Middle orders : all distributive lattices between weak and Bruhat orders

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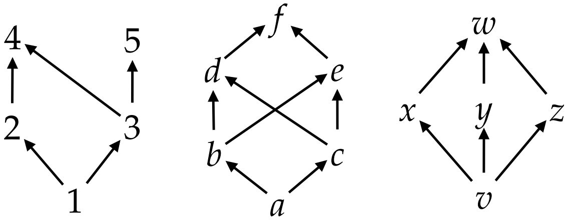
Séminaire combinatoire du LIX, January 19, 2026

A *poset* is a set endowed with a partial order relation $\leq$.

A *lattice* is a poset such that for all pairs x, y: there exists:

- a smallest element $x \vee y$ such that $x \leq x \vee y$ and $y \leq x \vee y$ (the *join*);
- a biggest element $x \wedge y$ such that $x \wedge y \leq x$ and $x \wedge y \leq y$ (the *meet*).

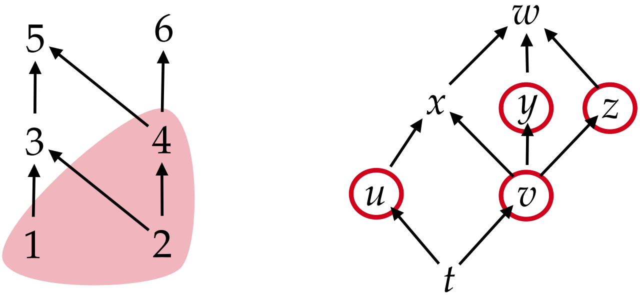
In particular finite lattices have a minimal and a maximal element.



A *lower set* (or *order ideal*) of a poset $\mathcal{P}$ is a subset $X \subset \mathcal{P}$ such that

$$\forall x \in X,\ y \leq x \implies y \in X.$$

A *join-irreducible* element of a lattice $\mathbb{L}$ is an element covering only one element.



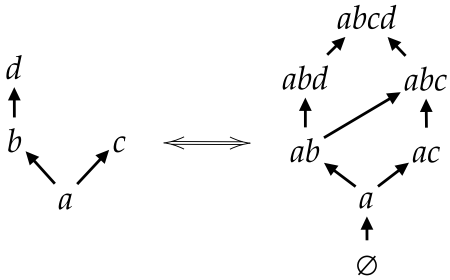
A lattice is *distributive* if the operations $\vee$ and $\wedge$ distribute over each other:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Distributive lattices can be encoded nicely by smaller posets:

Theorem (Birkhoff 1937)

Elements of a finite distributive lattice $\mathbb{L}$ are in bijection with lower sets of $\mathrm{Irr}(\mathbb{L})$, *i.e. the poset of join-irreducible elements of $\mathbb{L}$.*



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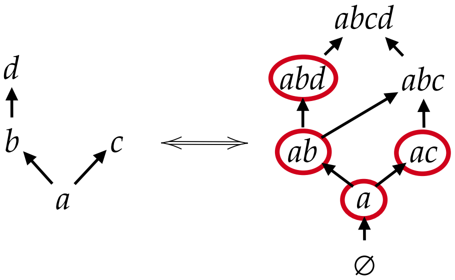


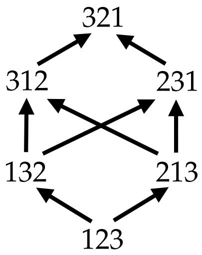
Many operations on distributive lattices translate nicely on their posets of irreducible elements:

- product of lattices $\iff$ disjoint union of posets;
- quotients of lattice $\mathbb{L}$ $\iff$ subposets of $\mathrm{Irr}(\mathbb{L})$;
- sublattices of $\mathbb{L}$ $\iff$ refinements of $\mathrm{Irr}(\mathbb{L})$.

Distributive lattices are great.
Be more distributive.

Definition (Bruhat order)

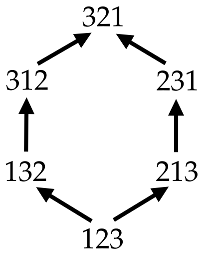
Let $\sigma, \tau \in \mathfrak{S}_n$. We let $\sigma \leq \tau$ if σ can be obtained from τ by exchanging two of its values, and if $\ell(\sigma) < \ell(\tau)$. We then define the Bruhat order on $\mathfrak{S}_n$ as the transitive closure of this relation, i.e. $\sigma \leq \tau$ if there exists $u_0, \dots, u_i \in \mathfrak{S}_n$ such that $\sigma = u_0 \leq u_1 \leq \dots \leq u_i = \tau$.



Definition (Weak order)

Let $\sigma, \tau \in \mathfrak{S}_n$. We let $\sigma \leq_R \tau$ if σ can be obtained from τ by exchanging two adjacent values, and if $\ell(\sigma) < \ell(\tau)$. We then define the (right) weak order on $\mathfrak{S}_n$ as the transitive closure of this relation.

The weak order on $\mathfrak{S}_n$ is a lattice (non distributive for $n \geq 3$).

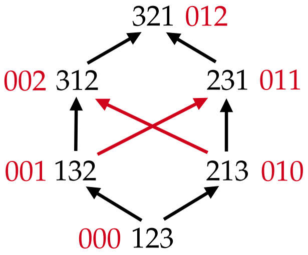


Definition (Inversion sequence)

For $\sigma \in \mathfrak{S}_n$, its inversion sequence is $I(\sigma) = (i_1, \ldots, i_n)$ where

$$i_k = \#\{j < i \mid \sigma^{-1}(j) > \sigma^{-1}(i)\}.$$

For example, $I(4317256) = (0, 0, 2, 3, 0, 0, 3)$. Inversions sequences are in bijection with permutations, and they are exactly the sequences $(i_1, \ldots, i_n)$ such that for all $1 \leq k \leq n$, $0 \leq i_k < k$.



