Heaps of rhombic dodecahedra, catalan congruences on alternating sign matrices, and bases of the Temperley–Lieb algebra

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The Bruhat order and the excedance relation

The **Bruhat order** is the poset on the symmetric group $\mathfrak{S}_n$ whose order is defined by the exchange of values in increasing order:

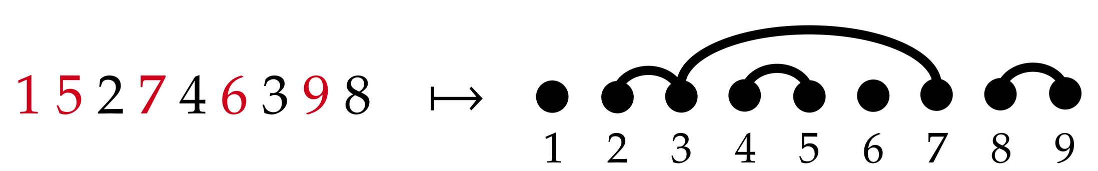
$$a < b \implies \cdots a \cdots b \cdots \leq \cdots b \cdots a \cdots$$

In [BG24], Bergeron and Gagnon defined the **excedance relation** on $\mathfrak{S}_n$ defined by $\sigma \equiv \tau$ if and only if $\mathrm{E}_{\mathrm{pos}}(\sigma) = \mathrm{E}_{\mathrm{pos}}(\tau)$ and $\mathrm{E}_{\mathrm{val}}(\sigma) = \mathrm{E}_{\mathrm{val}}(\tau)$, where $\mathrm{E}_{\mathrm{pos}}(\sigma) := \{i : i \leq \sigma_i\}$ and $\mathrm{E}_{\mathrm{val}}(\sigma) := \{\sigma_i : i \leq \sigma_i\}$.

Equivalence classes of $\equiv$ are in bijection with **noncrossing partitions**:

$$C_\lambda := \{\sigma \in \mathfrak{S}_n : \mathrm{E}_{\mathrm{pos}}(\sigma) = [n] \setminus \lambda^- \text{ and } \mathrm{E}_{\mathrm{val}}(\sigma) = [n] \setminus \lambda^+\}$$

where λ^- (resp. λ^+) contains positions of ingoing (resp. outgoing) arcs of λ.

