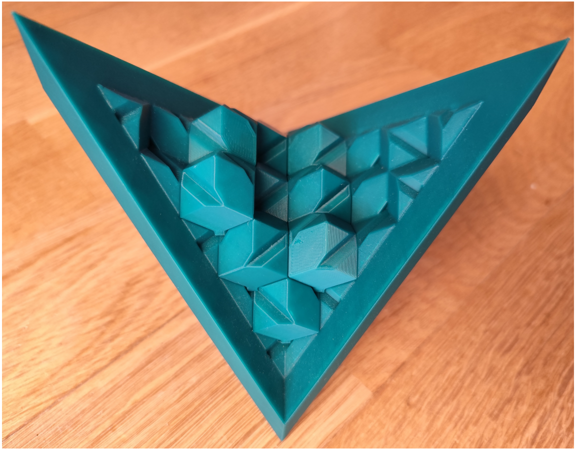
(a) An ASM seen as a lower set of $\mathcal{J}(\mathrm{ASM}_n)$.

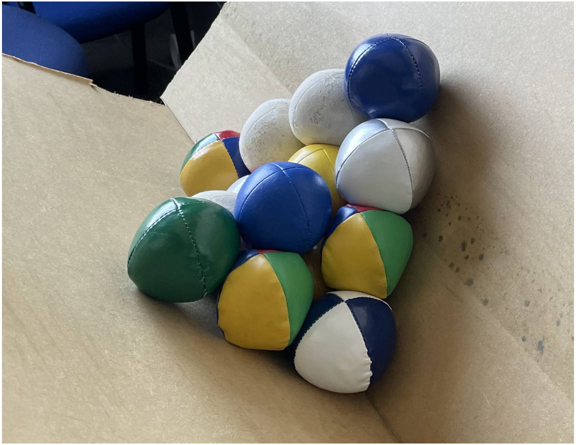


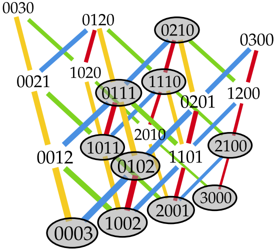
(b) A vertical projection of (a), with the corresponding ASM.


0 0 0 0 0 0
0 0 0 1 1 1
0 0 1 1 2 2
0 0 1 2 2 3
0 1 1 2 3 4
0 1 2 3 4 5
0 1 2 3 4 5
1 2 3 2 3 4
2 3 2 3 2 3
3 4 3 2 3 2
4 3 4 3 2 1
5 4 3 2 1 0

We can also see ASMs as stackings of rhombic dodecahedra:







$$\begin{array}{ccccccc} & & & 2 & & & \\ & & 2 & & 3 & & \\ & 1 & & 3 & & 3 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

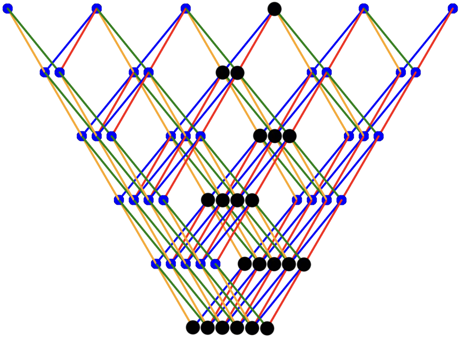
A *catalan triangle* of size n is a Gelfand-Tsetlin pattern with bottom row $12\cdots n$ which is *gapless* on columns and diagonals.

Proposition (Hivert, Pilaud, S. '25)

Quotients of ASM_n *isomorphic to* Dyck_n *are in bijection with catalan triangles of size* $n-1$.

Catalan triangles

The excedance quotient corresponds to an undulating subposet of $\mathcal{J}(\mathrm{ASM}_n)$:

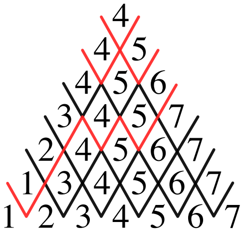


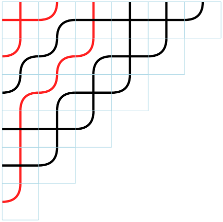
3
3 4
2 3 4
2 3 4 5
1 2 3 4 5
1 2 3 4 5 6

Catalan triangles

Proposition (Hivert, Pilaud, S. '25)

Catalan triangles of size n are in bijection with 2-colored pipe dreams of size $n-1$.