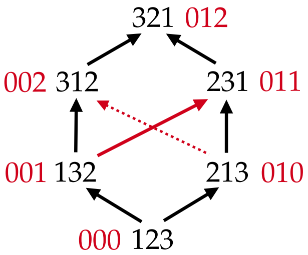
Proposition (Bouvel, Ferrari and Tenner 2024)

$\mathfrak{S}_n$ ordered by the coordinatewise comparison of inversion sequences is a distributive lattice.

Proposition (Bouvel, Ferrari and Tenner 2024)

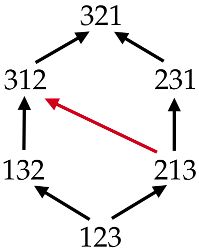
The middle order is contained in the Bruhat order and contains the weak order:

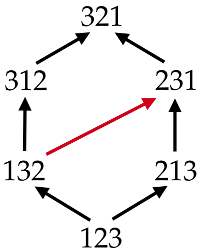
$$\sigma \leq_R \tau \implies I(\sigma) \leq I(\tau) \implies \sigma \leq \tau$$



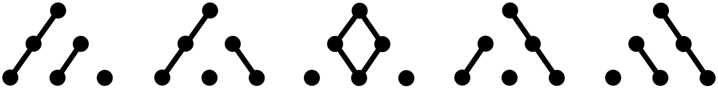
What about other middle orders ?

For $n = 3$, the exists another distributive lattice between the weak and Bruhat orders on $\mathfrak{S}_n$:





For $n = 4$, there are 5 of them, whose posets of irreducible elements are:



Computer explorations show there exists $1, 2, 5, 14...$ middle orders on $\mathfrak{S}_n$ for $n = 1, 2, 3, 4...$

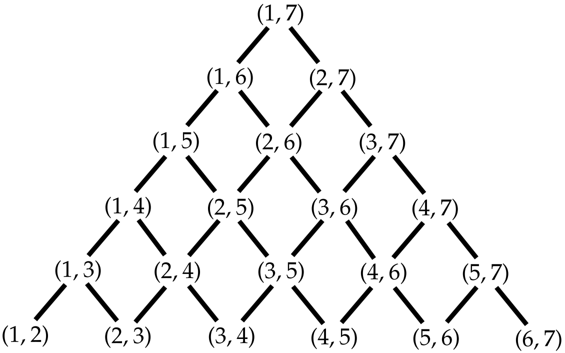
How can we construct these middle orders ? Why are they counted by Catalan numbers ? To explain this:

- We first construct irreducible posets of middle orders;
- We find a bijection between permutations and ideals of these posets.
- We then show these lattices are between the weak and Bruhat orders.

We will eventually show the middle orders we have constructed are the only distributive lattices between the weak and Bruhat orders.

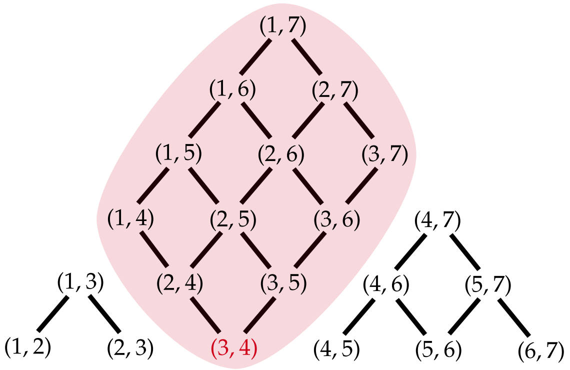
Our construction

We start by constructing the posets whose ideals will be in bijection with permutations. For this, consider the poset of pairs (i,j) ordered by $(i_1,j_1) \leq (i_2,j_2)$ if $i_1 \geq i_2$ and $j_1 \leq j_2$:

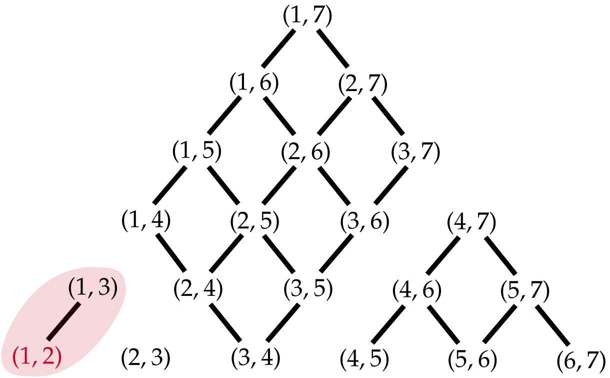


(This poset is the *root poset* of the Coxeter group A_n, more on it later)

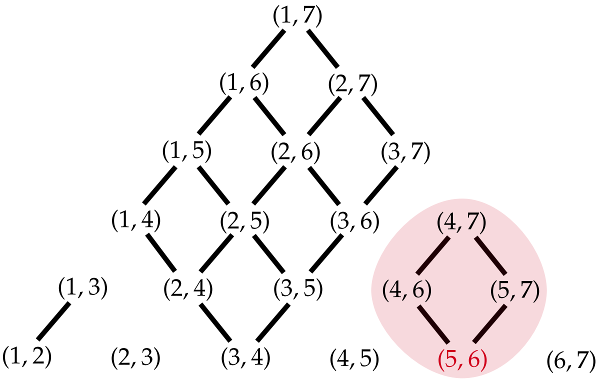
Pick any minimal element and separate the elements which are above it from the rest of the poset:



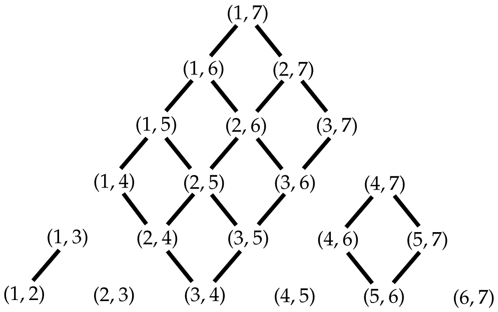
Do the same with another minimal element...



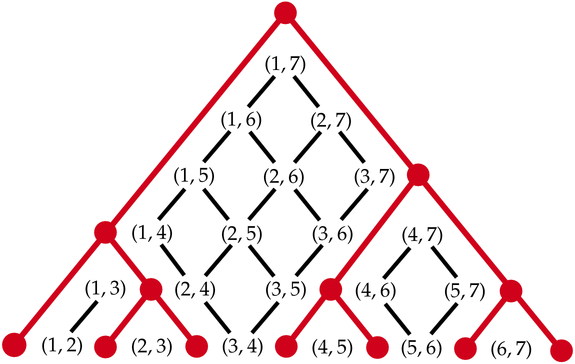
...until all minimal elements belong to different connected components.



The minimal elements can be picked in $(n-1)!$ different ways, but we get only C_{n-1} different rectangulations of the poset, for they are in bijection with permutations avoiding the pattern 231.



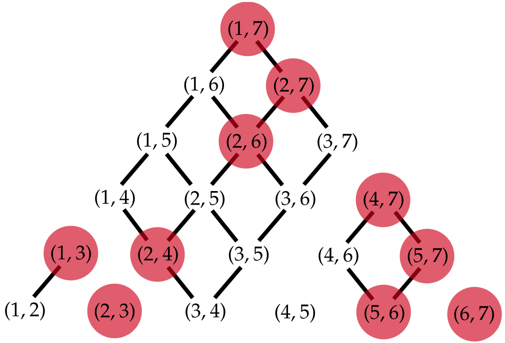
The minimal elements can be picked in $(n-1)!$ different ways, but we get only C_{n-1} different rectangulations of the poset, for they are in bijection with permutations avoiding the pattern 231 (and with binary trees).



We want to find a bijection between permutations and ideals of a given rectangulation. For this we consider inversion sets of permutations:

$$N(\sigma) := \{(i,j) : 1 \leq i < j \leq n, \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

Here is the inversion set of the permutation 3714625:

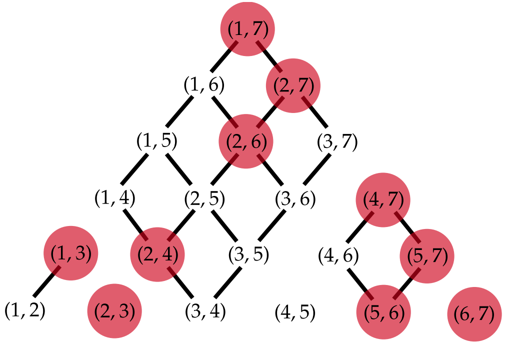


Note that the weak order is obtained by ordering permutations by the inclusion of their inversion sets:

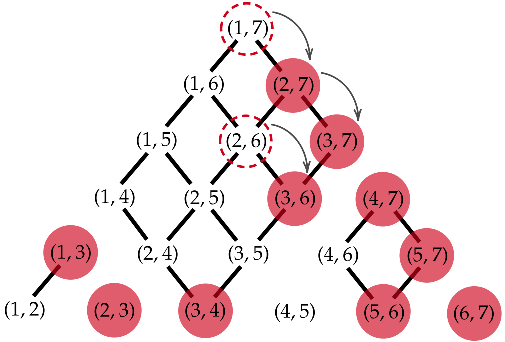
$$\sigma \leq_R \tau \iff N(\sigma) \subset N(\tau)$$

Inversion sets are characterized by the property that for all $i < j < k$:

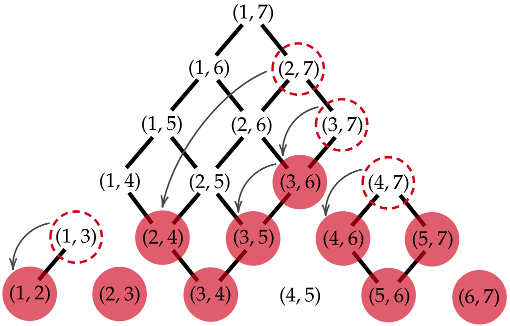
- $(i, j) \in I$ and $(j, k) \in I \implies (i, k) \in I$;
- $(i, k) \in I \implies (i, j) \in I$ or $(j, k) \in I$.



To turn the inversion set into an ideal, we let the inversions fall to the right inside each rectangular poset.