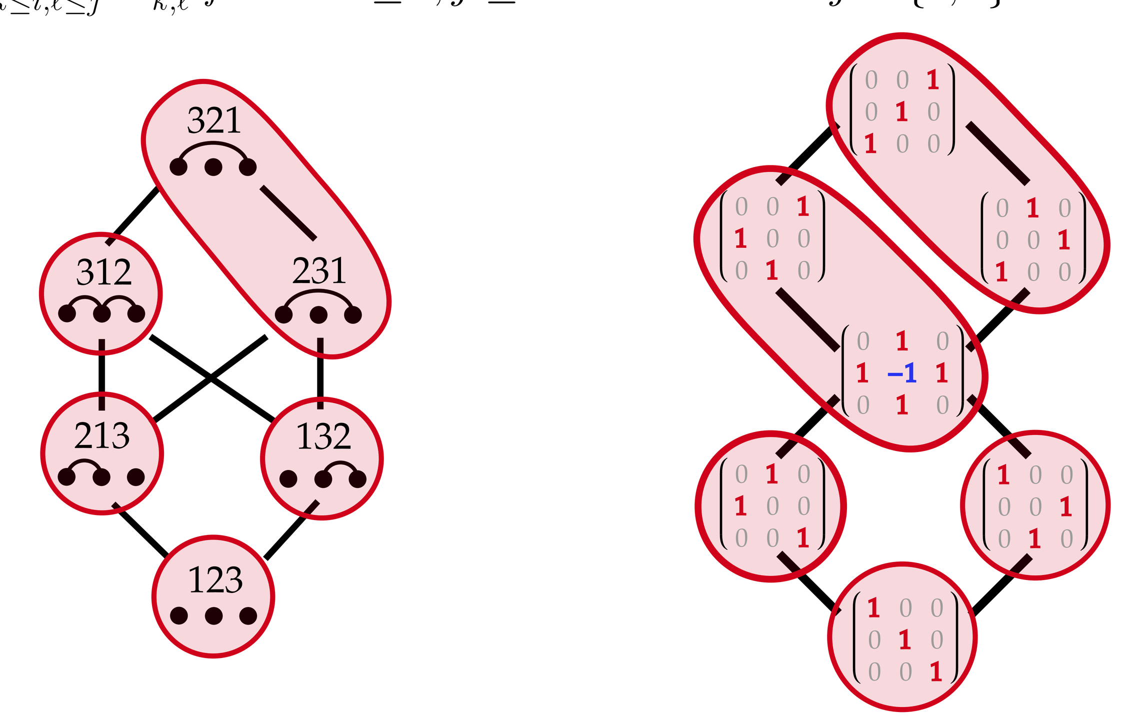


Bergeron and Gagnon proved that each class C_λ is an interval of the Bruhat order, and the induced order on these intervals is isomorphic to the Stanley lattice $\mathbf{Dyck}_n$, *i.e.*, the lattice of Dyck paths of length $2n$ ordered by inclusion.

The excedance relation on ASMs

An **alternating sign matrix (ASM)** is a matrix with entries in $\{-1, 0, 1\}$, such that nonzero entries alternate between 1 and -1 and sum to 1 on each row and column. ASMs of size n form a lattice $\mathbf{ASM}_n$ which is the Macneille-Dedekind completion of the Bruhat order on $\mathfrak{S}_n$.

Proposition. *The excedance relation $\equiv$ extends to a lattice congruence of $\mathbf{ASM}_n$, where $M \equiv M'$ if and only if $\sum_{k \leq i, \ell \leq j} M_{k,\ell} = \sum_{k \leq i, \ell \leq j} M'_{k,\ell}$ for all $1 \leq i, j \leq n$ such that $i - j \in \{0, 1\}$.*



