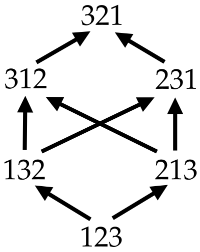
# Middle orders : all distributive lattices between weak and Bruhat orders

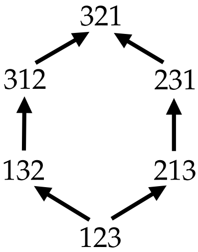
Ludovic Schwob

LIGM, Université Gustave Eiffel

SLC 94

We consider the Bruhat and (right) weak orders on $\mathfrak{S}_n$:




In the Bruhat order $a < b \implies \cdots a \cdots b \cdots \leq \cdots b \cdots a \cdots$
In the weak order $a < b \implies \cdots ab \cdots \leq_R \cdots ba \cdots$

**Definition (Inversion sequence)**

*For $\sigma \in \mathfrak{S}_n$, its inversion sequence is $I(\sigma) = (i_1, \dots, i_n)$ where*

$$i_k = \#\{j < k \mid \sigma^{-1}(j) > \sigma^{-1}(k)\}.$$

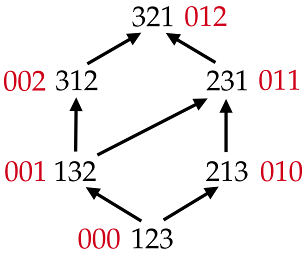
For example, $I(4317256) = (0,0,2,3,0,0,3)$. Inversions sequences are in bijection with permutations, and they are exactly the sequences $(i_1, \dots, i_n)$ such that for all $1 \leq k \leq n$, $0 \leq i_k < k$.

**Proposition (Bouvel, Ferrari and Tenner 2024)**

*$\mathfrak{S}_n$ ordered by the coordinatewise comparison of inversion sequences is a distributive lattice.*

The middle order is contained in the Bruhat order and contains the weak order:

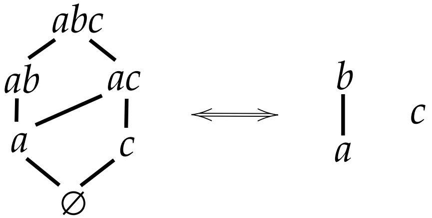
$$\sigma \leq_R \tau \implies I(\sigma) \leq I(\tau) \implies \sigma \leq \tau$$



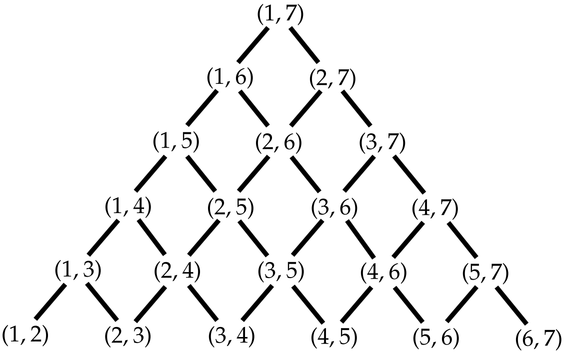
Distributive lattices can be encoded by posets:

**Theorem (Birkhoff 1937)**

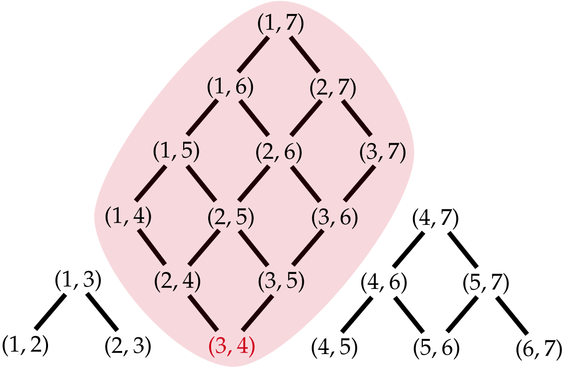
*Elements of a finite distributive lattice* L *are in bijection with lower sets of* Irr(L), *i.e. the poset of join-irreducible elements of* L.