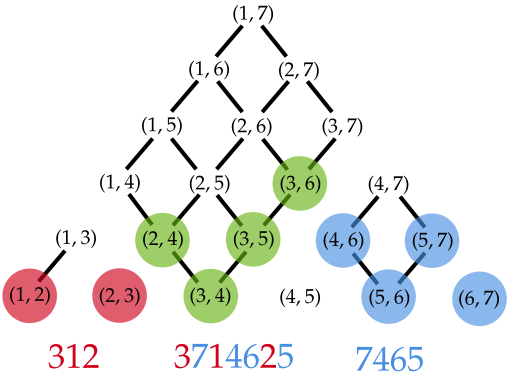


Then let the inversions fall to the left:



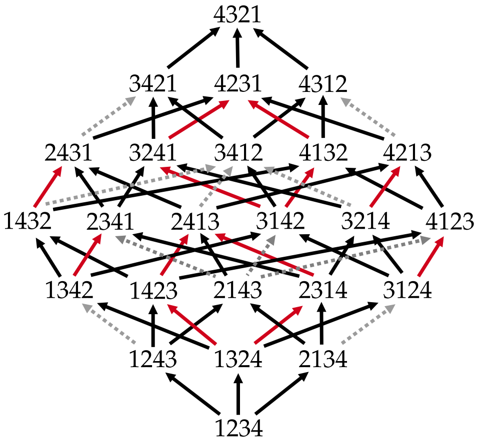
Letting the inversions fall to the left and then to the right would give the same result.

Ideals of rectangular posets encode how two subpermutations are shuffled:



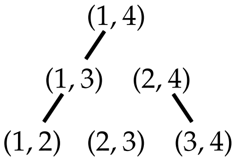
This gives us a bijection between permutations and ideals of the poset.

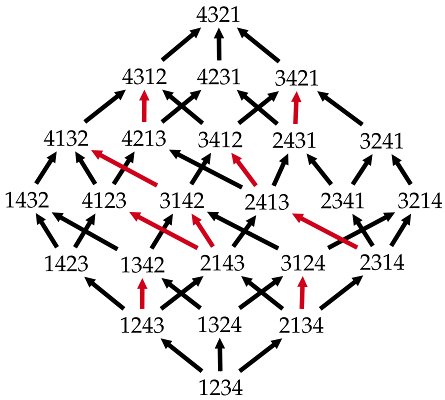
When ordering permutations by the inclusion of the corresponding ideals, we endow $\mathfrak{S}_n$ with the structure of a distributive lattice. This lattice contains the weak order and is contained in the Bruhat order.



Proposition

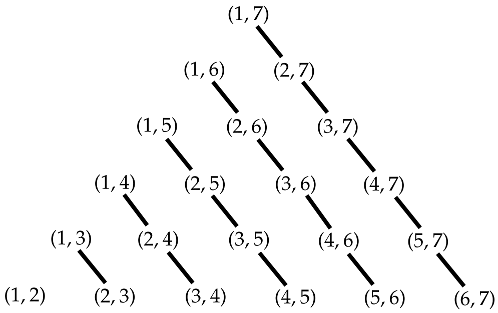
Let $\mathbb{L}$ be a middle order obtained from a rectangulation $\mathcal{R}$. We have $\sigma \lessdot \tau$ in $\mathbb{L}$ if σ and τ differ by a transposition (a, b) and for all $\sigma^{-1}(a) < k < \sigma^{-1}(b)$, $k \notin \{i, \ldots, j\}$ where (i, j) is the maximal element of the rectangle containing (a, b) in $\mathcal{R}$.



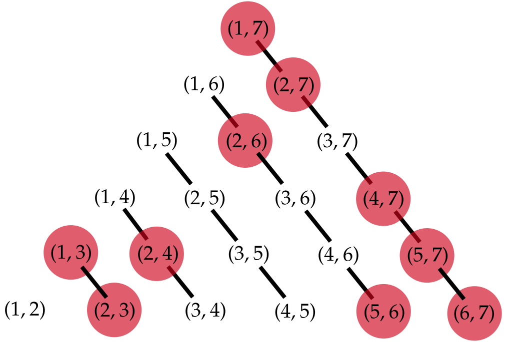


Back to the original middle order

The original middle order can be obtained with this rectangulation:



Inversion sequences give the number of inversions in each chain:

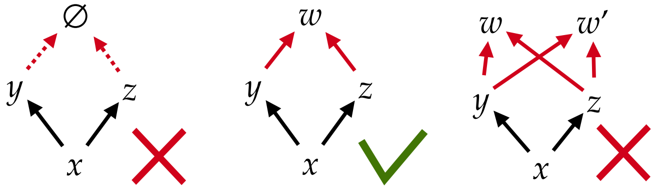


$$I(3714625) = (0,0,2,1,0,2,5)$$

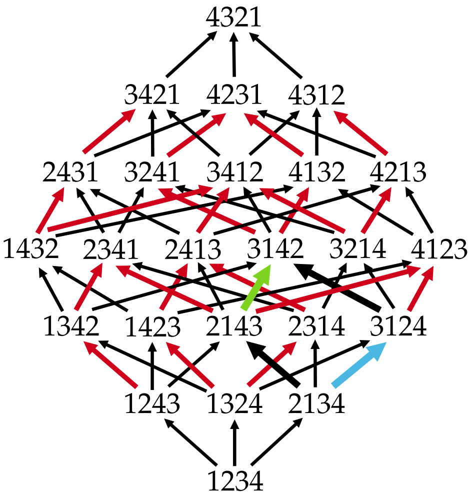
We now want to prove that our middle orders are the only distributive lattices between the weak and Bruhat orders.
For this we consider the edges of the Bruhat order which are not in the weak order, and how we can add them to make a distributive lattice.