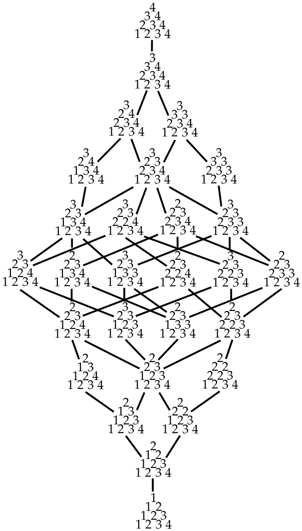
Proposition (Hivert, Pilaud, S. '25)

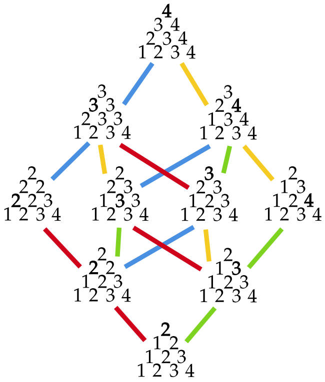
For all $C \in \mathrm{Cat}_n$, let $\mathrm{bcc}(C)$ be the number of bicolored crossings of C seen as a pipe dream. We have the equality

$$\sum_{C \in \mathrm{Cat}_n} 2^{\mathrm{bcc}(C)} = 2^{\binom{n}{2}}.$$

Lattice of catalan triangles

Catalan triangles ordered coordinatewise form a distributive lattice:



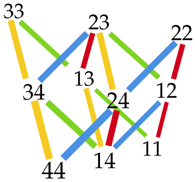


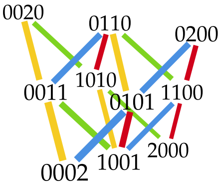
Let $\mathcal{P}$ be a poset. $\mathcal{P}^n/\mathfrak{S}_n$ is the poset of n-tuples of elements of $\mathcal{P}$ up to permutation.
Elements $\overline{x} \in \mathcal{P}^n/\mathfrak{S}_n$ can be encoded by vectors $(\alpha_k(\overline{x}))_{k\in\mathcal{P}}$ where $\alpha_k(\overline{x})$ is the number of occurences of k in x.

Proposition (Hivert, Pilaud, S. '25)

Let $\overline{x}, \overline{y} \in \mathcal{P}^n/\mathfrak{S}_n$. We have $\overline{x} \leq \overline{y}$ if and only if for all upper sets I of $\mathcal{P}$,

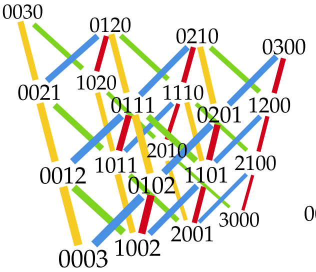
$$\sum_{k\in I} \alpha_k(\overline{x}) \leq \sum_{k\in I} \alpha_k(\overline{y}).$$

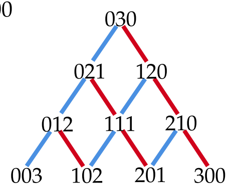


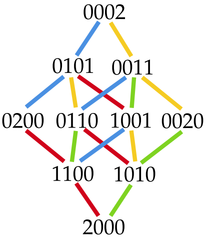


Proposition (Hivert, Pilaud, S. '25)

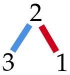
Posets $\mathcal{J}(\mathrm{ASM}_n)$, $\mathcal{J}(\mathrm{Dyck}_n)$ and $\mathcal{J}(\mathrm{Cat}_n)$ are isomorphic to $\mathcal{P}^{n-2}/\mathfrak{S}_{n-2}$ for some poset $\mathcal{P}$.





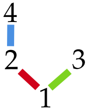


$\mathcal{P} =$ 

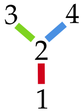
$\mathcal{P} =$ 

$\mathcal{P} =$ 

Other posets $\mathcal{P}$ give interesting lattices $\mathrm{Low}(\mathcal{P}^n/\mathfrak{S}_n)$:

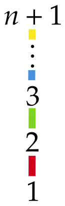


Magog triangles

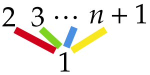


Gapless Gog and Magog triangles

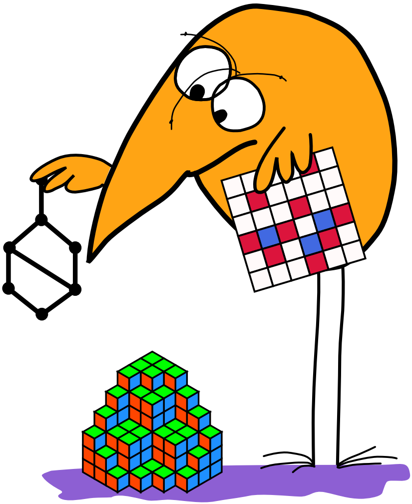
GT triangles with bottom row 12 $\cdots$ n



n – dimensional totally symetric partitions



Subpartitions of a n – dimensional stair – shaped partition



THANKS FOR YOUR ATTENTION!