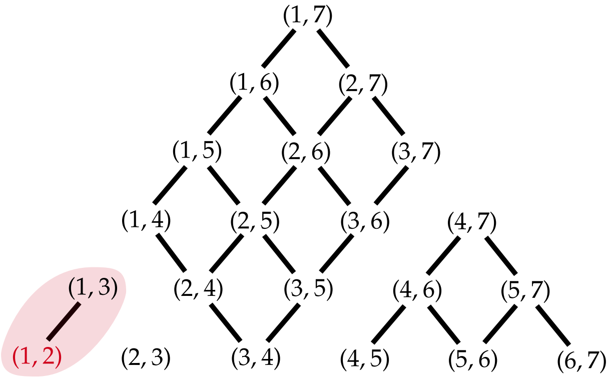
We start by constructing the posets whose ideals are in bijection with permutations. For this, consider the poset of inversions $(i, j)$ ordered by $(i_1, j_1) \leq (i_1, j_2)$ if $i_1 \geq i_2$ and $j_1 \leq j_2$:



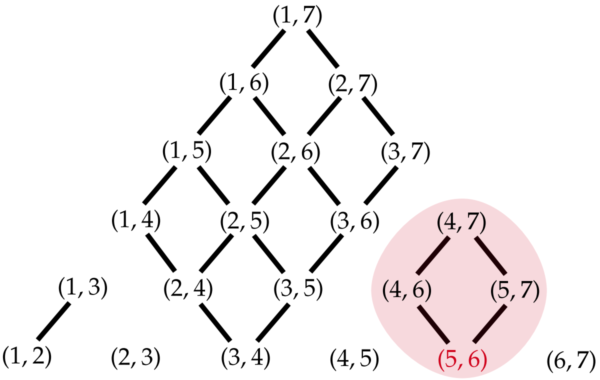
Pick any minimal element and separate the elements which are above it from the rest of the poset:



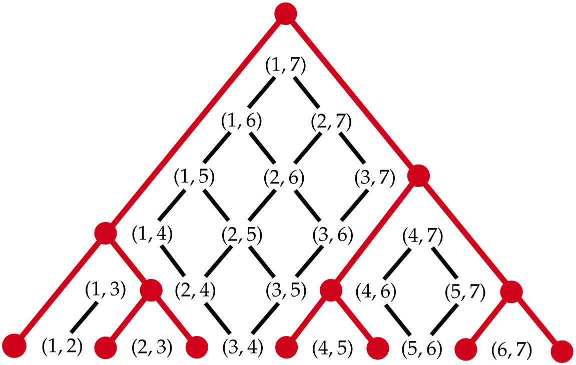
Do the same with another minimal element...



…until all minimal elements belong to different connected components.



The minimal elements can be picked in $(n-1)!$ different ways, but we get only $C_{n-1}$ different partitions of the poset, for they are in bijection with binary trees.



Permutations are encoded by their inversion sets, which are characterized by the property that for all $i < j < k$:

- $(i,j) \in I$ and $(j,k) \in I \implies (i,k) \in I$;
- $(i,k) \in I \implies (i,j) \in I$ or $(j,k) \in I$.

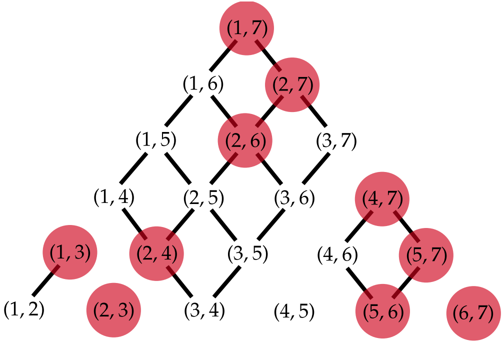
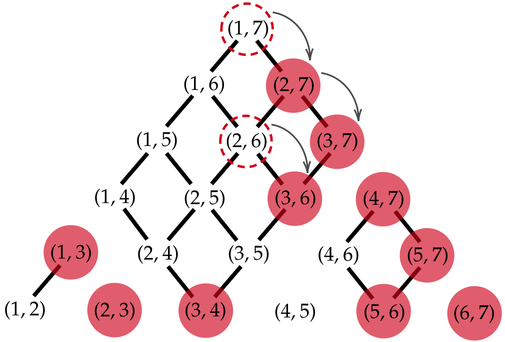
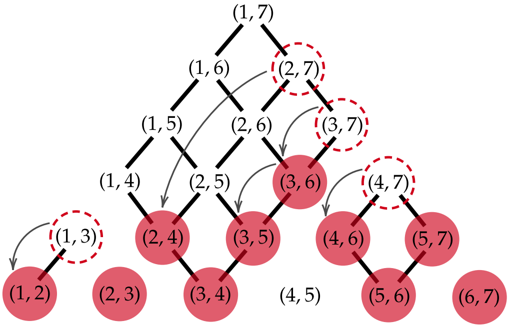