Lemma

Let $\mathbb{L}$ be a distributive lattice and $x, y, z \in \mathbb{L}$. If y and z cover x in $\mathbb{L}$, then there exists a unique $w \in \mathbb{L}$ covering y and z (If y and z are covered by w, then there exists a unique x covered by y and z).

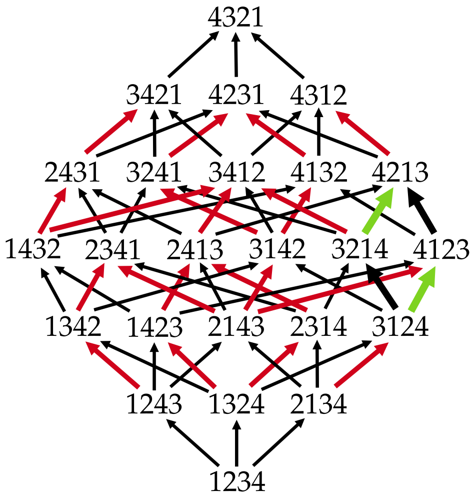


There are implication relations between edges. For example $(2143, 3142)$ implies $(2134, 3124)$:



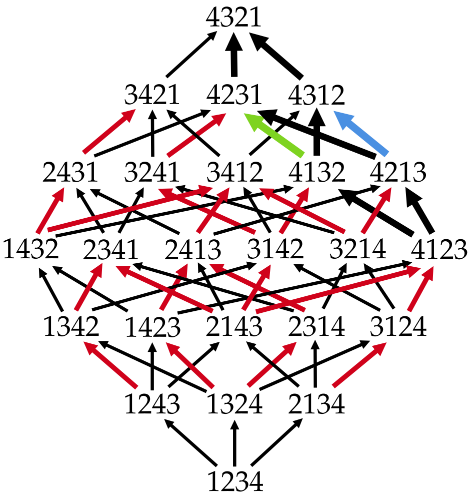
Proof of the exhaustivity of our construction

There are also equivalence relations between edges. For example $(3214, 4213)$ and $(3124, 4123)$ are equivalent:



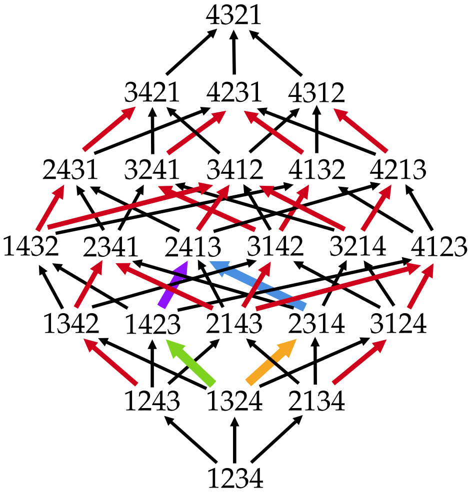
Proof of the exhaustivity of our construction

For some pairs of edges, exactly one must appear. For example one of the edges $(4132, 4231)$ and $(4213, 4312)$ must appear:



Finally, we have relations of the form

$$(1324, 1423) \wedge (1324, 2314) \iff (1423, 2413) \wedge (2314, 2413).$$



We describe equivalence classes of edges of the Bruhat order:

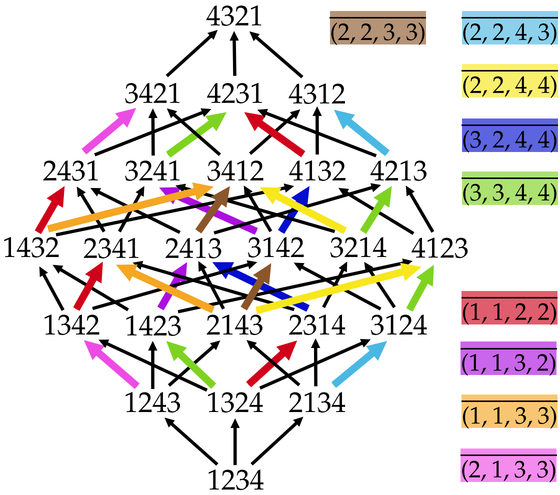
Definition

Let $1 \leq i \leq a < b \leq j \leq n$. $\overline{(a, i, j, b)}$ is the set of edges (v, w) of $(\mathfrak{S}_n, \leq)$ such that $w = v \circ (a, b)$ and $\{i, ..., j\}$ is the biggest interval containing $\{a, ..., b\}$ whose values are not between a and b in v (and w).

For example if $v = 142653$ and $w = 152643$, the edge (v, w) is in $\overline{(4, 3, 5, 5)}$. If $(v, w) \in \overline{(a, i, j, b)}$ with $i = 1$ and $j = n$, (v, w) is an edge of the weak order since no value lies between a and b.

Proposition

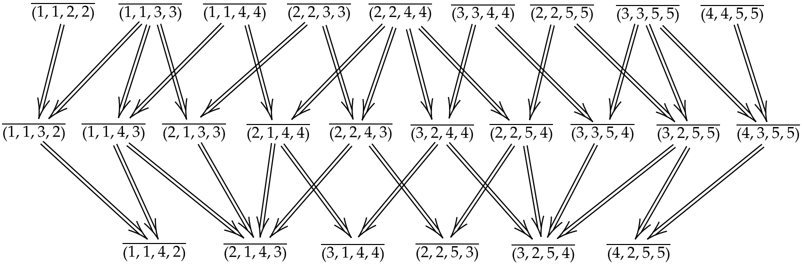
Let E be the set of edges of a middle order, $1 \leq i \leq a < b \leq j \leq n$, and $(v_1, w_1), (v_2, w_2) \in \overline{(a, i, j, b)}$. We have $(v_1, w_1) \in E$ if and only if $(v_2, w_2) \in E$.