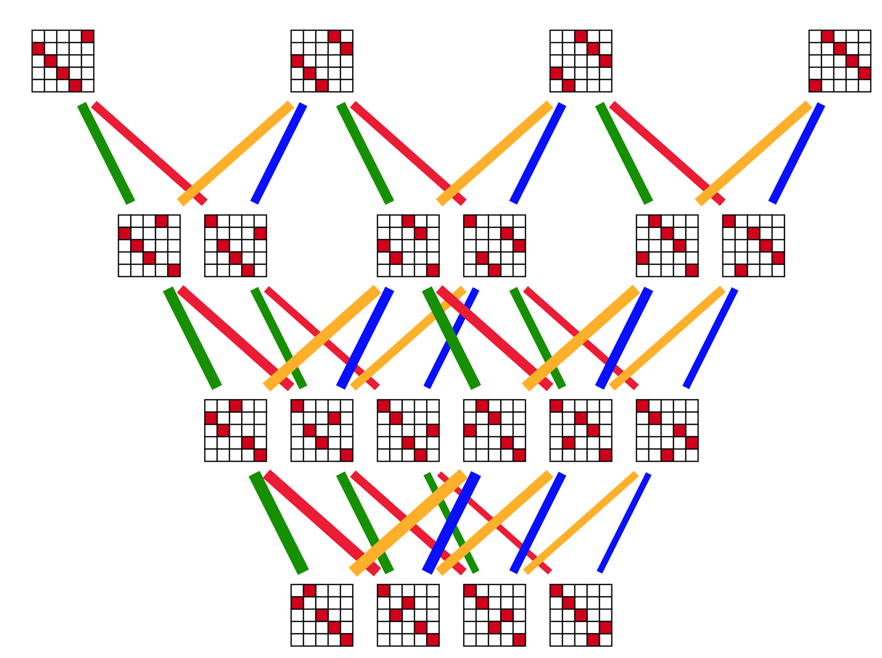
References

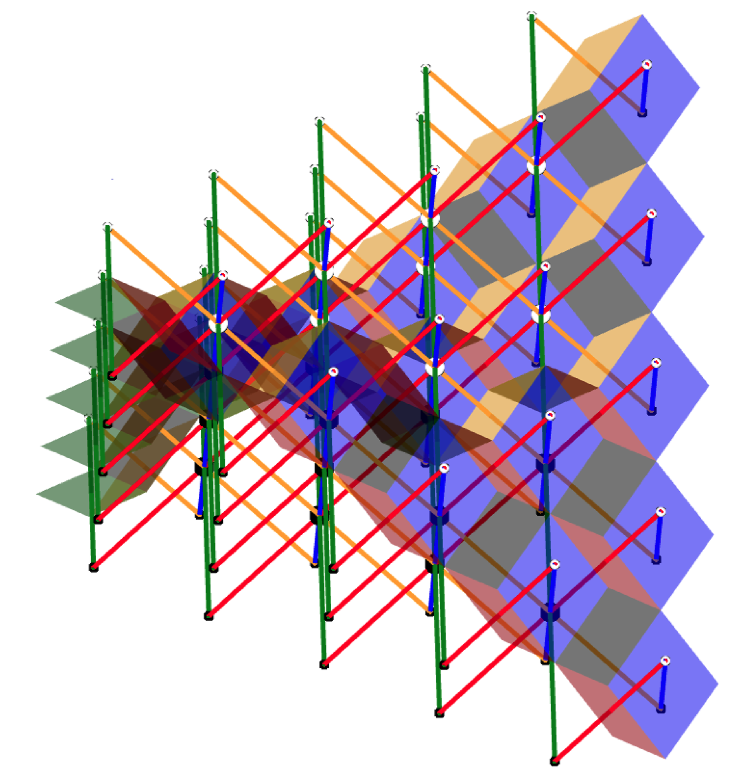
[BG24] N. Bergeron and L. Gagnon. The excedance quotient of the Bruhat order, quasisymmetric varieties, and Temperley-Lieb algebras. *J. Lond. Math. Soc. (2)*, 110(4):Paper No. e13007, 28, 2024.

[Str11] J. Striker. A unifying poset perspective on alternating sign matrices, plane partitions, Catalan objects, tournaments, and tableaux. *Adv. in Appl. Math.*, 46(1-4):583–609, 2011.