Figure: The inversion set of 3714625, inside the poset we constructed.

Let the inversions fall to the right inside each rectangular poset:

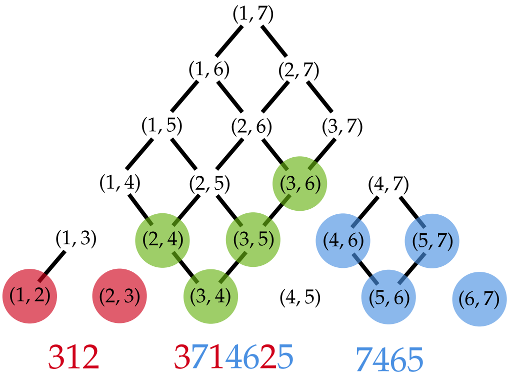


Then let the inversions fall to the left:



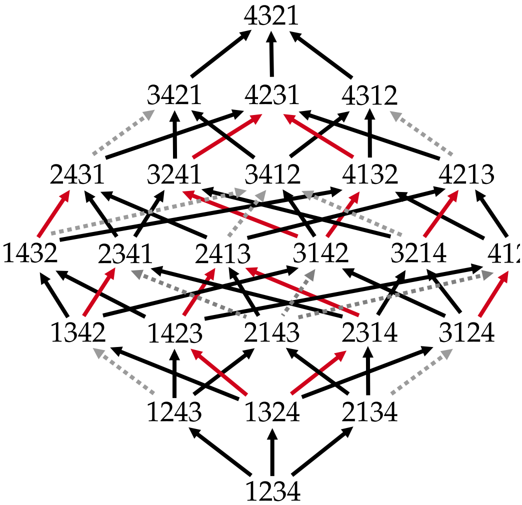

Letting the inversions fall to the left and then to the right would give the same result.

Lower sets of rectangular posets encode how two subpermutations are shuffled:

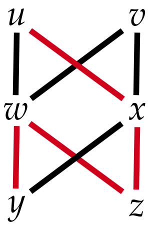



This gives us a bijection between permutations and ideals of $\mathcal{P}_T$.

When ordering permutations by the inclusion of the corresponding lower sets of $\mathcal{P}_T$, we endow $\mathfrak{S}_n$ with the structure of a distributive lattice. This lattice contains the weak order and is contained in the Bruhat order.



To prove that our middle orders are the only distributive lattices between the weak and Bruhat orders, we consider the edges of the Bruhat order which are not in the weak order, and how we can add them to make a distributive lattice.