4321
3421 4231 4312
2431 3241 3412 4132 4213
1432 2341 2413 3142 3214 4123
1342 1423 2143 2314 3124
1243 1324 2134
1234
(2, 2, 3, 3)
(2, 2, 4, 3)
(2, 2, 4, 4)
(3, 2, 4, 4)
(3, 3, 4, 4)
(1, 1, 2, 2)
(1, 1, 3, 2)
(1, 1, 3, 3)
(2, 1, 3, 3)

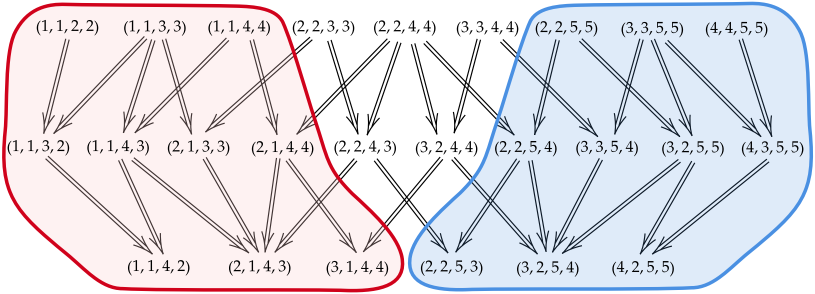
Proof of the exhaustivity of our construction

These equivalence classes are ordered by implications: if $a_1 \leq a_2$, $i_1 \geq i_2$, $j_1 \leq j_2$ and $b_1 \geq b_2$, then we have

$$\left[\overline{(a_1, i_1, j_1, b_1)} \subset E\right] \implies \left[\overline{(a_2, i_2, j_2, b_2)} \subset E\right].$$

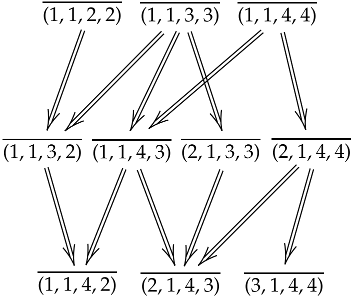


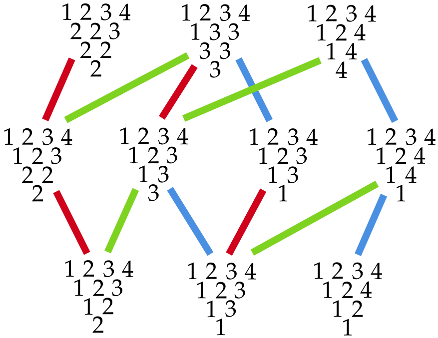
We distinguish two types of equivalence classes of edges, *left edges* and *right edges*:



For all $i < j < k$, we have $\overline{(i, 1, k-1, j)} \in E$ if and only if $\overline{(j, i+1, n, k)} \notin E$. This means that if we know the left edges of a middle order, we also know its right edges and conversely.

The posets of left and right edges are isomorphic to the poset of irreducible elements of the lattice of Gelfand-Tsetlin triangles with first row $12 \ldots n-1$:

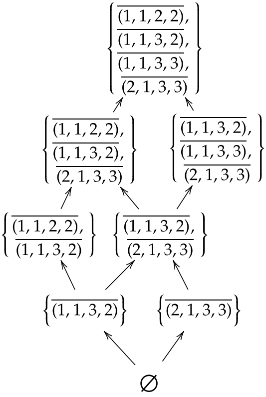


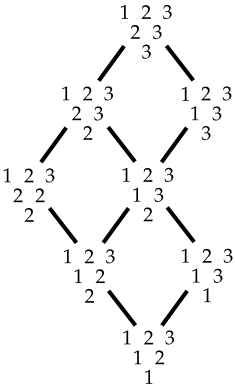


This means that sets of left (or right) edges satistfying implication relations are in bijection with Gelfand-Tsetlin triangles.

Proof of the exhaustivity of our construction

Sets of left (or right) edges satistfying implication relations are in bijection with Gelfand-Tsetlin triangles with first row $12 \ldots n-1$:





Proof of the exhaustivity of our construction

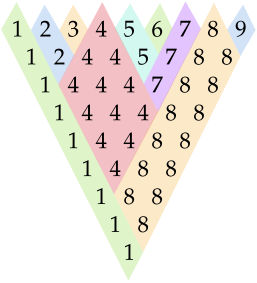
For all $i < j < k < \ell$, we have

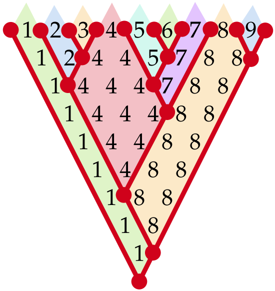
$$\overline{(i,1,k-1,j)} \wedge \overline{(k,j+1,n,\ell)} \iff \overline{(i,1,\ell-1,j)} \wedge \overline{(k,i+1,n,\ell)}$$

which translates on Gelfand-Tsetlin triangles as

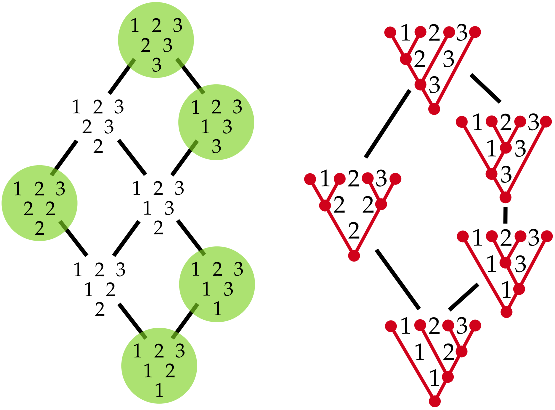
$$X_{i,j} = k \implies X_{k+i-j,k} = X_{j+n-k,j} = k.$$

This means that entries of $(X_{i,j})$ of the same value form parallelograms. Triangles with this property are in bijection with binary trees:





Hence there are C_{n-1} triangles with this property, and the induced order on these triangles is the Tamari lattice!