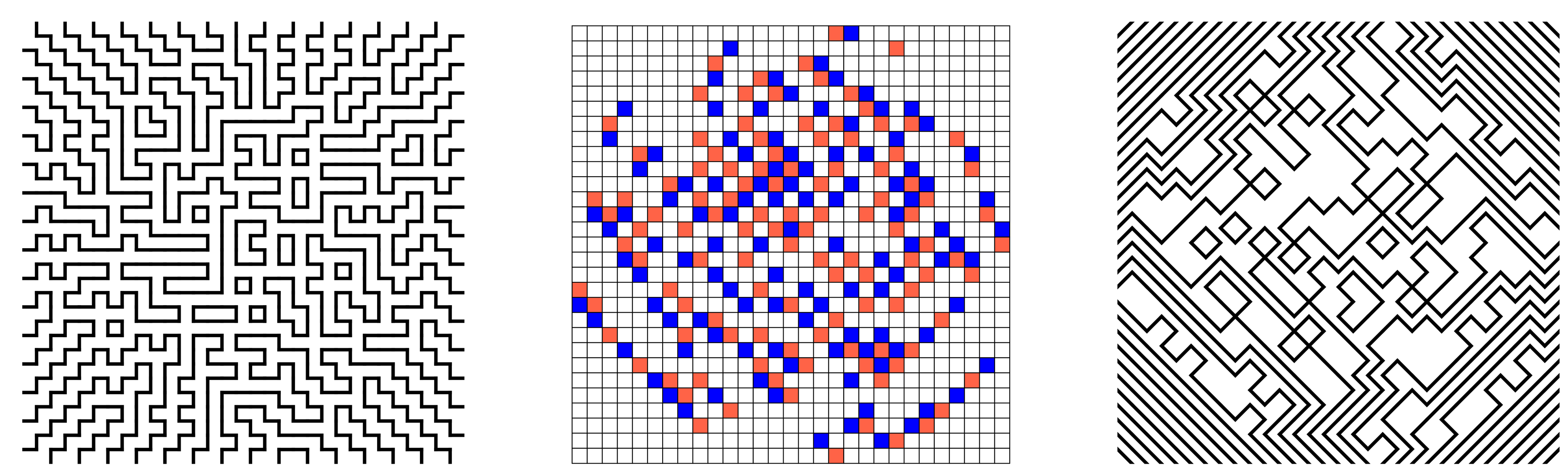
The lattice of ASMs

$\mathbf{ASM}_n$ is a distributive lattice whose join-irreducible elements form a tetrahedral poset $\mathcal{T}_n$. ASMs can be represented as the boundary between the corresponding lower set of $\mathcal{T}_n$ and its complement.





ASMs can then be seen as stackings of rhombic dodecahedra. The vertical projection of this 3D model gives objects in bijection with ASMs, such as the well-known fully packed loops (left), and new models obtained by recording peaks and pits (middle), or cliffs (right).

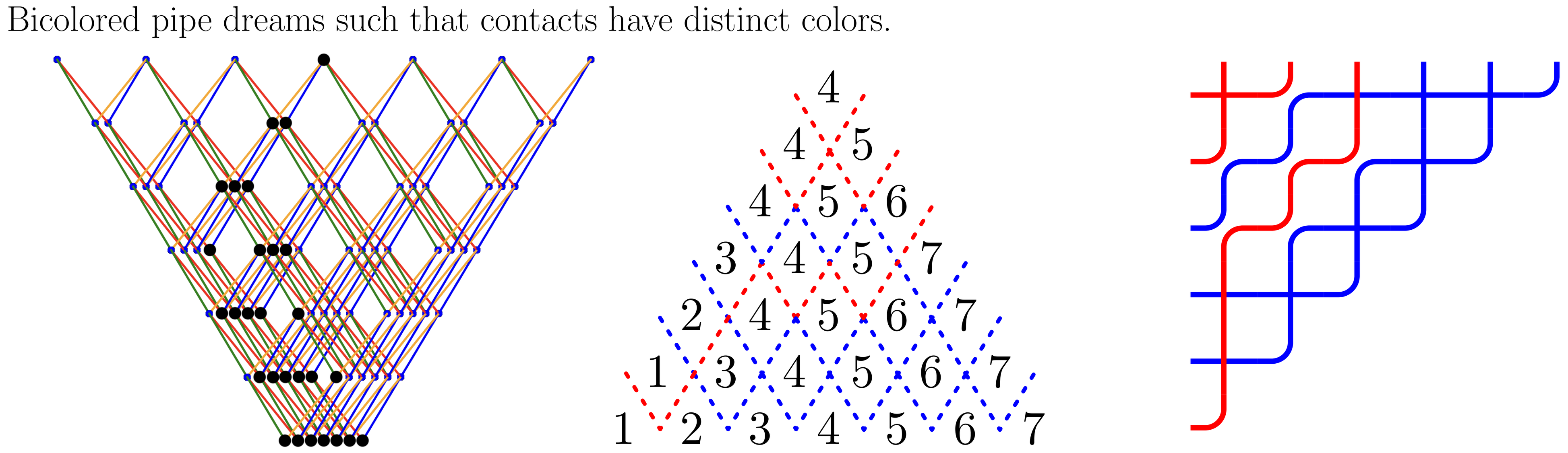


Catalan congruences

The lattice $\mathbf{Dyck}_n$ is also distributive, and its poset of join-irreducible elements is a triangular poset $\mathcal{D}_n$.

Catalan congruences, *i.e.*, quotients of $\mathbf{ASM}_n$ isomorphic to $\mathbf{Dyck}_n$, are in bijection with:

- Subposets of $\mathcal{T}_n$ isomorphic to $\mathcal{D}_n$;
- **Catalan triangles** of size $n-1$, *i.e.*, Gelfand-Tsetlin patterns $(X_{i,j})_{1 \leq i \leq j \leq n-1}$ with first row $12\ldots(n-1)$ which are doubly gapless: $X_{i,j} - X_{i+1,j} \leq 1$ and $X_{i+1,j+1} - X_{i,j} \leq 1$ for all $1 \leq i \leq j < n-1$;
- Bicolored pipe dreams such that contacts have distinct colors.