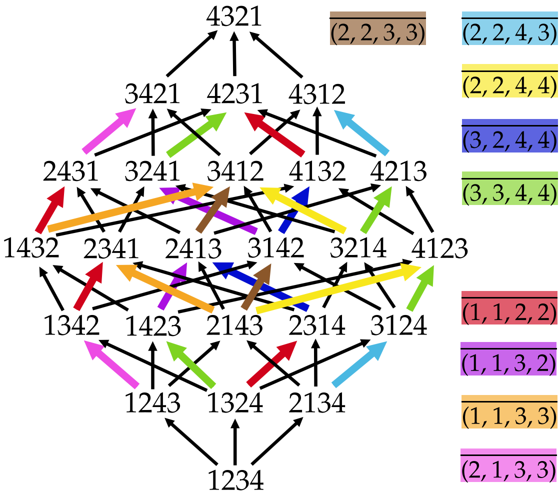

$$(x,u) \implies (z,w) \wedge (z,x) \oplus (y,w)$$

$$[(z,w) \wedge (z,x)] \oplus (y,w)$$

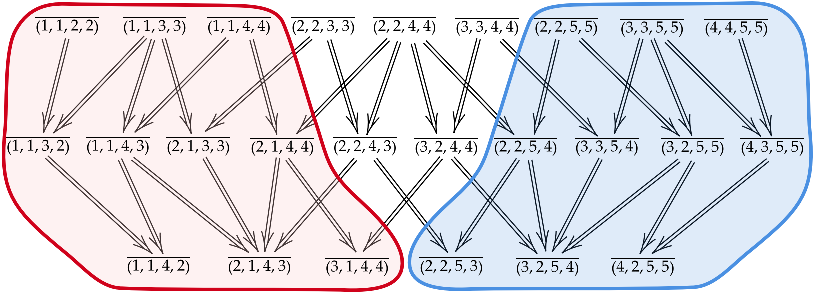
$$(y,w) \implies \neg(x,u)$$

$$(z,w) \wedge (z,x) \implies \neg(x,u)$$

Edges which can be added are grouped into equivalence classes:



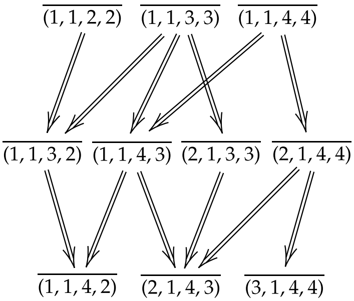
These equivalence classes are ordered by implications:

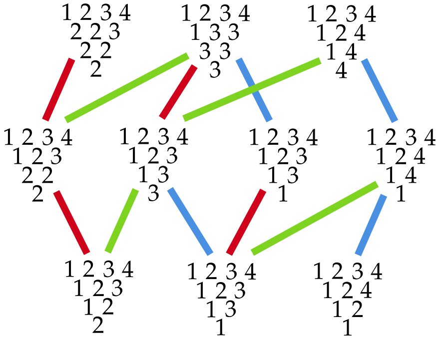



For all $i < j < k$, we have $\overline{(i, 1, k-1, j)} \oplus \overline{(j, i+1, n, k)}$.

# Proof of the exhaustivity of our construction

The posets of left and right edges are isomorphic to the poset of irreducible elements of the lattice of Gelfand-Tsetlin triangles with first line $12...n$:



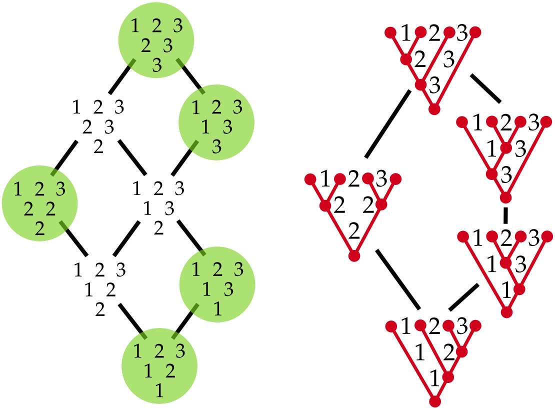



For all $i < j < k < \ell$, we have

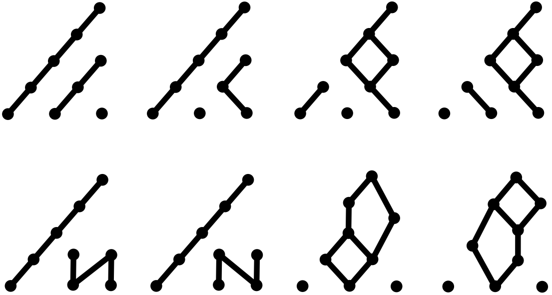
$$\overline{(i,1,k-1,j)} \wedge \overline{(k,j+1,n,\ell)} \iff \overline{(i,1,\ell-1,j)} \wedge \overline{(k,i+1,n,\ell)}$$

which translates on Gelfand-Tsetlin triangles as

$$X_{i,j} = k \implies X_{k+i-j,k} = X_{j+n-k,j} = k.$$



Irreducible posets of middle orders on $B_3$:

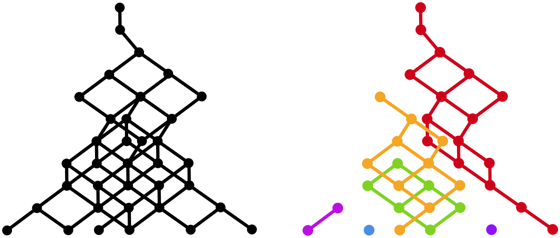


Let $(W, S)$ be a Coxeter group, and $W_J$ the subgroup of $W$ generated by $J \subset S$. We have the inclusions of orders

$$(W, \leq_R) \subset (W_J, \leq_R) \times (W^J, \leq_L) \subset (W_J, \leq) \times (W^J, \leq) \subset (W, \leq).$$

where $W^J$ is the set of minimal elements of cosets in $W/W_J$.

If $W$ is a Weyl group, $s$ a minuscule root and $J = S \setminus \{s\}$, the weak and Bruhat orders on $W^J$ are distributive lattices, isomorphic to $\mathrm{Low}(\mathcal{P}_s)$, where $\mathcal{P}_s$ is the upper set of $s$ in the root poset.
Using this we construct *minuscule middle orders* whose irreducible posets are partitions of the root poset into minuscule posets.



Let $\mathcal{P}$ be a finite poset, $F_m(\mathcal{P}, q)$ is the polynomial in $q$ giving the graduation of $\mathrm{Low}(\mathcal{P} \times [m])$ where $[m]$ is a chain with $m$ elements.

**Proposition**

*Let* L *be a minuscule middle order on* $W$ *with irreducibles poset* $\mathcal{R}$, *we have*

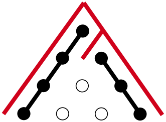
$$F_m(\mathcal{R}, q) = \prod_{\alpha \in \Phi^+} \frac{1 - q^{m + r_\alpha}}{1 - q^{r_\alpha}}$$

*where* $r_\alpha - 1$ *is the rank of* $\alpha$ *in the root poset.*

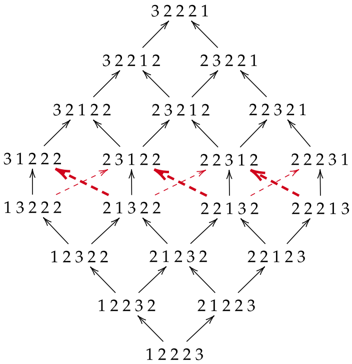
Minuscule middle orders are self-dual, and the number of self-dual ideals of $\mathcal{R} \times [m]$ is equal to $F_m(\mathcal{R}, -1)$.

# Other middle orders

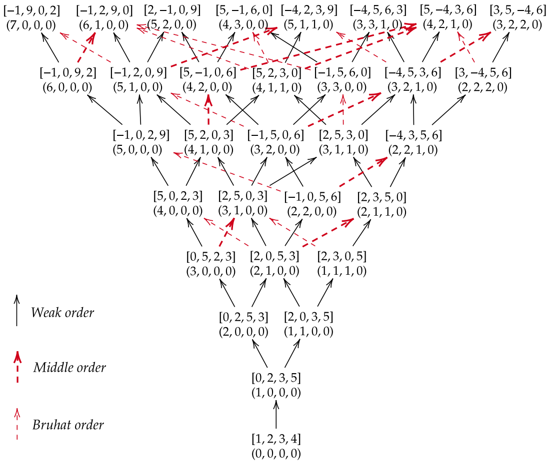
Middle orders on parabolic quotients of the symmetric group are also in bijection with binary trees:

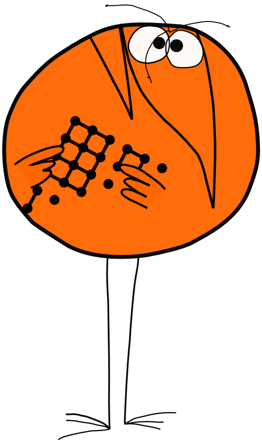


$\mathfrak{S}_5 \,/\, \langle s_2, s_3 \rangle$



We can also consider middle orders on affine Coxeter groups.





THANKS FOR YOUR ATTENTION!