A small summary of our main result:

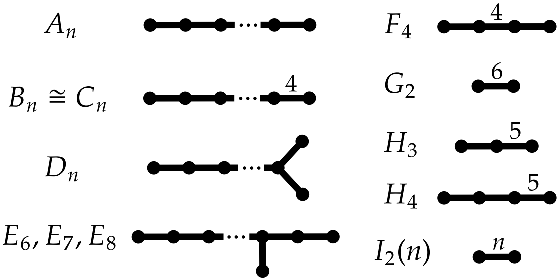
- We constructed C_{n-1} distributive lattices between the weak and Bruhat orders on $\mathfrak{S}_n$;
- We described equivalence classes of edges of the Bruhat order and implications between them;
- We used this to show there are at most C_{n-1} distributive lattices between the weak and Bruhat orders on $\mathfrak{S}_n$.

In conclusion, we have constructed all distributive lattices between the weak and Bruhat orders on $\mathfrak{S}_n$.

What about other Coxeter groups?

The symmetric groups are generalized by Coxeter groups, *i.e.* groups with a presentation $\langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$, where $m_{ij} = m_{ji}$, $m_{ii} = 1$ and $m_{ij} \geq 2$ if $i \neq j$.

Coxeter groups can be represented by their Coxeter diagrams, *i.e.* graphs whose vertices are generators, and whose edges give the order of the product of two generators.

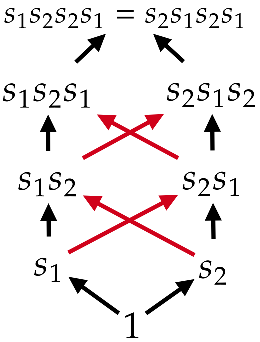


What about other Coxeter groups?

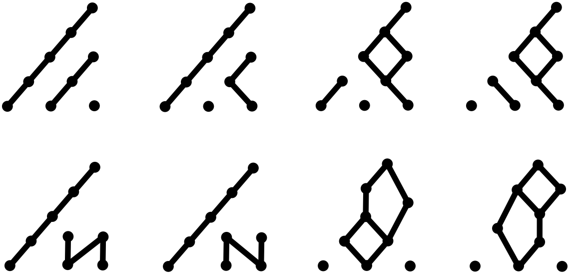
Let $\ell(u)$ be the length of any reduced expression $u = s_{i_1} \cdots s_{i_k}$. The weak and Bruhat orders are defined as the transitive closures of the following relations:

- $u \leq_R v$ if there exists $s \in S$ such that $us = v$ and $\ell(u) < \ell(v)$;
- $u \leq v$ if a reduced expression u is a subword of a reduced expression of v.

Here are the weak and Bruhat orders on B_2:



Here are the posets of irreducible elements of the middle orders on B_3:

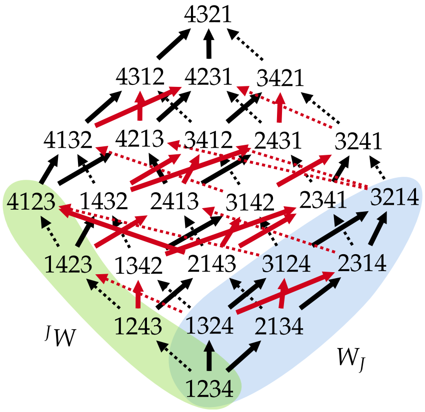


It seems difficult to classify all middle orders on Coxeter groups of other types.

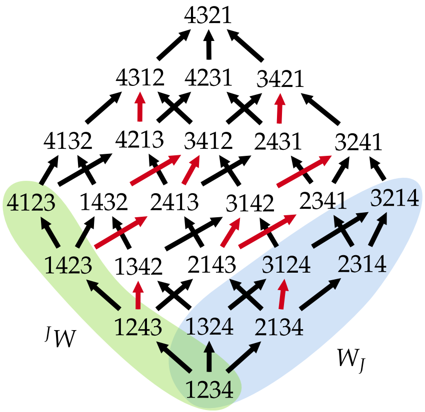
Let (W, S) be a Coxeter group, and W_J the subgroup of W generated by $J \subset S$. We have the inclusions of orders

$$(W, \leq_R) \subset (W_J, \leq_R) \times ({}^J W, \leq_R) \subset (W_J, \leq) \times ({}^J W, \leq) \subset (W, \leq).$$

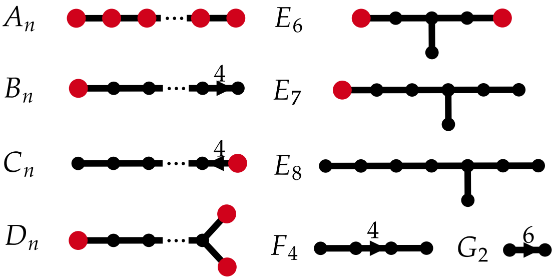
where ${}^J W$ is the set of minimal elements of cosets in $W_J \backslash W$.