The number of catalan congruences on $\mathbf{ASM}_n$ ($n \geq 1$) is: 1, 1, 2, 6, 28, 202, 2252, 38756, 1028964, 42127054...

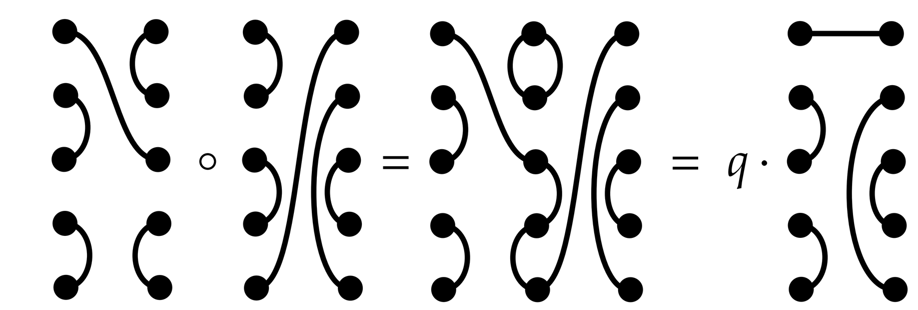
The excedance congruence corresponds to an undulating triangular subposet, or equivalently to a bicolored pipe dreams with only contacts.

Bases of the Temperley-Lieb algebra

Proposition. *Let $\equiv$ be a catalan congruence on $\mathbf{ASM}_n$. The maximal element of each congruence class of $\equiv$ is a **covexillary permutation*** (i.e., 3412-avoiding).

*Each congruence class of $\equiv$ contains a unique minimal permutation which is 321-**avoiding**.*

The **Temperley-Lieb algebra** $\mathrm{TL}_n(q)$ can be defined using this diagrammatic description:



It is well-known that $\mathrm{TL}_n(2)$ is isomorphic to

$$\mathbb{C}\mathfrak{S}_n / \langle () - (ij) - (jk) - (ik) + (ijk) + (ikj) \mid 1 \leq i < j < k \leq n \rangle,$$

and that 321-avoiding permutations form a basis of this algebra.