Since $(W, \leq_R) \subset (W_J, \leq_R) \times ({}^J W, \leq_R) \subset (W_J, \leq) \times ({}^J W, \leq) \subset (W, \leq)$, if we have middle orders on W_J and ${}^J W$, their product is isomorphic to a middle order on W:



For some Coxeter groups W and $J \subset S$, the weak and Bruhat orders on ${}^J W$ coincide and are distributive lattices. It is the case when W is a Weyl group and $J = S \setminus \{s\}$ where s a minuscule root:

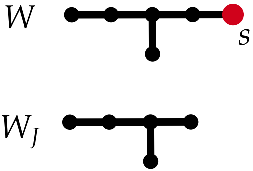


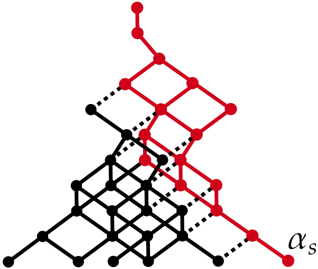
Minuscule middle orders

If W is a Weyl group, s a minuscule root and $J = S \setminus \{s\}$, the weak and Bruhat orders on ${}^JW$ are isomorphic to $\mathrm{Low}(\mathcal{P}_s)$, where $\mathcal{P}_s$ is the upper set of α_s in the root poset.

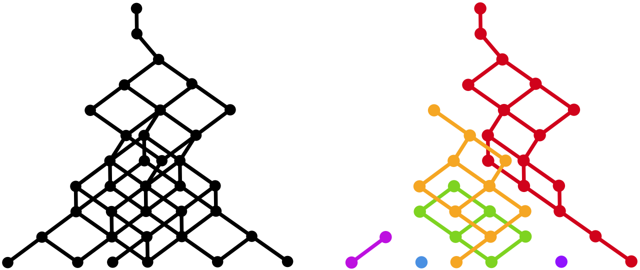
(The root poset generalizes the triangular poset we used for $\mathfrak{S}_n$.)

An example of a quotient of $W = E_6$ by $J = S \setminus \{s\}$:





By recursingly quotient by W_J where $J = S \setminus \{s\}$ and s is a minuscule root, we construct *minuscule middle orders* whose irreducible posets are partitions of the root poset into minuscule posets $\mathcal{P}_s$.



Let $\mathcal{P}$ be a finite poset, $F_m(\mathcal{P}, q)$ is the polynomial in q giving the graduation of $\mathrm{Low}(\mathcal{P} \times [m])$ where $[m]$ is a chain with m elements.
($F_m(\mathcal{P}, 1)$ gives the number of multichains of $\mathrm{Low}(\mathcal{P})$ with m elements.)

Proposition

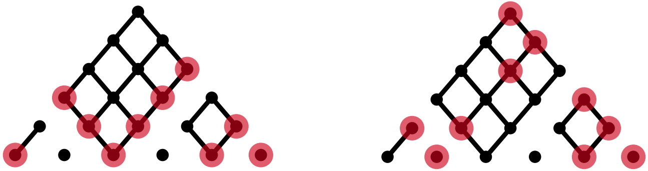
Let $\mathbb{L}$ be a minuscule middle order on W with irreducibles poset $\mathcal{R}$, we have

$$F_m(\mathcal{R}, q) = \prod_{\alpha \in \Phi^+} \frac{1 - q^{m+r_\alpha}}{1 - q^{r_\alpha}}$$

where $r_\alpha - 1$ is the rank of α in the root poset.

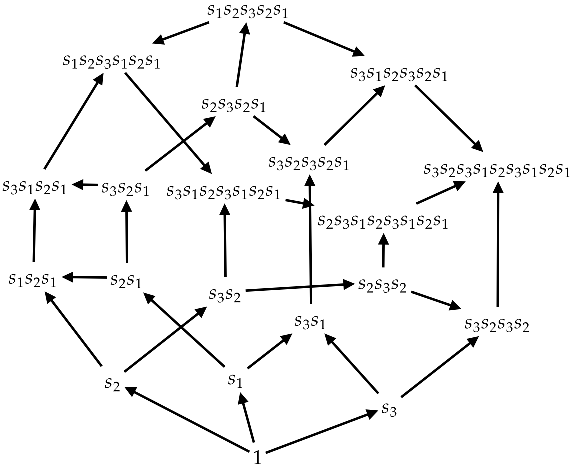
Minuscule middle orders are self-dual, and the number of self-dual ideals of $\mathcal{R} \times [m]$ is equal to $F_m(\mathcal{R}, -1)$.

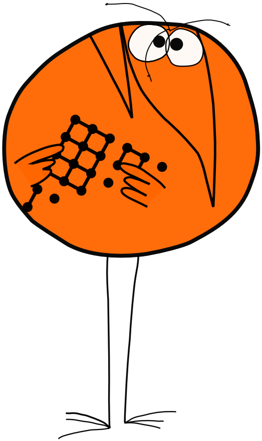
We constructed a bijection between inversion sets of permutations and ideals of rectangulations of the root poset. What are the inversion sets which are also ideals of a given rectangulation ?



- They are counted by Catalan numbers;
- When ordered by inclusion, they form a semidistributive and extremal lattice;
- This lattice is (conjecturally) the 1-skeleton of a polytope.

This can be generalized to minuscule middle orders of other types!





THANKS FOR YOUR ATTENTION!