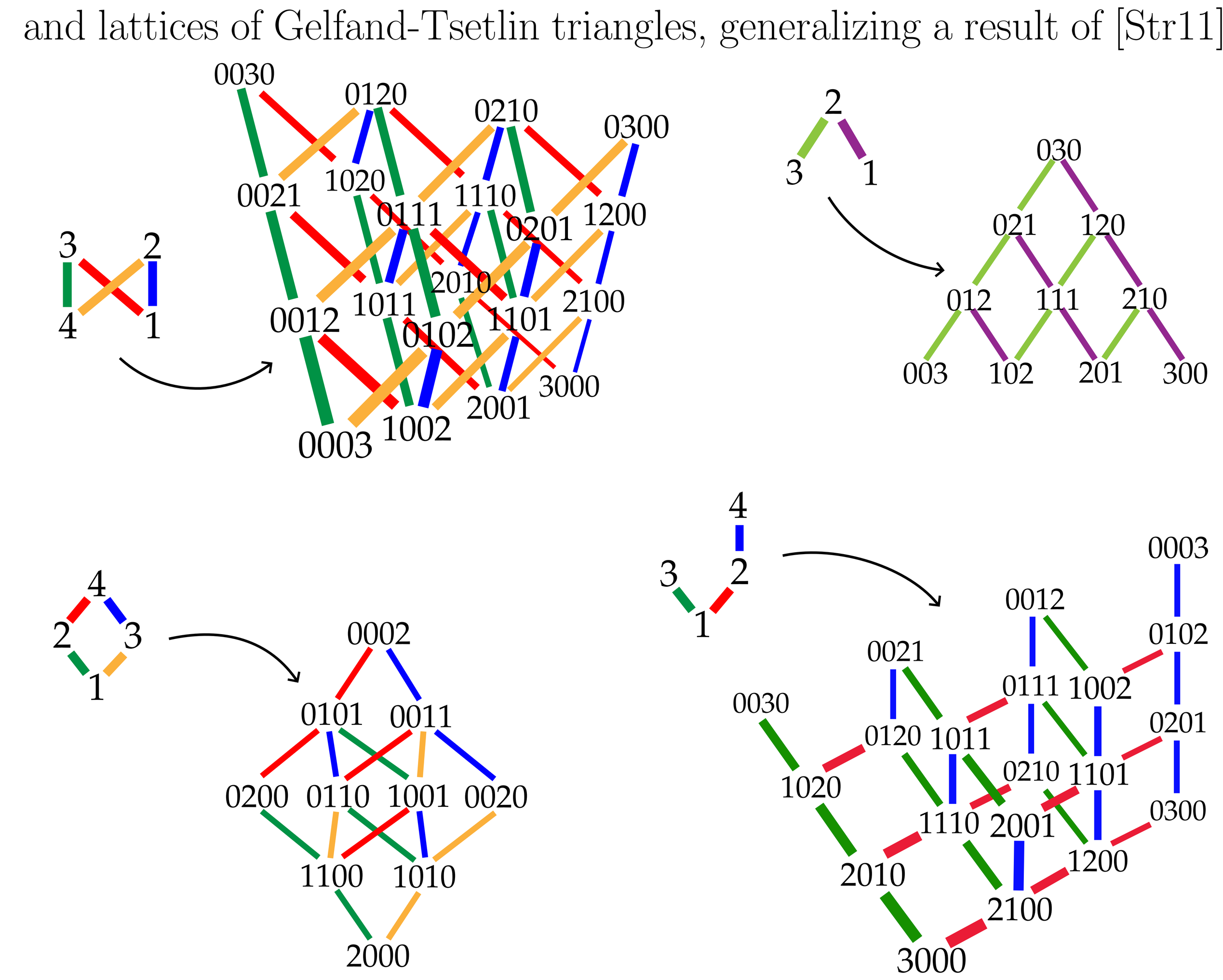
We generalize Theorem 5.3 of [BG24] to all catalan congruences:

Theorem. *Let $\equiv$ be a catalan congruence on $\mathbf{ASM}_n$. Any choice of representative permutations in the congruence classes of $\equiv$ forms a basis of* $\mathrm{TL}_n(2)$.

Posets $\mathcal{P}^n/\mathfrak{S}_n$

We consider the set $\mathcal{P}^n/\mathfrak{S}_n$ of tuples of $\mathcal{P}^n$ up to permutation, ordered by $\bar{\boldsymbol{x}} \leq \bar{\boldsymbol{y}}$ if there exists $\boldsymbol{x} \in \bar{\boldsymbol{x}}$ and $\boldsymbol{y} \in \bar{\boldsymbol{y}}$ such that $\boldsymbol{x} \leq \boldsymbol{y}$ coordinate-wise.

$\mathbf{ASM}_n$, $\mathbf{Dyck}_n$ and $\mathbf{Cat}_n$ are all isomorphic to $\mathbf{Low}(\mathcal{P}^{n-2}/\mathfrak{S}_{n-2})$ for some poset $\mathcal{P}$. Other examples include notably the lattice of TSSCPPs, and lattices of Gelfand-Tsetlin triangles, generalizing a result of [Str11].



Elements of $\mathcal{P}/\mathfrak{S}_n$ can be represented by barycentric coordinates $(|\boldsymbol{x}|_p)_{p \in \mathcal{P}}$, where $|\boldsymbol{x}|_p$ is the number of of occurences of p in $\boldsymbol{x}$. We have $\boldsymbol{x} \leq \boldsymbol{y}$ in $\mathcal{P}^n/\mathfrak{S}_n$ if and only if $\sum_{p \in U} |\boldsymbol{x}|_p \leq \sum_{p \in U} |\mathbf{y}|_p$ for all upper sets U of $\mathcal{P}$.